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Volume 1

MATHEMATICS FOR JEE (MAIN & ADVANCED)

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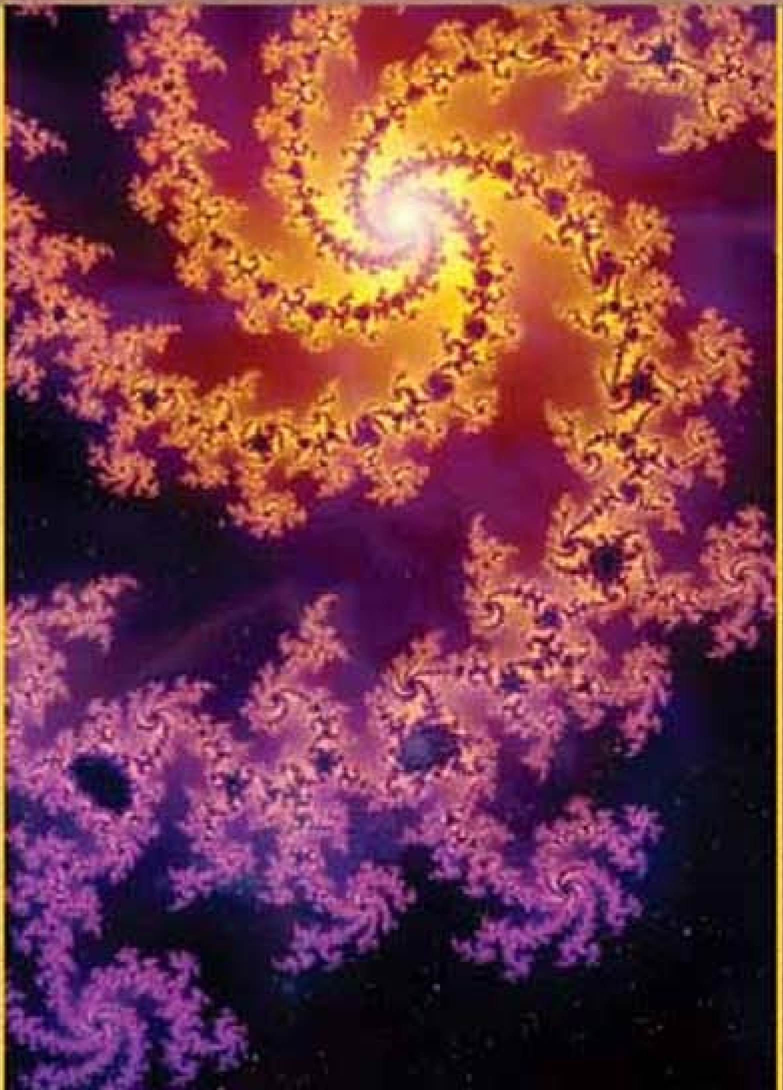

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Dr. U M SWAMY

Mathematics for IIT-JEE

ALGEBRA

VOL. 1

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Dedication

Dedicated to
my mother
Smt. Ganti Balamma
for her untiring efforts to bring up the family to a respectable stage in the society after our father's premature demise.

Dr. G. S. N. Murti

Acknowledgments

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1. Mr. U. V. Chalapati Rao, M. Sc. of erstwhile Gangadhar Tutorials, a pioneer in IIT-JEE coaching, for giving him the opportunity to teach for IIT coaching in 1985.
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Dr. U. M. Swamy would like to thank the following:

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2. The co-author Dr. G. S. N. Murti for his collaboration in this work.

Features and Benefits at a Glance

Feature	Benefit to student
Chapter Opener	Peaks the student's interest with the chapter opening vignette, definitions of the topic, and contents of the chapter.
Clear, Concise, and Inviting Writing Style, Tone and Layout	Students are able to **Read** this book, which reduces math anxiety and encourages student success.
Theory and Applications	Unlike other books that provide very less or no theory, here theory is well matched with solved examples.
Theorems	Relevant theorems are provided along with proofs to emphasize conceptual understanding.
Solved Examples	Topics are followed by solved examples for students to practice and understand the concept learned.
Examples	Wherever required, examples are provided to aid understanding of definitions and theorems.
Quick Look	Formulae/concepts that do not require extensive thought but can be looked at the last moment.
Try It Out	Practice problems for students in between the chapter.
Worked Out Problems	Based on IIT-JEE pattern problems are presented in the form of **Single Correct Choice Type Questions** **Multiple Correct Choice Type Questions** **Matrix-Match Type Questions** **Comprehension-Type Questions** **Assertion–Reasoning Type Questions** **Integer Answer Type Questions** In-depth solutions are provided to all problems for students to understand the logic behind.
Summary	Key formulae, ideas and theorems are presented in this section in each chapter.
Exercises	Offer self-assessment. The questions are divided into subsections as per requirements of IIT-JEE.
Answers	Answers are provided for all exercise questions for student's to validate their solution.

Note to the Students

The IIT-JEE is one of the hardest exams to crack for students, for a very simple reason – concepts cannot be learned by rote, they have to be absorbed, and IIT believes in strong concepts. Each question in the IIT-JEE entrance exam is meant to push the analytical ability of the student to its limit. That is why the questions are called brainteasers!

Students find Mathematics the most difficult part of IIT-JEE. We understand that it is difficult to get students to love mathematics, but one can get students to love succeeding at mathematics. In order to accomplish this goal, the book has been written in clear, concise, and inviting writing style. It can be used as a self-study text as theory is well supplemented with examples and solved examples. Wherever required, figures have been provided for clear understanding.

If you take full advantage of the unique features and elements of this textbook, we believe that your experience will be fulfilling and enjoyable. Let's walk through some of the special book features that will help you in your efforts to crack IIT-JEE.

To crack mathematics paper for IIT-JEE the five things to remember are:

1. Understanding the concepts

2. Proper applications of concepts

3. Practice

4. Speed

5. Accuracy

About the Cover Picture

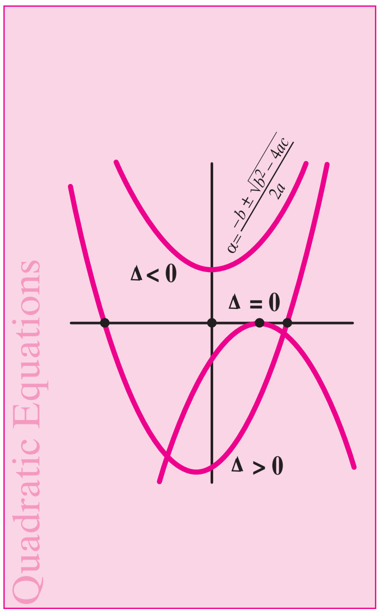
The **Mandelbrot set** is a mathematical set of points in the complex plane, the boundary of which forms a fractal. It is the set of complex values of c for which the orbit of 0 under iteration of the complex quadratic polynomial $z_{n+1} = z_n^2 + c$ remains bounded. The Mandelbrot set is named after **Benoît Mandelbrot**, who studied and popularized it.

A. PEDAGOGY

CHAPTER OPENER

Each chapter starts with an opening vignette, definition of the topic, and contents of the chapter that give you an overview of the chapter to help you see the big picture.

Quadratic Equations **4**



Contents

A polynomial equation of the second degree having the general form

$$ax^2 + bx + c = 0$$

is called a **quadratic equation**. Here x represents a variable, and a, b, and c, constants, with $a \neq 0$. The constants a, b, and c are called, respectively, the quadratic coefficient, the linear coefficient and the constant term or the free term.

The term "quadratic" comes from *quadratus*, which is the Latin word for "square". Quadratic equations can be solved by factoring, completing the square, graphing, Newton's method, and using the **quadratic formula** (explained in the chapter).

CLEAR, CONCISE, AND INVITING WRITING

Special attention has been paid to present an engaging, clear, precise narrative in the layout that is easy to use and designed to reduce math anxiety students may have.

DEFINITIONS

Every new topic or concept starts with defining the concept for students. Related examples to aid the understanding follow the definition.

4.1 | Quadratic Expressions and Equations

In this section, we discuss quadratic expressions and equations and their roots. Also, we derive various properties of the roots of quadratic equations and their relationships with the coefficients.

DEFINITION 4.1 A polynomial of the form $ax^2 + bx + c$, where a, b and c are real or complex numbers and $a \neq 0$, is called a ***quadratic expression*** in the variable x. In other words, a polynomial $f(x)$ of degree two over the set of complex numbers is called a *quadratic expression*. We often write $f(x) \equiv ax^2 + bx + c$ to denote a quadratic expression and this is known as the ***standard form***. In this case, a and b are called the coefficients of x^2 and x, respectively, and c is called the constant term. The term ax^2 is called the ***quadratic term*** and bx is called the ***linear term***.

DEFINITION 4.2 If $f(x) \equiv ax^2 + bx + c$ is a quadratic expression and α is a complex number, then we write $f(\alpha)$ for $a\alpha^2 + b\alpha + c$. If $f(\alpha) = 0$, then α is called a ***zero*** of the quadratic expression $f(x)$.

Examples

(1) Let $f(x) \equiv x^2 - 5x - 6$. Then $f(x)$ is a quadratic expression and 6 and -1 are zeros of $f(x)$.

(2) Let $f(x) \equiv x^2 + 1$. Then $f(x)$ is a quadratic expression and i and $-i$ are zeros of $f(x)$.

(3) Let $f(x) \equiv 2x^2 - ix + 1$ be a quadratic expression. In this case i and $-i/2$ are zeros of $f(x)$.

(4) The expression $x^2 + x$ is a quadratic expression and 0 and -1 are zeros of $x^2 + x$.

DEFINITION 4.3 If $f(x)$ is a quadratic expression, then $f(x) = 0$ is called a ***quadratic equation***. If α is a zero of $f(x)$, then α is called a ***root*** or a ***solution*** of the quadratic equation $f(x) = 0$. In other words, if $f(x) \equiv ax^2 + bx + c$, $a \neq 0$, then a complex number α is said to be a root or a solution of $f(x) = 0$, if $a\alpha^2 + b\alpha + c = 0$. The zeros of the quadratic expression $f(x)$ are same as the roots or solutions of the quadratic equation $f(x) = 0$. Note that α is a zero of $f(x)$ if and only if $x - \alpha$ is a factor of $f(x)$.

Examples

(1) 0 and $-i$ are the roots of $x^2 + ix = 0$.

(2) 2 is the only root of $x^2 - 4x + 4 = 0$.

(3) i and $-i$ are the roots of $x^2 + 1 = 0$.

(4) i is the only root of $x^2 - 2ix - 1 = 0$.

Example 4.1

Find the quadratic equation whose roots are 2 and $-i$.

Solution: The required quadratic expression is

$$(x-2)[x-(-i)] = (x-2)(x+i) = x^2 + (i-2)x - 2i$$

Hence the equation is $x^2 + (i-2)x - 2i = 0$.

Example 4.2

Find the quadratic equation whose roots are $1+i$ and $1-i$ and in which the coefficient of x^2 is 3.

Solution: The required quadratic expression is

$$\begin{aligned}3[x-(1+i)](x-(1-i)) &= 3[(x-1)-i)][(x-1)+i]\\ &= 3[(x-1)^2+1]\\ &= 3x^2-6x+6\end{aligned}$$

Hence the equation is $3x^2 - 6x + 6 = 0$.

Example 4.3

If α and β are roots of the quadratic equation $ax^2 + bx + c = 0$ and z is any complex number, then find the quadratic equation whose roots are $z\alpha$ and $z\beta$.

Solution: We have

$$\alpha + \beta = \frac{-b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a}$$

The equation whose roots are $z\alpha$ and $z\beta$ is

$$\begin{aligned}0 &= (x-z\alpha)(x-z\beta)\\ &= x^2-(z\alpha+z\beta)x+z\alpha\times z\beta\\ &= x^2+z[-(\alpha+\beta)]x+z^2\alpha\beta\\ &= x^2+z\left(\frac{b}{a}\right)x+z^2\frac{c}{a}\end{aligned}$$

that is,

$$ax^2 + zbx + z^2c = 0$$

Example 4.4

If α and β are the roots of a quadratic equation $ax^2 + bx + c = 0$, then find the quadratic equation whose roots are $\alpha + z$ and $\beta + z$, where z is any given complex number.

Solution: We have

Therefore, the required equation is

$$\begin{aligned}0 &= a[x-(\alpha+z)]\times[x-(\beta+z)]\\ &= ax^2+a[-(\alpha+z)-(\beta+z)]x+a(\alpha+z)(\beta+z)\\ &= ax^2+a\left(\frac{b}{a}-2z\right)x+a\left(\frac{c}{a}-\frac{b}{a}z+z^2\right)\end{aligned}$$

EXAMPLES

Examples pose a specific problem using concepts already presented and then work through the solution. These serve to enhance the students' understanding of the subject matter.

THEOREMS

Relevant theorems are provided along with proofs to emphasize conceptual understanding rather than rote learning.

THEOREM 4.5 If a, b and c are real numbers and $a \neq 0$, then $(4ac - b^2)/4a$ is the maximum or minimum value of quadratic equation of $f(x) \equiv ax^2 + bx + c$ according as $a < 0$ or $a > 0$, respectively.

PROOF We have

$$f(x) \equiv ax^2+bx+c \equiv a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)$$

$$\equiv a\left[\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right] \equiv a\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a}$$

If $a < 0$, then

$$f(x) \le \frac{4ac-b^2}{4a} = f\left(\frac{-b}{2a}\right) \quad \text{for all } x \in \mathbb{R}$$

Hence $(4ac - b^2)/4a$ is the maximum value of $f(x)$.
If $a > 0$, then

$$f\left(\frac{-b}{2a}\right) = \frac{4ac-b^2}{4a} \le f(x) \quad \text{for all } x \in \mathbb{R}$$

Hence $(4ac - b^2)/4a$ is the minimum value of $f(x)$. ■

QUICK LOOK 2

Let $f(x) \equiv ax^2 + bx + c = 0$ be a quadratic equation and α and β be its roots. Then the following hold good.

1. $f(x - z) = 0$ is an equation whose roots are $\alpha + z$ and $\beta + z$, for any given complex number z.
2. $f(x/z) = 0$ is an equation whose roots are $z\alpha$ and $z\beta$ for any non-zero complex number z.
3. $f(-x) = 0$ is an equation whose roots are $-\alpha$ and $-\beta$.
4. If $\alpha\beta \neq 0$ and $c \neq 0$, $f(1/x) = 0$ is an equation whose roots are $1/\alpha$ and $1/\beta$.
5. For any complex numbers z_1 and z_2 with $z_1 \neq 0$, $f[(x - z_2)/z_1] = 0$ is an equation whose roots are $z_1\alpha + z_2$ and $z_1\beta + z_2$.

QUICK LOOK

Some important formulae and concepts that do not require exhaustive explanation, but their mention is important, are presented in this section. These are marked with a **magnifying glass.**

TRY IT OUT

Within each chapter the students would find problems to reinforce and check their understanding. This would help build confidence as one progresses in the chapter. These are marked with a **pointed finger**.

Try it out Verify the following properties:

1. $((a, b) + (c, d)) + (s, t) = (a, b) + ((c, d) + (s, t))$
2. $(a, b) + (c, d) = (c, d) + (a, b)$
3. $(a, b) + (0, 0) = (a, b)$
4. $(a, b) + (-a, -b) = (0, 0)$
5. $(a, b) + (c, d) = (s, t) \Leftrightarrow (a, b) = (s, t) - (c, d)$
   $\Leftrightarrow (c, d) = (s, t) - (a, b)$

DEFINITION 3.2 For any complex numbers (a, b) and (c, d), let us define

$$(a, b)\cdot(c, d) = (ac - bd, ad + bc)$$

This is called the ***product*** of (a, b) and (c, d) and the process of taking products is called ***multiplication***.

Try it out Verify the following properties for any complex numbers (a, b), (c, d) and (s, t).

1. $[(a, b)\cdot(c, d)]\cdot(s, t) = (a, b)\cdot[(c, d)\cdot(s, t)]$
2. $(a, b)\cdot(c, d) = (c, d)\cdot(a, b)$
3. $(a, b)\cdot[(c, d) + (s, t)] = (a, b)\cdot(c, d) + (a, b)\cdot(s, t)$
4. $(a, b)\cdot(1, 0) = (a, b)$
5. $(a, 0)\cdot(c, d) = (ac, ad)$
6. $(a, 0)\cdot(c, 0) = (ac, 0)$
7. $(a, 0) + (c, 0) = (a + c, 0)$

SUMMARY

At the end of every chapter, a summary is presented that organizes the key formulae and theorems in an easy to use layout. The related topics are indicated so that one can quickly summarize a chapter.

SUMMARY

4.1 ***Quadratic expressions and equations:*** If a, b, c are real numbers and $a \neq 0$, the expression of the form $ax^2 + bx + c$ is called quadratic expression and $ax^2 + bx + c = 0$ is called quadratic equation.

4.2 Let $f(x) \equiv ax^2 + bx + c$ be a quadratic expression and α be a real (complex) number. Then we write $f(\alpha)$ for $a\alpha^2 + b\alpha + c$. If $f(\alpha) = 0$, the α is called a zero of $f(x)$ or a root of the equation $f(x) = 0$.

4.3 ***Roots:*** The roots of the quadratic equation $ax^2 + bx + c = 0$ are

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

4.4 ***Discriminant:*** $b^2 - 4ac$ is called the discriminant of the quadratic expression (equation) $ax^2 + bx + c = 0$.

4.5 ***Sum and product of the roots:*** If α and β are roots of the equation $ax^2 + bx + c = 0$, then

$$\alpha + \beta = \frac{-b}{2a} \quad \text{and} \quad \alpha\beta = \frac{c}{a}$$

4.6 Let $ax^2 + bx + c = 0$ be a quadratic equation and $\Delta = b^2 - 4ac$ be its discriminant. Then the following hold good.

(1) Roots are equal $\Leftrightarrow \Delta = 0$ (i.e., $b^2 = 4ac$).

(2) Roots are real and distinct $\Leftrightarrow \Delta > 0$.

(3) Roots are non-real complex (i.e., imaginary) $\Leftrightarrow \Delta > 0$.

4.7 ***Theorem:*** Two quadratic equations $ax^2 + bx + c = 0$ and $a'x^2 + b'x + c' = 0$ have same roots if and only if the triples (a, b, c) and (a', b', c') are proportional and in this case

$$ax^2 + bx + c = \frac{a}{a'}(a'x^2 + b'x + c')$$

4.8 ***Cube roots of unity:*** Roots of the equation $x^3 - 1 = 0$ are called cube roots of unity and they are

$$1, \quad \frac{-1}{2} \pm i\frac{\sqrt{3}}{2}$$

$-1/2 \pm i\sqrt{3}/2$ are called non-real cube roots of unity. Further each of them is the square of the other and the sum of the two non-real cube roots of unity is equal to -1. If $w \neq 1$ is a cube root of unity and n is any positive integer, then $1 + w^n + w^{2n}$ is equal to 3 or 0 according as n is a multiple of 3 or not.

4.9 ***Maximum and minimum values:*** If $f(x) \equiv ax^2 + bx + c$ and $a \neq 0$, then

$$f\left(\frac{-b}{2a}\right) = \frac{4ac - b^2}{4a}$$

is the maximum or minimum value of f according as $a < 0$ or $a > 0$.

4.10 ***Theorems (change of sign of $ax^2 + bx + c$):*** Let $f(x) \equiv ax^2 + bx + c$ where a, b, c are real and $a \neq 0$. If α and β are real roots of $f(x) = 0$ and $\alpha < \beta$, then

(1) (i) $f(x)$ and a (the coefficient of x^2) have the same sign for all $x < \alpha$ or $x > \beta$.

(ii) $f(x)$ and a will have opposite signs for all x such that $\alpha < x < \beta$.

(2) If $f(x) = 0$ has imaginary roots, then $f(x)$ and a will have the same sign for all real values of x.

4.11 If $f(x)$ is a quadratic expression and $f(p)f(q) < 0$ for some real numbers p and q, then the quadratic equation $f(x) = 0$ has a root in between p and q.

B. WORKED-OUT PROBLEMS AND ASSESSMENT – AS PER IIT-JEE PATTERN

Mere theory is not enough. It is also important to practice and test what has been proved theoretically. The worked-out problems and exercise at the end of each chapter are in resonance with the IIT-JEE paper pattern. Keeping the IIT-JEE pattern in mind, the worked-out problems and exercises have been divided into:

1. **Single Correct Choice Type Questions**
2. **Multiple Correct Choice Type Questions**
3. **Matrix-Match Type Questions**
4. **Comprehension-Type Questions**
5. **Assertion–Reasoning Type Questions**
6. **Integer Answer Type Questions**

WORKED-OUT PROBLEMS

In-depth solutions are provided to all worked-out problems for students to understand the logic behind and formula used.

WORKED-OUT PROBLEMS

Single Correct Choice Type Questions

1. If the equations

$$x^2 + ax + 1 = 0 \quad \text{and} \quad x^2 - x - a = 0$$

have a real common root, then the value of a is

(A) 0 (B) 1 (C) –1 (D) 2

Solution: Let α be a real common root. Then

$$\alpha^2 + a\alpha + 1 = 0$$

$$\alpha^2 - \alpha - a = 0$$

Therefore

$$\alpha(a+1) + (a+1) = 0$$

$$(a+1)(\alpha+1) = 0$$

If $a = -1$, then the equations are same and also cannot have a real root. Therefore $a + 1 \neq 0$ and hence $\alpha = -1$, so that $a = 2$.

Answer: (D)

$$m < 0 \quad \text{and} \quad 3m^2 + 4m - 4 > 0$$

$$\Rightarrow m < 0 \quad \text{and} \quad (3m-2)(m+2) > 0$$

This gives $m < -2$ and so

$$x^2 - 5x + 6 < 0 \Rightarrow (x-2)(x-3) < 0 \Rightarrow x \in (2, 3)$$

Answer: (C)

4. If p is prime number and both the roots of the equation $x^2 + px - (444)p = 0$ are integers, then p is equal to

(A) 2 (B) 3 (C) 31 (D) 37

Solution: Suppose the roots of $x^2 + px - (444)p = 0$ are integers. Then the discriminant

$$p^2 + 4(444)p = p\{p + 4 \times (444)\}$$

must be a perfect square. Therefore p divides $p + 4 \times (444)$. This implies

$$p \text{ divides } 4 \times (444) = 2^4 \times 3 \times 37$$

SINGLE CORRECT CHOICE TYPE QUESTIONS

These are the regular multiple choice questions with four choices provided. Only one among the four choices will be the correct answer.

MULTIPLE CORRECT CHOICE TYPE QUESTIONS

Multiple correct choice type questions have four choices provided, but one or more of the choices provided may be correct.

Multiple Correct Choice Type Questions

1. Suppose a and b are integers and $b \neq -1$. If the quadratic equation $x^2 + ax + b + 1 = 0$ has a positive integer root, then

(A) the other root is also a positive integer
(B) the other root is an integer
(C) $a^2 + b^2$ is a prime number
(D) $a^2 + b^2$ has a factor other than 1 and itself

Solution: Let α and β be the roots and α be a positive integer. Then

$$\alpha + \beta = -a \quad \text{and} \quad \alpha\beta = b + 1$$

$\beta = -a - \alpha$ implies β is an integer and

$$\begin{aligned} a^2 + b^2 &= (\alpha+\beta)^2 + (\alpha\beta - 1)^2 \\ &= \alpha^2 + \beta^2 + \alpha^2\beta^2 + 1 \\ &= (\alpha^2+1)(\beta^2+1) \end{aligned}$$

Since $\alpha^2 + 1 > 1$ and $\beta^2 + 1 > 1$, it follows that $\alpha^2 + 1$ is a factor of $a^2 + b^2$ other than 1 and itself.

Answers: (B), (D)

Solution:

Case 1: Suppose b is even, that is, $b = 2m$. Then $b^2 - 4ac = 4(m^2 - ac) = 4k$.

Case 2: Suppose b is odd, that is, $b = 2m - 1$. Then

$$\begin{aligned} b^2 - 4ac &= (2m-1)^2 - 4ac \\ &= 4m^2 + 4m + 1 - 4ac \\ &= 4(m^2 + m - ac) + 1 \\ &= 4k + 1 \end{aligned}$$

Answers: (A), (B)

3. If a and b are roots of the equation $x^2 + ax + b = 0$, then

(A) $a = 0, b = 1$ (B) $a = 0 = b$
(C) $a = 1, b = -1$ (D) $a = 1, b = -2$

Solution: If $a + b = -a$ and $ab = b$, then $a = 0 = b$ or $a = 1$, $b = -2$.

Answers: (B), (D)

MATRIX-MATCH TYPE QUESTIONS

These questions are the regular "Match the Following" variety. Two columns each containing 4 subdivisions or first column with four subdivisions and second column with more subdivisions are given and the student should match elements of column I to that of column II. There can be one or more matches.

Matrix-Match Type Questions

1. Match the items in Column I with those in Column II

Column I	*Column II*
(A) If $z = x + iy$, $z^{1/3} = a - ib$ and $\frac{x}{a} - \frac{y}{b} = \lambda(a^2 - b^2)$, then λ is	(p) 10
	(q) 14
(B) If $\lvert z - i\rvert < 1$, then the value of $\lvert z + 12 - 6i\rvert$ is less than	(r) 1
(C) If $\lvert z_1\rvert = 1$ and $\lvert z_2\rvert = 2$, then $\lvert z_1 + z_2\rvert^2 + \lvert z_1 - z_2\rvert^2$ is equal to	(s) 4
(D) If $z = 1 + i$, then $4(z^4 - 4z^3 + 7z^2 - 6z + 3)$ is equal to	(t) 5

Solution:

(A) $x + iy = z = (a - ib)^3 = a^3 - 3a^2bi + 3a(ib)^2 - i^3b^3$

$$= (a^3 - 3ab^2) + i(b^3 - 3a^2b)$$

Comparing the real parts we get

$$x = a^3 - 3ab^2 = a(a^2 - 3b^2)$$

$$\frac{x}{a} = a^2 - 3b^2$$

Comparing the imaginary parts we get

$$z^4 - 4z^3 + 7z^2 - 6z + 3 = z^2 - 2z + 3$$

$$= (z-1)^2 + 2 = i^2 + 2 = 1$$

$$4(z^4 - 4z^3 + 7z^2 - 6z + 3) = 4$$

Answer: (D) → (s)

2. Match the items in Column I with those in Column II. In the following, $w \neq 1$ is a cube root of unity.

Column I	*Column II*
(A) The value of the determinant $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1-w^2 & w^2 \\ 1 & w^2 & w^4 \end{vmatrix}$ is	(p) $3w(1-w)$
	(q) $3w(w-1)$
(B) The value of $4 + 5w^{2002} + 3w^{2009}$ is	(r) $-i\sqrt{3}$
(C) The value of the determinant $\begin{vmatrix} 1 & 1+i+w^2 & w^2 \\ 1-i & -1 & w^2-1 \\ -i & -i+w^2+1 & -1 \end{vmatrix}$ is	(s) $i\sqrt{3}$
(D) $w^{2n} + w^n + 1$ (n is a positive integer	(t) 0

COMPREHENSION-TYPE QUESTIONS

Comprehension-type questions consist of a small passage, followed by three multiple choice questions. The questions are of single correct answer type.

Comprehension-Type Questions

1. Passage: 4 Indians, 3 Americans and 2 Britishers are to be arranged around a round table. Answer the following questions.

(i) The number of ways of arranging them is

(A) 9! (B) $\frac{1}{2}9!$ (C) 8! (D) $\frac{1}{2}8!$

(ii) The number of ways arranging them so that the two Britishers should never come together is

(A) $7! \times 2!$ (B) $6! \times 2!$ (C) 7! (D) $6!\,{}^6P_2$

(iii) The number of ways of arranging them so that the three Americans should sit together is

(A) $7! \times 3!$ (B) $6! \times 3!$ (C) $6!\,{}^6P_3$ (D) $6!\,{}^7P_3$

Solution:

(i) n distinct objects can be arranged around a circular table in $(n-1)!$ ways. Therefore the number of ways of arranging $4+3+2$ people $= 8!$.

Answer: (C)

(ii) First arrange 4 Indians and 3 Americans around a round table in 6! ways. Among the six gaps, arrange the two Britishers in ${}^6P_2$ ways. Therefore the total number of arrangements in which Britishers are separated is $6! \times {}^6P_2$.

Answer: (D)

(iii) Treating the 3 Americans as a single object, 7 (= 4 + 1 + 2) objects can be arranged cyclically in 6! ways. In each of these, Americans can be arranged among themselves in 3! ways. Therefore, the number of required arrangements is $6! \times 3!$.

(ii) The number of ways in which all the four prizes can be given to any one of the 6 students = 6. Therefore the required number of ways is $6^4 - 6 = 1290$.

Answer: (B)

(iii) Give a set of two prizes to the particular student. Then the remaining 2 can be distributed among 5 students in 5^2 ways. There are ${}^4C_2$ sets, each containing 2 prizes. Therefore the required number of ways of distributing the prizes is

$$5^2 \times {}^4C_2 = 25 \times 6 = 150$$

Answer: (C)

3. Passage: A security of 12 persons is to form from a group of 20 persons. Answer the following questions.

(i) The number of times that two particular persons are together on duty is

(A) $\frac{20!}{12!\,8!}$ (B) $\frac{18!}{10!\,8!}$ (C) $\frac{20!}{10!\,8!}$ (D) $\frac{20!}{10!\,10!}$

(ii) The number of times that three particular persons are together on duty is

(A) $\frac{17!}{8!\,9!}$ (B) $\frac{17!}{8!\,8!}$ (C) $\frac{20!}{17!\,3!}$ (D) $\frac{20!}{9!\,8!}$

(iii) The number of ways of selecting 12 guards such that two particular guards are out of duty and three particular guards are together on duty is

(A) $\frac{(20)!}{(15)!\,5!}$ (B) $\frac{(18)!}{9!\,3!}$ (C) $\frac{(15)!}{9!\,6!}$ (D) $\frac{(15)!}{5!\,(10)!}$

ASSERTION–REASONING TYPE QUESTIONS

These questions check the analytical and reasoning skills of the students. Two statements are provided – Statement I and Statement II. The student is expected to verify if (a) both statements are true and if both are true, verify if statement I follows from statement II; (b) both statements are true and if both are true, verify if statement II is not the correct reasoning for statement I; (c), (d) which of the statements is untrue.

Assertion–Reasoning Type Questions

In the following set of questions, a Statement I is given and a corresponding Statement II is given just below it. Mark the correct answer as:

(A) Both I and II are true and II is a correct reason for I
(B) Both I and II are true and II is not a correct reason for I
(C) I is true, but II is false
(D) I is false, but II is true

1. Statement I: Let a, b and c be real numbers and $a \neq 0$. If $4a + 3b + 2c$ and a have same sign, then not both the roots of the equation $ax^2 + bx + c = 0$ belong to the open interval $(1, 2)$.

Statement II: A quadratic equation $f(x) = 0$ will have a root in the interval (a, b) if $f(a)\,f(b) < 0$.

Solution: Let $f(x) = px^2 + qx + r$. If $f(a)$ and $f(b)$ are of opposite sign, the curve (parabola) $y = f(x)$ must intersect x-axis at some point. This implies that $f(x)$ has a root in (a, b). Therefore, the Statement II is true.

Let α and β be roots of $ax^2 + bx + c = 0$. Then,

$$\alpha + \beta = \frac{-b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a}$$

By hypothesis,

$$\frac{4a + 3b + 2c}{a} > 0$$

2. Statement I: If $P(x) = ax^2 + bx + c$ and $Q(x) = -ax^2 + dx + c$, where $ac \neq 0$, then the equation $P(x)\,Q(x) = 0$ has at least two real roots.

Statement II: A quadratic equation with real coefficients has real roots if and only if the discriminant is greater than or equal to zero.

Solution: Let $px^2 + qx + r = 0$ be a quadratic equation. The roots are

$$\frac{-q \pm \sqrt{q^2 - 4pr}}{2p}$$

These are real $\Leftrightarrow q^2 - 4pr \geq 0$. Therefore Statement II is true.

In Statement I, $ac \neq 0$. Therefore $ac > 0$ or $ac < 0$. If $ac < 0$, then $b^2 - 4ac > 0$, so that $P(x) = 0$ has two real roots. If $ac > 0$, then $d^2 + 4ac > 0$ so that $Q(x) = 0$ has two real roots. Further, the roots of $P(x) = 0$ and $Q(x) = 0$ are also the roots of $P(x)Q(x) = 0$. Therefore, Statement I is true and Statement II is a correct reason for Statement I.

Answer: (A)

3. Statement I: If a, b and c are real, then the roots of the equation $(x-a)(x-b) + (x-b)(x-c) + (x-c)(x-a) = 0$ are imaginary.

Statement II: If p, q and r are real and $p \neq 0$, then the roots of the equation $px^2 + qx + r = 0$ are real or imaginary according as $q^2 - 4pr \geq 0$ or $q^2 - 4pr < 0$.

INTEGER-TYPE QUESTIONS

The questions in this section are numerical problems for which no choices are provided. The students re required to find the exact answers to numerical problems and enter the same in OMR sheets. Answers can be one-digit or two-digit numerals.

Integer Answer Type Questions

The answer to each of the questions in this section is a non-negative integer. The appropriate bubbles below the respective question numbers have to be darkened. For example, as shown in the figure, if the correct answer to the question number Y is 246, then the bubbles under Y labeled as 2, 4, 6 are to be darkened.

X	Y	Z	W
0	0	0	0
1	1	1	1
2		2	2
3	3	3	3
4		4	4
5	5	5	5
6		6	6
7	7	7	7
8	8	8	8
9	9	9	9

2. The number of negative integer solutions of $x^2 \times 2^{x+1} + 2^{|x-3|+2} = x^2 \times 2^{|x-3|+4} + 2^{x-1}$ is _____.

3. If $(\alpha + 5i)/2$ is a root of the equation $2x^2 - 6x + k = 0$, then the value of k is _____.

4. If the equation $x^2 - 4x + \log_{1/2} a = 0$ does not have distinct real roots, then the minimum value of $1/a$ is _____.

5. If a is the greatest negative integer satisfying

$$x^2 - 4x - 77 < 0 \quad \text{and} \quad x^2 > 4$$

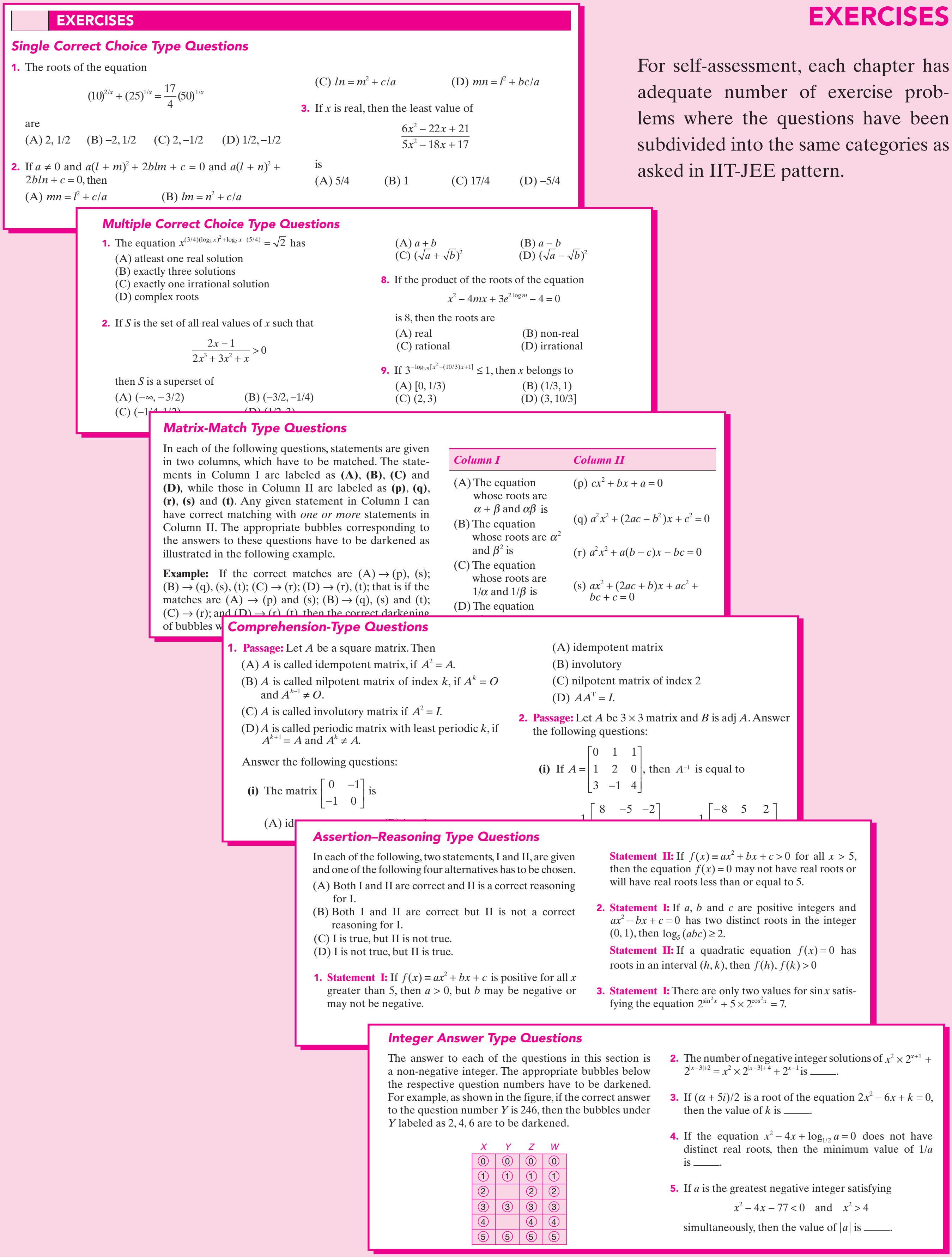
simultaneously, then the value of $|a|$ is _____.

6. The number of values of k for which the quadratic equations $(2k-5)x^2 - 4x - 15 = 0$ and $(3k-8)x^2 - 5x - 21 = 0$ have a common root is _____.

7. The number of real roots of the equation $2x^2 - 6x - 5\sqrt{x^2 - 3x - 6} = 0$ is _____.

EXERCISES

For self-assessment, each chapter has adequate number of exercise problems where the questions have been subdivided into the same categories as asked in IIT-JEE pattern.



EXERCISES

Single Correct Choice Type Questions

1. The roots of the equation

$$(10)^{2/x} + (25)^{1/x} = \frac{17}{4}(50)^{1/x}$$

are

(A) 2, 1/2 (B) −2, 1/2 (C) 2, −1/2 (D) 1/2, −1/2

2. If $a \neq 0$ and $a(l+m)^2 + 2blm + c = 0$ and $a(l+n)^2 + 2bln + c = 0$, then

(A) $mn = l^2 + c/a$ (B) $lm = n^2 + c/a$

(C) $ln = m^2 + c/a$ (D) $mn = l^2 + bc/a$

3. If x is real, then the least value of

$$\frac{6x^2 - 22x + 21}{5x^2 - 18x + 17}$$

is

(A) 5/4 (B) 1 (C) 17/4 (D) −5/4

Multiple Correct Choice Type Questions

1. The equation $x^{(3/4)(\log_2 x)^2 + \log_2 x - (5/4)} = \sqrt{2}$ has

(A) atleast one real solution

(B) exactly three solutions

(C) exactly one irrational solution

(D) complex roots

2. If S is the set of all real values of x such that

$$\frac{2x-1}{2x^3 + 3x^2 + x} > 0$$

then S is a superset of

(A) $(-\infty, -3/2)$ (B) $(-3/2, -1/4)$

(C) (−1/4, 1/2) (D) (1/2, 3)

(A) $a+b$ (B) $a-b$

(C) $(\sqrt{a}+\sqrt{b})^2$ (D) $(\sqrt{a}-\sqrt{b})^2$

8. If the product of the roots of the equation

$$x^2 - 4mx + 3e^{2\log m} - 4 = 0$$

is 8, then the roots are

(A) real (B) non-real

(C) rational (D) irrational

9. If $3^{-\log_{1/9}[x^2 - (10/3)x + 1]} \leq 1$, then x belongs to

(A) [0, 1/3) (B) (1/3, 1)

(C) (2, 3) (D) (3, 10/3]

Matrix-Match Type Questions

In each of the following questions, statements are given in two columns, which have to be matched. The statements in Column I are labeled as **(A)**, **(B)**, **(C)** and **(D)**, while those in Column II are labeled as **(p)**, **(q)**, **(r)**, **(s)** and **(t)**. Any given statement in Column I can have correct matching with *one or more* statements in Column II. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following example.

Example: If the correct matches are (A) → (p), (s); (B) → (q), (s), (t); (C) → (r); (D) → (r), (t); that is if the matches are (A) → (p) and (s); (B) → (q), (s) and (t); (C) → (r); and (D) → (r), (t), then the correct darkening of bubbles w

Column I	Column II
(A) The equation whose roots are $\alpha+\beta$ and $\alpha\beta$ is	(p) $cx^2 + bx + a = 0$
(B) The equation whose roots are α^2 and β^2 is	(q) $a^2x^2 + (2ac - b^2)x + c^2 = 0$
(C) The equation whose roots are $1/\alpha$ and $1/\beta$ is	(r) $a^2x^2 + a(b-c)x - bc = 0$
(D) The equation	(s) $ax^2 + (2ac+b)x + ac^2 + bc + c = 0$

Comprehension-Type Questions

1. **Passage:** Let A be a square matrix. Then

(A) A is called idempotent matrix, if $A^2 = A$.

(B) A is called nilpotent matrix of index k, if $A^k = O$ and $A^{k-1} \neq O$.

(C) A is called involutory matrix if $A^2 = I$.

(D) A is called periodic matrix with least periodic k, if $A^{k+1} = A$ and $A^k \neq A$.

Answer the following questions:

(i) The matrix $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ is

(A) id

(A) idempotent matrix

(B) involutory

(C) nilpotent matrix of index 2

(D) $AA^T = I$.

2. **Passage:** Let A be 3×3 matrix and B is adj A. Answer the following questions:

(i) If $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 3 & -1 & 4 \end{bmatrix}$, then A^{-1} is equal to

Assertion–Reasoning Type Questions

In each of the following, two statements, I and II, are given and one of the following four alternatives has to be chosen.

(A) Both I and II are correct and II is a correct reasoning for I.

(B) Both I and II are correct but II is not a correct reasoning for I.

(C) I is true, but II is not true.

(D) I is not true, but II is true.

1. **Statement I:** If $f(x) \equiv ax^2 + bx + c$ is positive for all x greater than 5, then $a > 0$, but b may be negative or may not be negative.

Statement II: If $f(x) \equiv ax^2 + bx + c > 0$ for all $x > 5$, then the equation $f(x) = 0$ may not have real roots or will have real roots less than or equal to 5.

2. **Statement I:** If a, b and c are positive integers and $ax^2 - bx + c = 0$ has two distinct roots in the integer (0, 1), then $\log_5(abc) \geq 2$.

Statement II: If a quadratic equation $f(x) = 0$ has roots in an interval (h, k), then $f(h), f(k) > 0$

3. **Statement I:** There are only two values for $\sin x$ satisfying the equation $2^{\sin^2 x} + 5 \times 2^{\cos^2 x} = 7$.

Integer Answer Type Questions

The answer to each of the questions in this section is a non-negative integer. The appropriate bubbles below the respective question numbers have to be darkened. For example, as shown in the figure, if the correct answer to the question number Y is 246, then the bubbles under Y labeled as 2, 4, 6 are to be darkened.

X	Y	Z	W
0	0	0	0
1	1	1	1
2		2	2
3	3	3	3
4		4	4
5	5	5	5

2. The number of negative integer solutions of $x^2 \times 2^{x+1} + 2^{|x-3|+2} = x^2 \times 2^{|x-3|+4} + 2^{x-1}$ is ______.

3. If $(\alpha + 5i)/2$ is a root of the equation $2x^2 - 6x + k = 0$, then the value of k is ______.

4. If the equation $x^2 - 4x + \log_{1/2} a = 0$ does not have distinct real roots, then the minimum value of $1/a$ is ______.

5. If a is the greatest negative integer satisfying

$$x^2 - 4x - 77 < 0 \quad \text{and} \quad x^2 > 4$$

simultaneously, then the value of $|a|$ is ______.

ANSWERS

The Answer key at the end of each chapter contains answers to all exercise problems.

ANSWERS

Single Correct Choice Type Questions

1. (D)
2. (B)
3. (C)
4. (C)
5. (A)
6. (D)
7. (D)
8. (A)
9. (D)
10. (D)
11. (C)
12. (B)
13. (A)
14. (B)
15. (C)
16. (A)
17. (A)
18. (B)
19. (B)
20. (D)
21. (C)
22. (D)
23. (A)
24. (A)
25. (C)

Multiple Correct Choice Type Questions

1. (B), (C)
2. (B), (D)
3. (B), (C)
4. (A), (B)
5. (B), (D)
6. (A), (B), (C)
7. (A), (B), (C), (D)
8. (A), (B), (C), (D)
9. (A), (B), (C), (D)
10. (B), (D)
11. (A), (B), (C)
12. (A), (B), (C), (D)
13. (A), (B)
14. (A), (B), (C), (D)
15. (A), (D)

Matrix-Match Type Questions

1. (A) → (p), (B) → (p), (C) → (r), (D) → (r)
2. (A) → (p), (B) → (q), (C) → (p), (D) → (q)
3. (A) → (q), (B) → (s), (C) → (p), (D) → (r)
4. (A) → (r), (B) → (r), (C) → (q), (D) → (p)
5. (A) → (q), (r), (s) (B) → (s), (C) → (p), (D) → (q), (s)

Comprehension-Type Question

1. (i) (B); (ii) (A); (iii) (C)
2. (i) (B); (ii) (A); (iii) (C)
3. (i) (A); (ii) (B); (iii) (A)
4. (i) (D); (ii) (C); (iii) (D)

Assertion–Reasoning Type Questions

1. (A)
2. (A)
3. (D)
4. (C)
5. (A)

Integer Answer Type Questions

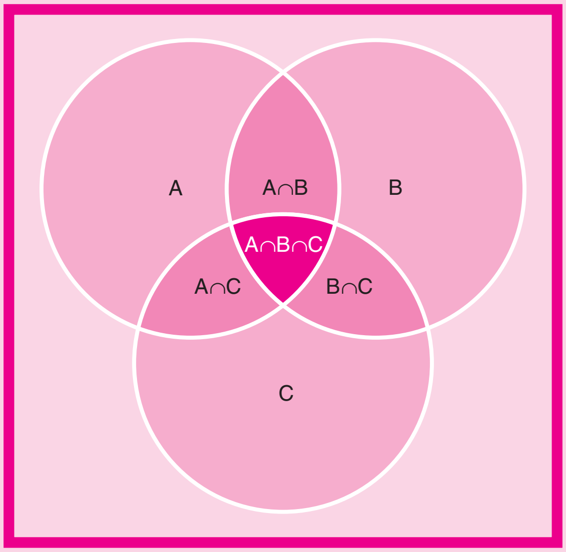
1. 2
2. 3
3. 6
4. 16
5. 0

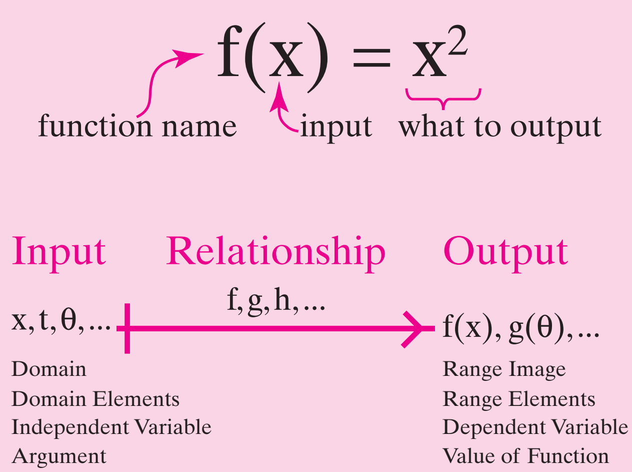
Contents

Sets, Relations and Functions

1

Sets, Relations and Functions





Contents

Sets: Any collection of well-defined objects.

Relations: For any two sets A and B, any subset of $A \times B$ is called a relation from A to B.

Functions: A relation f from a set A to a set B is called a function from A to B if for each $a \in A$, there exists unique $b \in B$ such that $(a, b) \in f$.

Mankind has been using the number concept as an abstraction without expressely formulating what, in precise terms, a number is. The first precise formulation was made by the Swiss mathematician George Cantor during the years 1874–1897 while working on number aggregates. To start with one has to realize that the abstraction that is the number "five", say, is the commonality that exists between all sets which can be put into one-to-one correspondence with the set of fingers on a normal human hand. In olden days a shepherd would carry a bag of pebbles just to say that he has that many sheep with him or, equivalently, there is a one-to-one correspondence between the pebbles in the bag and the sheep he possesses. The concept of set and the concept of one-to-one correspondence of sets were introduced by George Cantor for the first time into the world of mathematics. For a number like five or for any finite number, Cantor's approach through one-to-one correspondence of sets may appear to be a triviality. But if we turn to infinite sets, we feel the difference. First of all, what is a set? The precise mathematical definition of a set had to wait for more than three decades after Cantor's proposal: It is a collection of objects and several paradoxes that followed the Cantor's viewpoint.

1.1 | Sets: Definition and Examples

For our present discussion we can be content with what most introductory mathematics texts are content with: the intuitive concept of a set. *A set is just a well-defined collection of objects*, well-defined in the sense that given any object in the world, one can say this much: Either the object belongs to the set or it does not. It cannot happen both ways. Let us consider a counterexample first and an example of a set later.

Counter Example

Let X be the collection of all sets A such that A is not an object in A or, A does not belong to A. We shall argue that X is not a set. Suppose, on the contrary, that X is a set.

If X belongs to X, then X does not belong to X.

If X does not belong to X, then X belongs to X. Either way, we get a contradiction. Therefore, we cannot decide whether X is an object in X. Thus, X is not a well-defined collection of objects and hence X is not a set.

Example

A positive integer greater than one is called a *prime number* if it has exactly two positive divisors, namely 1 and itself. Let P be the collection of all prime numbers. This is a well-defined collection of objects. For, given any object in the world, the question whether it belongs to this set or not has a unique answer. First recognize that if the given object is other than a positive integer, one can answer the question in the negative without any thinking. If the object is a positive integer, the question arises whether it is a prime number or not. For example, consider the number 2009. We may not be able to answer whether it is a prime number or not. But this much is certain that either 2009 is a prime or it is not. It can never be both. This is the property of being a well-defined collection.

DEFINITION 1.1 **Set** Any well-defined collection of objects is called a *set*.

DEFINITION 1.2 **Element** Let X be any set. The objects belonging to X are called *elements* of X, or *members* of X. If x is an element X, then we say that x belongs to X and denote this by $x \in X$. If x does not belong to X, then we write $x \notin X$.

The sets are usually denoted by capital letters of English alphabet while the elements are denoted in general by small letters. A set is represented by listing all its elements between the brackets { } and by separating them from each other by commas, if there are more than one element. Here are some examples of sets and the usual notations used to denote them.

QUICK LOOK 1

1. The set of all natural numbers (i.e., the set of all positive integers) is denoted by $\mathbb{N}$ or $\mathbb{Z}^+$. That is, $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$.
2. The set of all non-negative integers is denoted by $\mathbb{W}$; that is $\mathbb{W} = \{0, 1, 2, 3, \ldots\}$.
3. $\mathbb{Z}$ denotes the set of all integers.
4. $\mathbb{Q}$ denotes the set of all rational numbers.
5. The set of all real numbers is denoted by $\mathbb{R}$.
6. The set of all positive real numbers is denoted by $\mathbb{R}^+$.
7. The set of all positive rational numbers is denoted by $\mathbb{Q}^+$.
8. $\mathbb{C}$ denotes the set of all complex numbers.

Example 1.1

Verify whether the following are sets:

(1) The collection of all intelligent persons in Visakhapatnam.

(2) The collection of all prime ministers of India.

(3) The collection of all negative integers.

(4) The collection of all tall persons in India.

Note that the collections given in (1) and (4) are not sets because, if we select a person in Visakhapatnam, we cannot say with certainty whether he/she belongs to the collection or not, as there is no stand and scale for the evaluation of intelligence or for being tall. However, the collections given in (2) and (3) are sets.

A set may be represented with the help of certain property or properties possessed by all the elements of that set. Such a property is a statement which is either true or false. Any object which does not possess this property will not be an element of that set. In order to represent a set by this method we write between the brackets { } a variable x which stands for each element of the set. Then we write the property (or properties) possessed by each element x of the set. We denote this property by $p(x)$ and seperate x and $p(x)$ by a symbol: or |, read as "such that". Thus, we write

$$\{x \mid p(x)\} \quad \text{or} \quad \{x : p(x)\}$$

to represent the set of all objects x such that the statement $p(x)$ is true. This representation of a set is called "*set builder form*" representation.

Examples

(1) Let P be the collection of all prime numbers. Then it can be represented in the set builder form as

$$P = \{x \mid x \text{ is a prime number}\}$$

(2) Let X be the set of all even positive integers which are less than 15. Then

$$X = \{x \mid x \text{ is even integer and } 0 < x < 15\}$$
$$= \{2, 4, 6, 8, 10, 12, 14\}$$

(3) Let X be the set given above in (2) and

$$Y = \left\{ y \mid y = 0 \text{ or } \frac{1}{y} \in X \right\}$$

Then

$$Y = \left\{0, \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \frac{1}{12}, \frac{1}{14}\right\}$$

DEFINITION 1.3 **Empty Set** The set having no elements belonging to it is called the *empty set* or *null set* and is denoted by the symbol ϕ.

Examples

(1) Let $X = \{x \mid x \text{ is an integer and } 0 < x < 1\}$. Then X is a set and there are no elements in X, since there is no integer x such that $0 < x < 1$. Therefore, X is the empty set.

(2) Let $X = \{a \mid a \text{ is a rational number and } a^2 = 2\}$. Then X is the empty set, since there is no rational number a for which $a^2 = 2$.

Notation: The symbol $\Rightarrow$ is read as "implies". Thus $a \Rightarrow b$ is read as "a implies b". The symbol $\Leftrightarrow$ is read as "implies and is implied by" or as "if and only if". Thus $a \Leftrightarrow b$ is read as "a implies and implied by b" or "a if and only if b".

Examples

(1) x is an integer and $0 < x < 2 \Leftrightarrow x = 1$.

(2) a is an integer and $a^2 = a \Leftrightarrow a = 0$ or $a = 1$.

DEFINITION 1.4 **Equal Sets** Two sets A and B are defined to be equal if they contain the same elements, in the sense that,

$$x \in A \Leftrightarrow x \in B$$

In this case, we write $A = B$. If A and B are not equal, then we denote it by $A \neq B$.

Examples

(1) Let $A = \{1, 2, 3, 4\}$ and $B = \{4, 2, 3, 1\}$. Then $A = B$.

(2) Let

$$X = \{n \mid n \in \mathbb{Z} \text{ and } 1 \leq n^2 \leq 16\}$$

$$Y = \{n \mid n \in \mathbb{Z} \text{ and } 1 \leq n \leq 4\}$$

and $$Z = \{n \mid n \in \mathbb{Z}^+ \text{ and } 1 \leq n^2 \leq 16\}$$

Then $Y = Z$ and $X \neq Y$, since $-1 \in X$ and $-1 \notin Y$. Note that $X = \{-4, -3, -2, -1, 1, 2, 3, 4\}$.

DEFINITION 1.5 **Finite and Infinite Sets** A set having a definite number of elements is called a *finite set*. A set which is not finite is called an *infinite set*.

Examples

(1) The set $\mathbb{Z}^+$ of positive integers is an infinite set.

(2) $\{a, b, c, d\}$ is a finite set, since it has exactly four elements.

(3) The set $\mathbb{R}$ of real numbers is an infinite set.

(4) $\{x \mid x \in \mathbb{Z} \text{ and } 0 < x \leq 100\}$ is a finite set.

(5) $\{x \mid x \in \mathbb{Q} \text{ and } 0 < x < 1\}$ is an infinite set.

DEFINITION 1.6 **Family of Sets** A set whose members are sets is called a *family of sets* or *class of sets*.

Note that a family of sets is also a set. Usually families of sets are denoted by script letters $\mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{D}$, etc.

Examples

(1) For any integer n, let $A_n = \{x \mid x$ is an integer and $x \geq n\}$. Then $\{A_n \mid n$ is an integer$\}$ is a family of sets.

(2) For any house h, let

$$X_h = \text{The set of persons belonging to the house } h$$

Then $\{X_h \mid h$ is a house in Visakhapatnam$\}$ is a family of sets.

DEFINITION 1.7 **Indexed Family of Sets** A family $\mathscr{C}$ of sets is called an *indexed family* if there exists a set I such that for each element $i \in I$, there exists a unique member A_i in $\mathscr{C}$ associated with i and $\mathscr{C} = \{A_i : i \in I\}$. In this case, the set I is called the *index set*.

For example, the family of sets $\mathbb{Z}^+$ of positive integers is an indexed family of sets, the index set being $\mathbb{Z}$, the set of integers. In the example X_h = The set of persons belonging to the house h where $\{X_h \mid h$ is a house in Visakhapatnam$\}$ also we have an indexed family of sets, where the index set is the set of houses in Visakhapatnam. If $\mathscr{A}$ is an indexed family of sets with the index set I, then we usually write

$$\mathscr{A} = \{A_i\}_{i \in I} \quad \text{or} \quad \{A_i \mid i \in I\}$$

DEFINITION 1.8 **Intervals in $\mathbb{R}$** For any real numbers a and b, we define the *intervals* as the sets given below:

1. $(a, b) = \{x \mid x \in \mathbb{R} \text{ and } a < x < b\}$
2. $(a, b] = \{x \mid x \in \mathbb{R} \text{ and } a < x \leq b\}$
3. $[a, b) = \{x \mid x \in \mathbb{R} \text{ and } a \leq x < b\}$
4. $[a, b] = \{x \mid x \in \mathbb{R} \text{ and } a \leq x \leq b\}$

Examples

(1) $[2, 4] = \{x \mid x \in \mathbb{R} \text{ and } 2 < x < 4\}$

(2) $[0, 1] = \{x \mid x \in \mathbb{R} \text{ and } 0 \leq x \leq 1\}$

Note that, for any two real numbers a and b, the intervals $[a, b]$ or $[a, b)$ or $(a, b]$ is empty if and only if $a \geq b$. Also (a, b) is empty if and only if $a > b$. Further $[a, b]$ has exactly one element if and only if $a = b$. Thus these intervals become non-trivial only if $a < b$. Usually (a, b) is called an *open interval*, $(a, b]$ is called *left open and right closed interval*, $[a, b)$ is called the *left closed and right open interval* and $[a, b]$ is called a *closed interval*.

1.2 | Set Operations

We define certain operations between sets. These are closely related to the logical connectives "and", "or" and "not". To begin with, we have the following.

DEFINITION 1.9 **Subset** For any two sets A and B, we say that *A is a subset of B* or *A is contained in B* if every element of A is an element of B; in this case we denote it by $A \subseteq B$. A is not a subset of B is denoted by $A \not\subseteq B$.

If $A \subseteq B$ we also say that B is a *super set* of A or B *contains* A or B is *larger than* A or A is *smaller than* B. Sometimes, we write $B \supseteq A$ instead of $A \subseteq B$. If A is a subset of B and $A \neq B$, then we say that A is a *proper subset* of B and denote this by $A \subset B$. Note that, for any sets A and B, $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

QUICK LOOK 2

1. The set $\mathbb{Z}^+$ of positive integers is a proper subset of the set $\mathbb{Z}$ of integers.
2. $\mathbb{Z}$ is a proper subset of the set $\mathbb{Q}$ of rational numbers.
3. $\mathbb{Q}$ is a proper subset of the set $\mathbb{R}$ of real numbers.
4. $\mathbb{R}$ is a proper subset of the set $\mathbb{C}$ of complex numbers.
5. The set of Indians is a subset of the set of human beings.
6. If $A = \{1, 2, 3, 4, 5\}$ and $B = \{x \mid x \in \mathbb{R} \text{ and } x^2 - 5x + 6 = 0\}$, then $B \subset A$.

DEFINITION 1.10 **Power Set** For any set X, the collection of all subsets of X is also a set and is called the *power set* of X. It is denoted by $P(X)$.

Note that the empty set ϕ and the set X are always elements in the power set $P(X)$. Also, $X = \phi$ if and only if $P(X)$ has only one element. Infact, X has exactly n elements if and only if $P(X)$ has exactly 2^n elements, as proved in Theorem 1.1. First, let us consider certain examples.

Examples

(1) If $X = \{a\}$, then $P(X) = \{\phi, X\}$

(2) If $X = \{a, b\}$, then $P(X) = \{\phi, \{a\}, \{b\}, X\}$

(3) If $X = \{1, 2, 3\}$, then

$$P(X) = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{3, 1\}, X\}$$

(4) If $X = \{1, 2, 3, 4, 5\}$, then $P(X)$ has $32\,(=2^5)$ elements

(5) If X is a set such that $P(X)$ has 128 elements then X has 7 elements, since $2^7 = 128$

DEFINITION 1.11 **Cardinality** If X is a finite set, then the number of elements in X is denoted by $|X|$ or $n(X)$ and this number is called the *cardinality* of X.

THEOREM 1.1 Let X be any set. Then X is a finite set with n elements if and only if the power set $P(X)$ is a finite set with 2^n elements.

PROOF Suppose that X is a finite set with n elements. We apply induction on n. If $n = 0$, then $X = \phi$ and $P(X) = \{\phi\}$ which is a set with $1\,(=2^0)$ element. Now, let $n > 0$ and assume that the result is true for all sets with $n-1$ elements; that is, if Y is a set with $n-1$ elements, then $P(Y)$ has exactly 2^{n-1} elements.

Since $n > 0$, X is a non-empty set and hence we can choose an element a in X. Let Y be the set of all elements in X other than a. Then $|Y| = n-1$ and therefore $|P(Y)| = 2^{n-1}$. Clearly $P(Y) \subseteq P(X)$. Also, if $A \in P(X)$ and $A \notin P(Y)$, then $A \subseteq X$ and $A \not\subset Y$ and hence $a \in A$. Therefore, the number of subsets of X which are not subsets of Y is equal to the number of subsets of X containing a which in turn coincides with $|P(Y)|$. Hence,

$$|P(X)| = |P(Y)| + |P(Y)| = 2^{n-1} + 2^{n-1} = 2^n$$

Converse is clear; since each element $x \in X$ produces an element $\{x\} \in P(X)$, therefore X must be finite if $P(X)$ is finite. Also, note that, for non-negative integers n and m, $2^n = 2^m$ if and only if $n = m$. ■

COROLLARY 1.1 For any finite set X, $|X| < |P(X)|$.

DEFINITION 1.12 **Intersection of Sets** For any two sets A and B, we define the intersection of A and B to be the set of all elements belonging to both A and B. It is denoted by $A \cap B$. That is,

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

Example 1.2

Let $A = \{x \mid x$ is an odd prime and $x < 20\}$ and $B = \{x \mid x$ is an integer and $x > 6\}$. Find $A \cap B$.

Solution: By hypothesis

$A = \{3, 5, 7, 11, 13, 17, 19\}$ and $B = \{7, 8, 9, 10, 11, 12, \ldots\}$

Therefore

$$A \cap B = \{7, 11, 13, 17, 19\}$$

Example 1.3

Let X = The set of all circles in the plane whose radii is 5 cm and Y = The set of all line segments of length 5 cm in the plane. Find $X \cap Y$.

Solution: $X \cap Y = \phi$, the empty set, since no circle of positive radius can be a line segment.

Example 1.4

Let F = The set of all boys in a school who can play football and C = The set of all boys in the school who can play cricket. Find $F \cap C$.

Solution: $F \cap C$ = The set of all boys in the school who can play both football and cricket.

Example 1.5

Let A = The set of all non-negative integers and B = The set of all non-positive integers. Find $A \cap B$.

Solution: $A \cap B = \{x \mid x \text{ is an integer}, x \geq 0 \text{ and } x \leq 0\} = \{0\}$.

The following can be proved easily.

Try it out

THEOREM 1.2 The following hold for any sets, A, B and C.

1. $A \subseteq B \Leftrightarrow A = A \cap B$
2. $A \cap A = A$
3. $A \cap B = B \cap A$
4. $(A \cap B) \cap C = A \cap (B \cap C)$
5. $A \cap \phi = \phi$, where ϕ is the empty set.
6. For any set X, $X \subseteq A \cap B$ if and only if $X \subseteq A$ and $X \subseteq B$.

In view of (4) above, we write simply $A \cap B \cap C$ for $(A \cap B) \cap C$ or $A \cap (B \cap C)$. In general, if $A_1, A_2, \ldots, A_n$ are sets, we write

$$\bigcap_{i=1}^{n} A_i \quad \text{for } A_1 \cap A_2 \cap \cdots \cap A_n$$

More generally, for any indexed family $\{A_i\}_{i \in I}$ of sets, we write $\bigcap_{i \in I} A_i$ for the set of all elements common to all A_i's, $i \in I$ and express this by

$$\bigcap_{i \in I} A_i = \{x \mid x \in A_i \text{ for all } i \in I\}$$

DEFINITION 1.13 **Disjoint Sets** Two sets A and B are called *disjoint* if $A \cap B$ is the empty set. In this case we say that A *is disjoint with* B or B *is disjoint with* A.

Examples

(1) Let E be the set of even integers and O the set of all odd integers. Then E and O are disjoint sets.

(2) Let $A = \{\sqrt{p} \mid p \text{ is a prime number}\}$. Then $A \cap \mathbb{Q} = \phi$, where $\mathbb{Q}$ is the set of rational numbers, since it is known that $\sqrt{p}$ is an irrational number for any prime p.

(3) $\bigcap_{n=1}^{\infty} \left[0, \frac{1}{n}\right] = \{0\}$

(4) $\bigcap_{n=1}^{\infty} \left(0, \frac{1}{n}\right) = \phi$

since, for any given $a > 0$, we can find an integer n such that $0 < 1/n < a$ and hence $a \notin (0, 1/n)$.

DEFINITION 1.14 **Union of Sets** For any two sets A and B, we define the *union* of A and B as the set of all elements belonging to A or B and denote this by $A \cup B$; that is,

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

Note that the statement "$x \in A$ or $x \in B$" does not exclude the case "$x \in A$ and $x \in B$". Therefore

$$A \cup B = \{x \mid x \in A \text{ or } x \in B \text{ or both}\}$$

Example

Let E be the set of even integers and O the set of all odd integers. Then $E \cup O = \mathbb{Z}$, the set of integers. In this case, E and O are disjoint and hence we do not come across the case "$x \in E$ and $x \in O$".

Example 1.6

Let A be the interval [0, 1] and B the interval [1/2, 2]. Then find $A \cup B$ and $A \cap B$.

Solution: We have

$$A \cup B = \{x | x \in A \text{ or } x \in B\}$$

$$= \left\{x | x \in \mathbb{R} \text{ and ‘}0 \le x \le 1 \text{ or } \frac{1}{2} \le x \le 2\text{’}\right\}$$

$$= \{x | x \in \mathbb{R} \text{ and } 0 \le x \le 2\}$$

$$= [0, 2]$$

Also,

$$A \cap B = \left[\frac{1}{2}, 1\right]$$

Example 1.7

Let $A = [0, 1] \cap \mathbb{Q}$ and $B = (1, 2) \cap \mathbb{Q}$. Find $A \cup B$.

Solution: $A \cup B = \{x | x \in A \text{ or } x \in B\}$

$$= \{x | x \in \mathbb{Q} \text{ and ‘}x \in [0, 1] \text{ or } x \in (1, 2)\text{’}\}$$

$$= \{x | x \in \mathbb{Q} \text{ and } 0 \le x < 2\}$$

$$= [0, 2) \cap \mathbb{Q}$$

Example 1.8

Let A be the set of all even primes and B the interval $(2, 3)$. Find $A \cup B$.

Solution: $A \cup B = \{x | x \text{ is an even prime or } x \in (2, 3)\}$

$$= \{x | x = 2 \text{ or } x \in \mathbb{R} \quad \text{such that } 2 < x < 3\}$$

$$= \{x | x \in \mathbb{R} \text{ and } 2 \le x < 3\}$$

$$= [2, 3)$$

The following can be easily proved.

Try it out

THEOREM 1.3 For any sets A, B and C the following hold.

1. $A \cap B \subseteq A \cup B$
2. For any set X, $A \cup B \subseteq X$ if and only if $A \subseteq X$ and $B \subseteq X$
3. $A \cup A = A$
4. $A \cup B = B \cup A$
5. $(A \cup \text{B}) \cup C = A \cup (B \cup C)$
6. $A \subseteq B \Leftrightarrow A \cup B = B$
7. $A \cup \phi = A$
8. $A = A \cap B \Leftrightarrow A \subseteq B \Leftrightarrow A \cup B = B$
9. $A \cap (A \cup B) = A$
10. $A \cup (A \cap B) = A$

THEOREM 1.4 DISTRIBUTIVE LAWS The following hold for any sets A, B and C.

1. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
2. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

These are called the distributive laws for intersection $\cap$ and union $\cup$.

PROOF **1.** $x \in A \cap (B \cup C) \Rightarrow x \in A \quad \text{and} \quad x \in B \cup C$

$$\begin{aligned}
&\Rightarrow x \in A \quad \text{and} \quad (x \in B \text{ or } x \in C) \\
&\Rightarrow (x \in A \text{ and } x \in B) \quad \text{or} \quad (x \in A \text{ and } x \in C) \\
&\Rightarrow x \in A \cap B \quad \text{or} \quad x \in A \cap C \\
&\Rightarrow x \in (A \cap B) \cup (A \cap C)
\end{aligned}$$

Therefore

$$A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C) \tag{1.1}$$

On the other hand, we have

$$\begin{aligned}
x \in (A \cap B) \cup (A \cap C) &\Rightarrow x \in A \cap B \quad \text{or} \quad x \in A \cap C \\
&\Rightarrow (x \in A \text{ and } x \in B) \quad \text{or} \quad (x \in A \text{ and } x \in C) \\
&\Rightarrow x \in A \quad \text{and} \quad (x \in B \text{ or } x \in C) \\
&\Rightarrow x \in A \quad \text{and} \quad x \in B \cup C \\
&\Rightarrow x \in A \cap (B \cup C)
\end{aligned}$$

Therefore

$$(A \cap B) \cup (A \cup C) \subseteq A \cap (B \cup C) \tag{1.2}$$

From Eqs. (1.1) and (1.2), we have $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

2. It can be proved similarly and is left as an exercise for the reader. ■

Try it out $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

THEOREM 1.5 For any sets A, B and C,

$$A \cap B = A \cap C \quad \text{and} \quad A \cup B = A \cup C \quad \text{imply} \quad B = C$$

PROOF Suppose that $A \cap B = A \cap C$ and $A \cup B = A \cup C$. Consider

$$\begin{aligned}
B &= B \cap (A \cup B) && \text{[by part (9) of Theorem 1.3]} \\
&= B \cap (A \cup C) && (\text{since } A \cup B = A \cup C) \\
&= (B \cap A) \cup (B \cap C) && (\text{by the distributive laws}) \\
&= (C \cap A) \cup (C \cap B) && (\text{since } A \cap B = A \cap C) \\
&= C \cap (A \cup B) && (\text{by the distributive laws}) \\
&= C \cap (A \cup C) && (\text{since } A \cup B = A \cup C) \\
&= C && \text{[by part (9) of Theorem 1.3]}
\end{aligned}$$

Therefore $B = C$.

Since $(A \cup B) \cup C = A \cup (B \cup C)$ for any sets A, B and C, we simply write $A \cup B \cup C$ without bothering about the brackets. In general, if $A_1, A_2, \ldots, A_n$ are any sets, then we write

$$\bigcup_{i=1}^{n} A_i \quad \text{for } A_1 \cup A_2 \cup \cdots \cup A_n$$

For any indexed family $\{A_i\}_{i \in I}$ of sets, we write $\bigcup_{i \in I} A_i$ for the set of all elements belonging to at least one A_i and express this by

$$\bigcup_{i \in I} A_i = \{x \mid x \in A_i \text{ for some } i \in I\}$$

■

Examples

(1) For any positive integer n, let

$$A_n = (-n, n) = \{x \mid x \in \mathbb{R} \text{ and } -n < x < n\}$$

Then

$$\bigcup_{n=1}^{\infty} A_n = \{x \mid x \in \mathbb{R} \text{ and } -n < x < n \text{ for some } n \in \mathbb{Z}^+\} = \mathbb{R}$$

since, for any real number x, there exists a positive integer n such that $|x| < n$ and hence $-n < x < n$, so that $x \in A_n$.

(2) For any positive integer n, let

$$P_n = \{p \mid p \text{ is a prime number and } p < n\}$$

Note that $P_1 = \phi = P_2$, $P_3 = \{2\}$ and $P_4 = \{2, 3\}$. Now

$$\bigcup_{n=1}^{\infty} P_n = \text{The set of all prime numbers}$$

since, for any prime p, we have $p \in A_{p+1}$.

(3) For any positive real number a, let

A_a = The set of human beings on the Earth whose height is less than or equal to a cm

Then

$$\bigcup_{a \in \mathbb{R}^+} A_a = \text{The set of all human beings on the Earth}$$

(4) For any positive integer n, let

$$X_n = \left(-\frac{1}{n}, \frac{1}{n}\right) = \left\{x \mid x \in \mathbb{R} \text{ and } -\frac{1}{n} < x < \frac{1}{n}\right\}$$

Then

$$\bigcap_{n=1}^{\infty} X_n = \{0\} \quad \text{and} \quad \bigcup_{n=1}^{\infty} X_n = (-1, 1)$$

since $X_n \subseteq X_1$ for all $n \in \mathbb{Z}^+$.

DEFINITION 1.15 For any two sets A and B, the difference of A and B is defined as the set

$$A - B = \{x \mid x \in A \text{ and } x \notin B\}$$

Example 1.9

Find the difference of the following sets.

(1) $A = (0, 1) = \{x \mid x \in \mathbb{R} \text{ and } 0 < x < 1\}$ and $B = \left\{x \mid x \in \mathbb{R}^+ \text{ and } \frac{1}{x} \in \mathbb{Z}\right\}$

(2) $\mathbb{R} - \mathbb{Z}$ where the symbols have there usual meaning.

(3) A = The set of all students in a school and B = The set of all girls

(4) $\mathbb{Z} - \mathbb{Z}^+$ where the symbols have the usual meaning.

Solution:

(1) By hypothesis

$A = (0, 1) = \{x \mid x \in \mathbb{R} \text{ and } 0 < x < 1\}$ and $B = \{x \mid x \in \mathbb{R}^+$ and $1/x \in \mathbb{Z}\}$. We have

$$B = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\right\}$$

Now

$$\begin{aligned} A - B &= \{x \mid x \in A \text{ and } x \notin B\} \\ &= \left\{x \mid x \in \mathbb{R}, 0 < x < 1 \text{ and } \frac{1}{x} \notin \mathbb{Z}\right\} \\ &= \bigcup_{n=1}^{\infty} \left(\frac{1}{n+1}, \frac{1}{n}\right) \end{aligned}$$

(2)
$$\begin{aligned} \mathbb{R} - \mathbb{Z} &= \{x \mid x \in \mathbb{R} \text{ and } x \notin \mathbb{Z}\} \\ &= \{x \mid x \text{ is a real number and not an integer}\} \\ &= \bigcup_{n \in \mathbb{Z}} (n, n+1) \\ &= \cdots \cup (-2, -1) \cup (-1, 0) \cup (0, 1) \cup (1, 2) \cup \end{aligned}$$

(3) $A - B$ = The set of all boys in the school

(4) $\mathbb{Z} - \mathbb{Z}^+$ = The set of all non-positive integers

$$= \{x \mid x \in \mathbb{Z} \text{ and } x \leq 0\}$$

THEOREM 1.6 DE MORGAN'S LAWS For any sets A, B and C, the following hold:

1. $A - (B \cup C) = (A - B) \cap (A - C)$
2. $A - (B \cap C) = (A - B) \cup (A - C)$

PROOF

1. $$\begin{aligned} x \in A - (B \cup C) &\Rightarrow x \in A \quad \text{and} \quad x \notin B \cup C \\ &\Rightarrow x \in A \quad \text{and} \quad (x \notin B \text{ and } x \notin C) \end{aligned}$$

$$\Rightarrow (x \in A \text{ and } x \notin B) \quad \text{and} \quad (x \in A \text{ and } x \notin C)$$
$$\Rightarrow x \in A - B \quad \text{and} \quad x \in A - C$$
$$\Rightarrow x \in (A - B) \cap (A - C)$$

and therefore, $A - (B \cup C) \subseteq (A - B) \cap (A - C)$. Also,

$$\begin{aligned} x \in (A - B) \cap (A - C) &\Rightarrow x \in A - B \quad \text{and} \quad x \in A - C \\ &\Rightarrow (x \in A \text{ and } x \notin B) \quad \text{and} \quad (x \in A \text{ and } x \notin C) \\ &\Rightarrow x \in A \quad \text{and} \quad (x \notin B \text{ and } x \notin C) \\ &\Rightarrow x \in A \quad \text{and} \quad x \notin B \cup C \\ &\Rightarrow x \in A - (B \cup C) \end{aligned}$$

and therefore $(A - B) \cap (A - C) \subseteq A - (B \cup C)$. Thus

$$A - (B \cup C) = (A - B) \cap (A - C)$$

2. It can be similarly proved and is left as an exercise for the reader. ■

Try it out

THEOREM 1.7 The following hold for any sets A, B and C.

1. $B \subseteq C \Rightarrow A - C \subseteq A - B$
2. $A \subseteq B \Rightarrow A - C \subseteq B - C$
3. $(A \cup B) - C = (A - C) \cup (B - C)$
4. $(A \cap B) - C = (A - C) \cap (B - C)$
5. $(A - B) - C = A - (B \cup C) = (A - B) \cap (A - C)$
6. $A - (B - C) = (A - B) \cup (A \cap C)$

THEOREM 1.8 GENERALIZED DE MORGAN'S LAWS Let $\{A_i\}_{i \in I}$ be any family of sets and B and C any sets. Then the following hold:

1. $B - \left(\bigcup_{i \in I} A_i \right) = \bigcap_{i \in I} (B - A_i)$
2. $B - \left(\bigcap_{i \in I} A_i \right) = \bigcup_{i \in I} (B - A_i)$
3. $\left(\bigcup_{i \in I} A_i \right) - B = \bigcup_{i \in I} (A_i - B)$
4. $\left(\bigcap_{i \in I} A_i \right) - B = \bigcap_{i \in I} (A_i - B)$

PROOF These follow from the facts that

$$x \in \bigcup_{i \in I} A_i \Leftrightarrow x \in A_i \quad \text{for some } i \in I$$

$$x \in \bigcap_{i \in I} A_i \Leftrightarrow x \in A_i \quad \text{for all } i \in I$$

$$x \notin \bigcup_{i \in I} A_i \Leftrightarrow x \notin A_i \quad \text{for all } i \in I$$

and

$$x \notin \bigcap_{i \in I} A_i \Leftrightarrow x \notin A_i \quad \text{for some } i \in I$$

■

Examples

(1) $\mathbb{R} - \mathbb{Z} = \mathbb{R} - \left(\bigcup_{n \in \mathbb{Z}} \{n\} \right)$

$= \bigcap_{n \in \mathbb{Z}} (\mathbb{R} - \{n\})$

(2) $\mathbb{R} - \mathbb{Z} = \left(\bigcup_{n \in \mathbb{Z}} [n, n+1] \right) - \mathbb{Z}$

$= \bigcup_{n \in \mathbb{Z}} ([n, n+1] - \mathbb{Z})$

$= \bigcup_{n \in \mathbb{Z}} (n, n+1)$

Note that, here we have used the fact that, for any integer n, there is no integer m such that $n < m < n+1$.

(3) For any integer n,

$$\mathbb{R} - (n, n+1) = (-\infty, n] \cup [n+1, \infty)$$

Here $(-\infty, n)$ stands for the set of real numbers x such that $x \le n$ and $[n+1, \infty)$ for the set of real numbers x such that $n+1 \le x$.

(4) Let

$$A = \{x \in \mathbb{Z}^+ \mid x < 10\} = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

and $B =$ The set of all prime numbers

Then

$$A - B = \{1, 4, 6, 8, 9\}$$

It is convenient to write B' for the set of all elements not belonging to B and to write $A - B$ as $A \cap B'$. But the problem here is that B' may not be a set at all. However, if X is a superset of B, then certainly $X - B$ is a set, which can be imagined as B'. For any two sets A and B, we can take $X = A \cup B$ and then

$$A - B = A \cap (X - B) = A \cap B'$$

When we are dealing with a family $\{A_i\}_{i \in I}$ of sets (or set of sets), we can assume that each A_i is a subset of some set X; for example, we can take $X = \bigcup_{i \in I} A_i$. This common superset is called a *universal set*. Therefore, when we discuss about difference set $A - B$, we can treat A and B as subsets of a universal set X and treat $A - B$ as $A \cap B'$, where

$$B' = \{x \mid x \in X \text{ and } x \notin B\}$$

B' is certainly a set, since X and B are sets and so is $X - B$. This B' is called the *complement of B in X* or, simply, the *complement of B*, when there is no ambiguity about X. Note that $A - B = A - (A \cap B)$ and $A \cap B$ is a subset of A. Therefore, we can call $A - B$ is the complement of B in A. With this understanding, the properties proved above can be restated as follows:

$$A - B = A \cap B'$$

$$A - B = A - (A \cap B)$$

$$(B \cup C)' = B' \cap C' \qquad \text{[Part (1), Theorem 1.6]}$$

$$(B \cap C)' = B' \cup C' \qquad \text{[Part (2), Theorem 1.6]}$$

$$B \subseteq C \Rightarrow C' \subseteq B' \qquad \text{[Part (1), Theorem 1.7]}$$

$$\left(\bigcup_{i \in I} A_i \right)' = \bigcap_{i \in I} A_i' \qquad \text{[Part (1), Theorem 1.8]}$$

$$\left(\bigcap_{i \in I} A_i \right)' = \bigcup_{i \in I} A_i' \qquad \text{[Part (2), Theorem 1.8]}$$

$$A - (A - B) = A \cap B$$

$$B \subseteq A \Rightarrow A - (A - B) = B \quad \text{or} \quad (B')' = B$$

$$A \cap A' = \phi$$

$$A \cup A' = X, \text{ the universal set}$$

DEFINITION 1.16 **Symmetric Difference** For any sets A and B, the *symmetric difference* of A and B is defined as the set

$$A \Delta B = (A - B) \cup (B - A) = (A \cap B') \cup (B \cap A')$$

That is, $A \Delta B$ is the set all elements belonging to exactly one of A and B.

Example 1.10

Find the symmetric difference of the following:

(1) $A = \{1, 2, 3, 4\}$ and $B = \{4, 5, 6\}$

(2) $A = \{a, b, c, d, e\}$ and $B = \{b, c, f, g\}$

Solution:

(1) We have $A = \{1, 2, 3, 4\}$ and $B = \{4, 5, 6\}$. Then

$$A - B = \{1, 2, 3\} \quad \text{and} \quad B - A = \{5, 6\}$$

Therefore

$$A \Delta B = \{1, 2, 3\} \cup \{5, 6\} = \{1, 2, 3, 5, 6\}$$

(2) From the given sets we have

$$A - B = \{a, d, e\} \text{ and } B - A = \{f, g\}$$

Therefore

$$A \Delta B = \{a, d, e\} \cup \{f, g\} = \{a, d, e, f, g\}$$

THEOREM 1.9 The following hold for any sets A, B and C.

1. $A \Delta B = B \Delta A$
2. $(A \Delta B) \Delta C = A \Delta (B \Delta C)$
3. $A \Delta \phi = A$
4. $A \Delta A = \phi$

PROOF

1. $A \Delta B = (A - B) \cup (B - A)$
$\quad = (B - A) \cup (A - B)$
$\quad = B \Delta A$

2. $(A \Delta B) \Delta C = [(A \Delta B) \cap C'] \cup [C \cap (A \Delta B)']$

$$= [\{(A \cap B') \cup (B \cap A')\} \cap C'] \cup [C \cap \{(A \cap B') \cup (B \cap A')\}']$$
$$= [(A \cap B' \cap C') \cup (B \cap A' \cap C')] \cup [C \cap (A' \cup B) \cap (B' \cup A)]$$
$$= (A \cap B' \cap C') \cup (B \cap A' \cap C') \cup [C \cap \{(A' \cap B') \cup (A' \cap A) \cup (B \cap B') \cup (B \cap A)\}]$$
$$= (A \cap B' \cap C') \cup (B \cap A' \cap C') \cup [\{C \cap (A' \cap B') \cup (A \cap B)\}]$$
$$= (A \cap B' \cap C') \cup (B \cap A' \cap C') \cup (C \cap A' \cap B') \cup (C \cap A \cap B)$$

Therefore, we have

$$(A \Delta B) \Delta C = (A \cap B' \cap C') \cup (A' \cap B \cap C') \cup (A' \cap B' \cap C) \cup (A \cap B \cap C)$$

This is symmetric in A, B and C; that is, if we take B, C and A for A, B and C, respectively, the resultant is same. Therefore,

$$(A \Delta B) \Delta C = (B \Delta C) \Delta A = A \Delta (B \Delta C)$$

3. $A \Delta \phi = (A - \phi) \cup (\phi - A) = A \cup \phi = A$
4. $A \Delta A = (A - A) \cup (A - A) = \phi \cup \phi = \phi$ ■

1.3 | Venn Diagrams

A set is represented by a closed curve, usually a circle, and its elements by points within it. This facilitates better understanding and a good insight. A statement involving sets can be easily understood with pictorial representation of the sets. The diagram showing these sets is called the *Venn diagram* of that statement, named after the British logician John Venn (1834–1883).

Usually the universal set is represented by a rectangle and the given sets are represented by circles or closed geometrical figures inside the rectangle representing the universal set. An element of set A is represented by a point within the circle representing A.

In Figure 1.1, the rectangle represents the universal set S, A and B represent two disjoint sets contained in S and a and b represent arbitrary elements in A and B, respectively.

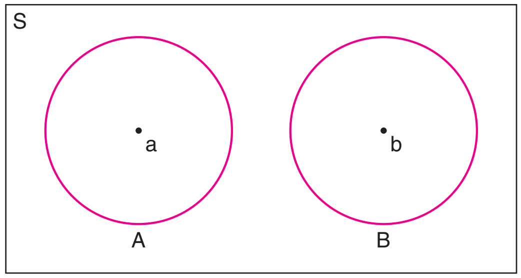


FIGURE 1.1 A Venn diagram.

In Figure 1.2, two intersecting sets A and B are represented by the intersecting circles, indicating that the common area of the circles represents the intersection $A \cap B$. Figure 1.3 represents the statement "A is a subset of B".

The shaded parts in Figures 1.4–1.6 represent the union of two sets A and B, namely $A \cup B$ in the cases $A \cap B = \phi$, $A \cap B \neq \phi$ and $A \subseteq B$, respectively. Figures 1.7–1.9 represent the intersection $A \cap B$ in these cases.

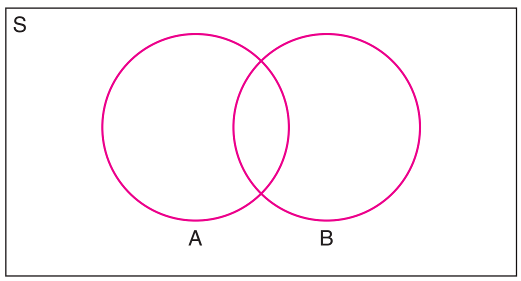


FIGURE 1.2 Two intersecting sets A and B.

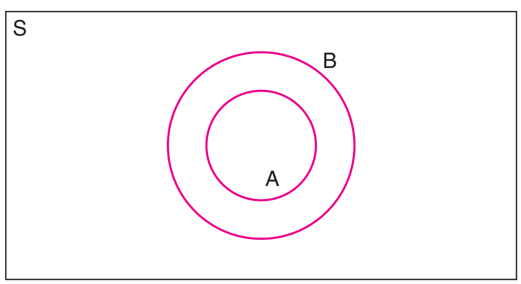


FIGURE 1.3 Representation of "A is a subset of B".

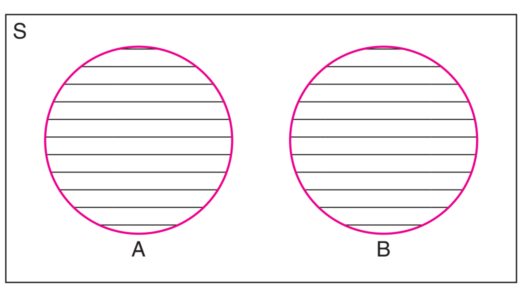


FIGURE 1.4 Representation of $A \cup B$ when $A \cap B = \phi$.

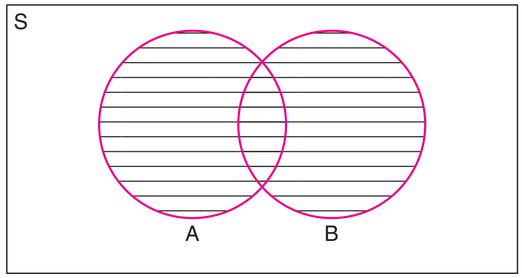


FIGURE 1.5 Representation of $A \cup B$ when $A \cap B \neq \phi$.

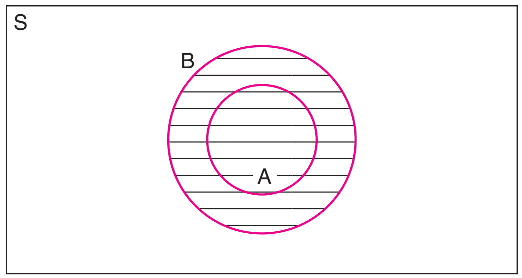


FIGURE 1.6 Representation of $A \cup B$ when $A \subseteq B$.

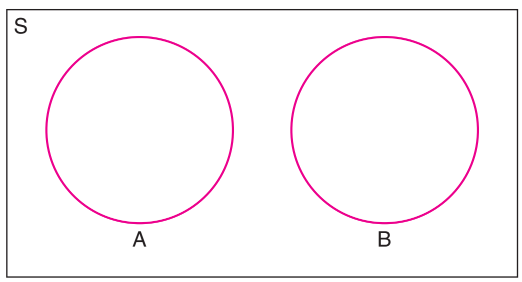


FIGURE 1.7 Representation of $A \cap B$ when $A \cap B = \phi$.

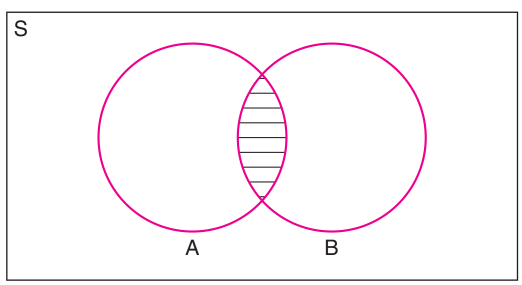


FIGURE 1.8 Representation of $A \cap B$ when $A \cap B \neq \phi$.

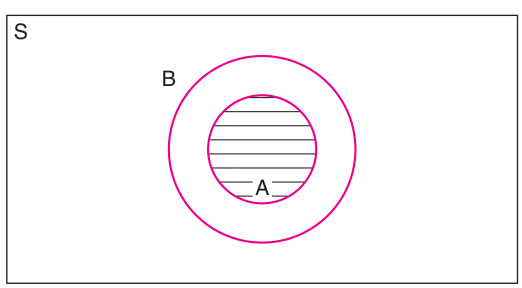


FIGURE 1.9 Representation of $A \cap B$ when $A \subseteq B$.

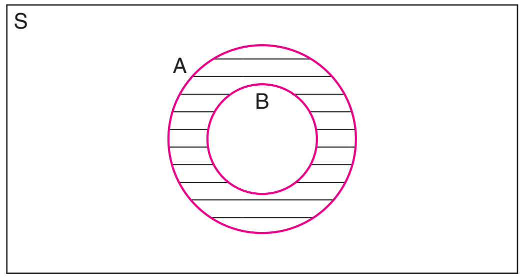


FIGURE 1.10 Representation of $A - B$ when $B \subseteq A$.

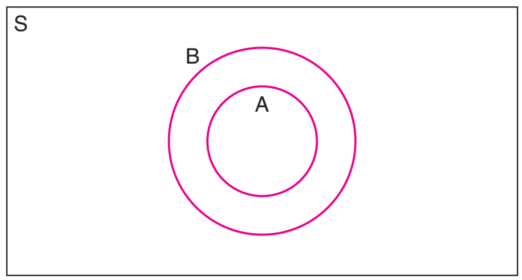


FIGURE 1.11 Representation of $A - B$ when $A \subseteq B$. In this case $A - B = \phi$.

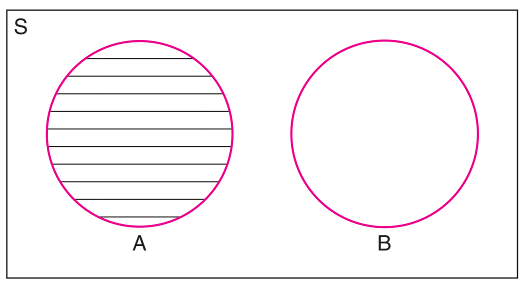


FIGURE 1.12 Representation of $A - B$ when $A \cap B = \phi$.

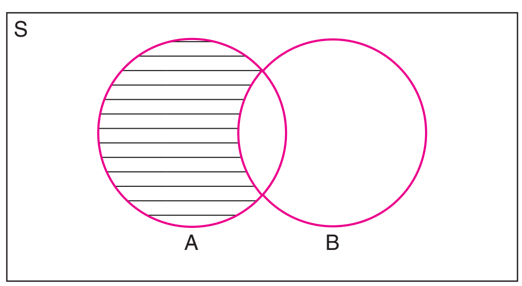


FIGURE 1.13 Representation of $A - B$ when $A \not\subset B$ and $B \not\subset A$.

The shaded parts in Figures 1.10–1.13 represent the difference $A - B$ in various cases. The symmetric differences $A \Delta B [= (A - B) \cup (B - A)]$ are represented by the shaded parts in the Figures 1.14–1.17 in these cases.

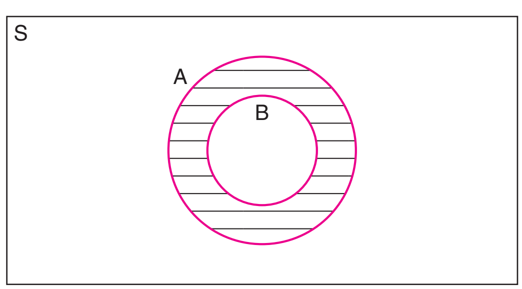


FIGURE 1.14 Representation of $A \Delta B$ when $B \subseteq A$.

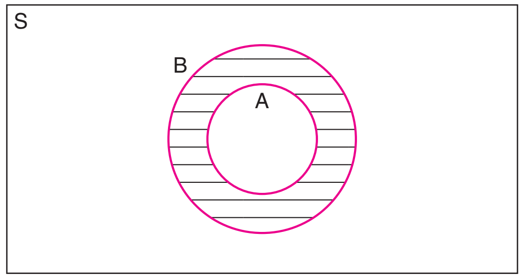


FIGURE 1.15 Representation of $A \Delta B$ when $A \subseteq B$.

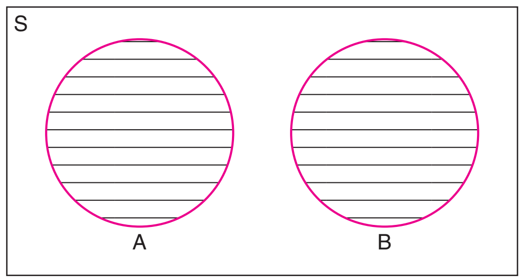


FIGURE 1.16 Representation of $A \Delta B$ when $A \cap B = \phi$.

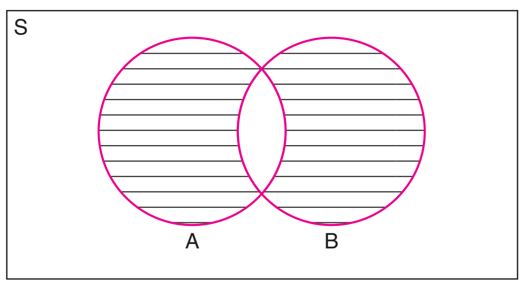


FIGURE 1.17 Representation of $A \Delta B$ when $A \not\subset B$ and $B \not\subset A$.

Figure 1.18 represents the complement of a set A in a universal set S. Figures 1.19–1.21 illustrate the cases $A \Delta B$, $(A \Delta B) - C$ and $C - (A \Delta B)$, respectively. $(A \Delta B) \Delta C$ is represented by Figure 1.22. From this one can easily see that $(A \Delta B) \Delta C = (A \Delta B) \Delta C$.

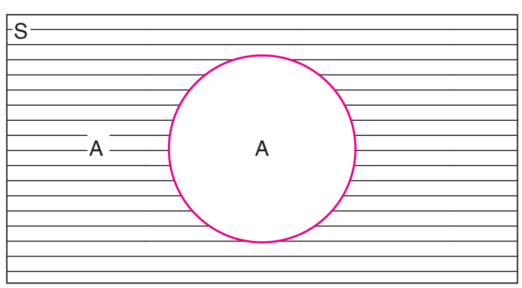


FIGURE 1.18 Complement of a set A.

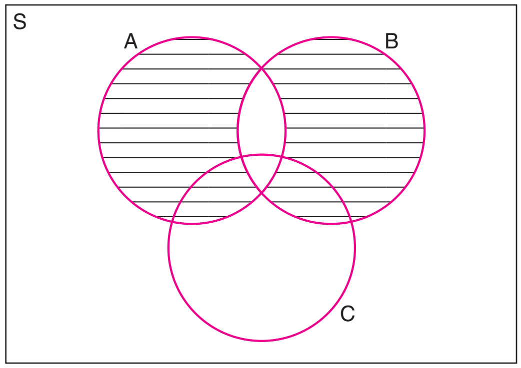


FIGURE 1.19 Representation of $A \Delta B$.

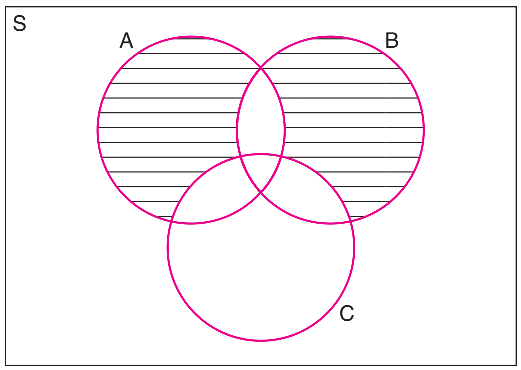


FIGURE 1.20 Representation of $(A \Delta B) - C$.

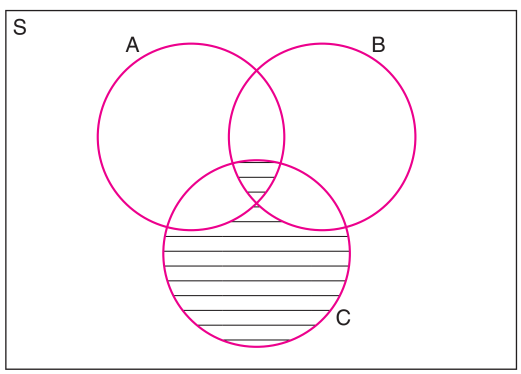


FIGURE 1.21 Representation of $C - (A \Delta B)$.

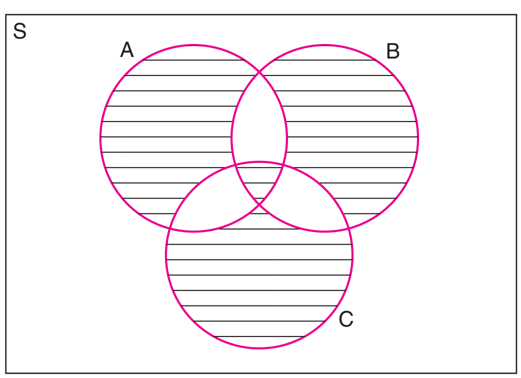


FIGURE 1.22 Representation of $(A \Delta B) \Delta C$.

Figures 1.23 and 1.24 represent the property

$$A - (B \cup C) = (A - B) \cap (A - C)$$

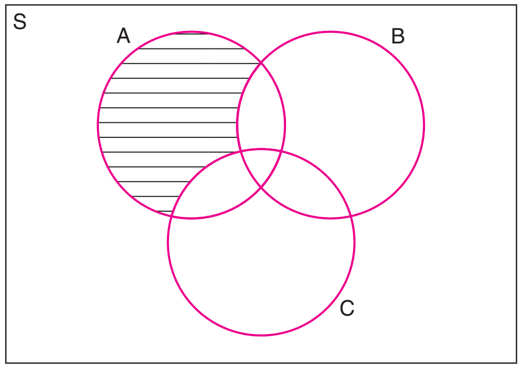


FIGURE 1.23 Representation of $A-(B\cup C)$.

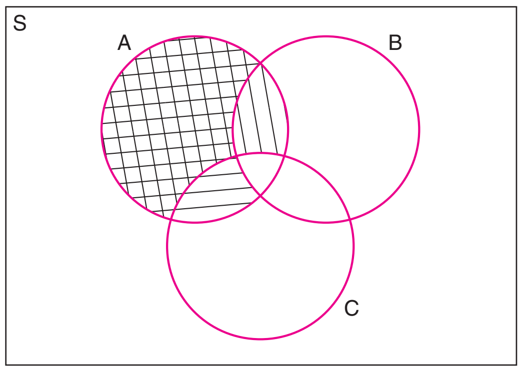


FIGURE 1.24 Representation of $(A-B)\cap(A-C)$.

In the following, we derive certain formulas for the number of elements in the intersection, union, difference and symmetric difference of two given finite sets. First, recall that, for any finite set A, $n(A)$ or $|A|$ denotes the number of elements in A.

Examples

(1) Let $A=\{a,b,c,d\}$, then $n(A)=4$.

(2) If $A=\{2,3,5,7\}$, then $n(A)=4$.

(3) If X is a finite set and $n(X)=m$, then $n[P(X)]=2^m$, where $P(X)$ is the set of all subsets of X.

(4) If $X=\{m\,|\,m\in\mathbb{Z}$ and $m^2=1\}$, then $n(X)=2$, since $X=\{1,-1\}$.

THEOREM 1.10 For any two disjoint sets A and B,

$$n(A\cup B)=n(A)+n(B)$$

PROOF Any element of $A\cup B$ is in exactly one of A and B and therefore $n(A\cup B)=n(A)+n(B)$ In Figure 1.25, the shaded part represents $A\cup B$ when A and B are disjoint sets.

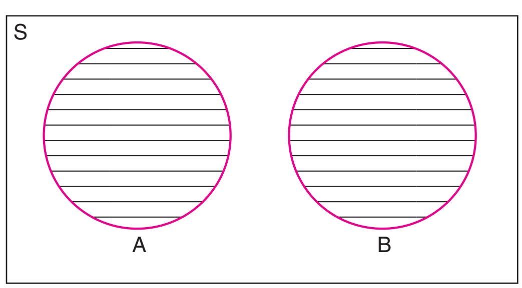


FIGURE 1.25 Representation of $A\cup B$ when A and B are disjoint sets. ■

Corollary 1.2 If $A_1, A_2, \ldots, A_n$ are pairwise disjoint sets, then

$$n(A_1 \cup A_2 \cup \cdots \cup A_n) = n(A_1) + n(A_2) + \cdots + n(A_n)$$

Theorem 1.11 For any finite sets A and B,

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

Proof Let A and B be finite sets, $n(A) = a, n(B) = b$ and $n(A \cap B) = m$. If $A \cap B$ is empty then $m = 0$ and, by Theorem 1.10,

$$n(A \cup B) = n(A) + n(B) = n(A) + n(B) - n(A \cap B)$$ ■

Suppose that $A \cap B \neq \phi$. Then $A - B$, $B - A$ and $A \cap B$ are pairwise disjoint sets (Figure 1.26) and hence we have

$$n(A \cup B) = n[(A - B) \cup (B - A) \cup (A \cap B)] = n(A - B) + n(B - A) + n(A \cap B)$$
$$= n(A) + n(B) - n(A \cap B)$$

since $n(A) = n(A - B) + n(A \cap B)$ and $n(B) = n(B - A) + n(A \cap B)$.

We have earlier proved that $n(A \cup B) = n(A) + n(B)$, if A and B are disjoint sets. The converse of this is also true.

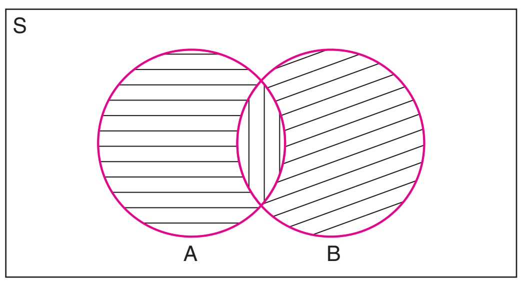


FIGURE 1.26 Representation of pairwise disjoint sets.

Corollary 1.3 If A and B are finite sets such that $n(A \cup B) = n(A) + n(B)$, then A and B are disjoint.

Proof If $n(A \cup B) = n(A) + n(B)$, then by Theorem 1.11 $n(A \cap B) = 0$ and hence $A \cap B = \phi$. ■

Corollary 1.4 For any finite sets A and B,

$$n(A - B) = n(A) - n(A \cap B)$$

Corollary 1.5 If A is a subset of a finite set B, then

$$n(B) = n(A) + n(B - A)$$

Theorem 1.12 For any finite sets A, B and C,

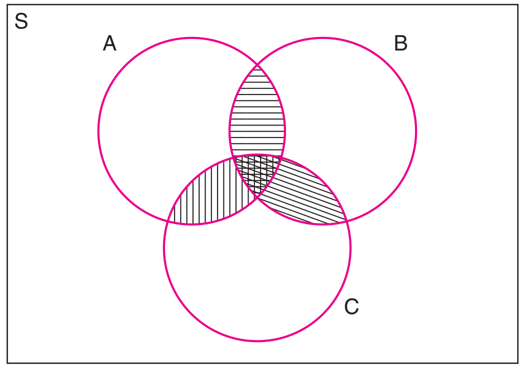
$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$$

Proof Let A, B and C be any finite sets. Then

$$n(A \cup B \cup C) = n(A \cup B) + n(C) - n[(A \cup B) \cap C]$$
$$= n(A) + n(B) - n(A \cap B) + n(C) - n[(A \cap C) \cup (B \cap C)]$$

$$= n(A) + n(B) + n(C) - n(A \cap B) - [n(A \cap C) + n(B \cap C) - n(A \cap C \cap B \cap C)]$$

$$= n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$$



THEOREM 1.13 Let A, B and C be finite sets. Then the number of the elements belonging to exactly two of the sets A, B and C is

$$n(A \cap B) + n(B \cap C) + n(C \cap A) - 3n(A \cap B \cap C)$$

PROOF The required number is

$$n[(A \cap B) - C] + n[(B \cap C) - A] + n[(C \cap A) - B] = [n(A \cap B) - n(A \cap B \cap C)]$$

$$+ [n(B \cap C) - n(B \cap C \cap A)] + [n(C \cap A) - n(C \cap A \cap B)]$$

$$= n(A \cap B) + n(B \cap C) + n(C \cap A) - 3n(A \cap B \cap C)$$

THEOREM 1.14 Let A, B and C be any finite sets. Then the number of elements belonging to exactly one of the sets A, B and C is

$$n(A) + n(B) + n(C) - 2n(A \cap B) - 2n(B \cap C) - 2n(C \cap A) + 3n(A \cap B \cap C)$$

PROOF The number of elements belonging only to A is

$$\begin{aligned} n[A - (B \cup C)] &= n(A) - n[A \cap (B \cup C)] \\ &= n(A) - n[(A \cap B) \cup (A \cap C)] \\ &= n(A) - [n(A \cap B) + n(A \cap C) - n(A \cap B \cap A \cap C)] \\ &= n(A) - n(A \cap B) - n(A \cap C) + n(A \cap B \cap C) \end{aligned}$$

Similarly, the number of elements belonging only to B is

$$n(B) - n(B \cap C) - n(B \cap A) + n(A \cap B \cap C)$$

Also, the number of the elements belonging only to C is

$$n(C) - n(C \cap A) - n(C \cap B) + n(A \cap B \cap C)$$

Thus the number of elements belonging to exactly one of the sets A, B and C is

$$[n(A) - n(A \cap B) - n(A \cap C) + n(A \cap B \cap C)] + [n(B) - n(B \cap C) - n(B \cap A) + n(A \cap B \cap C)]$$

$$+ [n(C) - n(C \cap A) - n(C \cap B) + n(A \cap B \cap C)]$$

$$= n(A) + n(B) + n(C) - 2n(A \cap B) - 2n(B \cap C) - 2n(C \cap A) + 3n(A \cap B \cap C)$$

QUICK LOOK 3

Summary of the formulas

Let A, B and C be given finite sets and S a universal finite set containing A, B and C. Then the following hold:

1. $n(A \cup B) + n(A \cap B) = n(A) + n(B)$
2. $n(A \cup B) = n(A - B) + n(B - A) + n(A \cap B)$
3. $n(A \cup B) = n(A) + n(B) \Leftrightarrow A \cap B = \phi$
4. $n(A) = n(A - B) + n(A \cap B)$
5. The number of the elements belonging to exactly one of A and B is
$$\begin{aligned} n(A \Delta B) &= n(A - B) + n(B - A) \\ &= n(A) + n(B) - 2n(A \cap B) \\ &= n(A \cup B) - n(A \cap B) \end{aligned}$$
6. The number of elements belonging to exactly one of A, B and C is
$$n(A) + n(B) + n(C) - 2n(A \cap B) - 2n(B \cap C) - 2n(C \cap A) + 3n(A \cap B \cap C)$$
7. The number of elements belonging to exactly two of A, B and C is
$$n(A \cap B) + n(B \cap C) + n(C \cap A) - 3n(A \cap B \cap C)$$
8. $n(A' \cup B') = n(S) - n(A \cap B)$
9. $n(A' \cap B') = n(S) - n(A \cup B)$

Example 1.11

If A and B are sets such that $n(A) = 9$, $n(B) = 16$ and $n(A \cup B) = 25$, find $A \cap B$.

Solution: We have
$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$
Therefore, substituting the values we get
$$\begin{aligned} 25 &= 9 + 16 - n(A \cap B) \\ &= 25 - n(A \cap B) \\ 0 &= n(A \cap B) \end{aligned}$$
Hence $A \cap B = \phi$.

Example 1.12

If A and B are sets such that $n(A) = 14$, $n(A \cup B) = 26$ and $n(A \cap B) = 8$, then find $n(B)$.

Solution: We have
$$\begin{aligned} n(B) &= n(A \cup B) + n(A \cap B) - n(A) \\ &= 26 + 8 - 14 = 20 \end{aligned}$$

Example 1.13

If A, B, C are sets such that $n(A) = 12, n(B) = 16, n(C) = 18$, $n(A \cap B) = 6, n(B \cap C) = 8, n(C \cap A) = 10$ and $n(A \cap B \cap C) = 4$, then find the number of elements belonging to exactly one of A, B and C.

Solution: The number of elements belonging to exactly one of A, B and C is
$$\begin{aligned} &n(A) + n(B) + n(C) - 2n(A \cap B) - 2n(B \cap C) \\ &- 2n(C \cap A) + 3n(A \cap B \cap C) \\ &= 12 + 16 + 18 - 2 \times 6 - 2 \times 8 - 2 \times 10 + 3 \times 4 \\ &= 10 \end{aligned}$$

Example 1.14

In Example 1.13, find the number of elements belonging to exactly two of A, B and C.

Solution: The number is
$$\begin{aligned} &n(A \cap B) + n(B \cap C) + n(C \cap A) - 3n(A \cap B \cap C) \\ &= 6 + 8 + 10 - 3 \times 4 = 12 \end{aligned}$$

Example 1.15

If A, B and C are sets defined as $A = \{x | x \in \mathbb{Z}^+ \text{ and } x \le 16\}$, $B = \{x | x \in \mathbb{Z} \text{ and } -3 < x < 8\}$ and $C = \{x | x \text{ is a prime number}\}$, then find the number of elements belonging to exactly two of A, B and C, even though C is an infinite set.

Solution: We have

$$n(A) = 16, \quad n(B) = 10 \quad \text{and} \quad n(C) = \infty$$

Now

$$A \cap B = \{1, 2, 3, 4, 5, 6, 7\}$$
$$B \cap C = \{2, 3, 5, 7\}$$
$$C \cap A = \{2, 3, 5, 7, 11, 13\}$$

and $$A \cap B \cap C = \{2, 3, 5, 7\}$$

Therefore, the required number is

$$n(A \cap B) + n(B \cap C) + n(C \cap A) - 3n(A \cap B \cap C)$$
$$= 7 + 4 + 6 - 3 \times 4 = 5$$

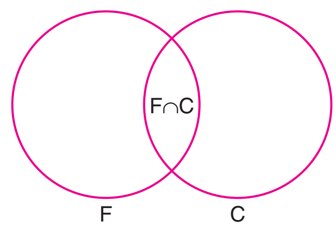
Example 1.16

In a group of 80 students, 50 play football, 45 play cricket and each student plays either football or cricket. Find the number of students who play both the games.

Solution: Let F be the set of the students who play football and C be the set of students who play cricket. Then $n(F) = 50$ and $n(C) = 45$.

Since each of the 80 students play at least one of the two games, we have $n(F \cup C) = 80$. Therefore,

$$n(F \cap C) = n(F) + n(C) - n(F \cup C)$$
$$= 50 + 45 - 80 = 15$$



Example 1.17

If 65% of people in a town like apples and 78% like mangoes, then find out the percentage of people who like both apples and mangoes and the percentage of people who like only mangoes.

Solution: Let the total number of people in the village be 100. Let A be the set of people who like apples and M the set of people who like mangoes. Then $n(A) = 65$, $n(M) = 78$ and $n(A \cup M) = 100$. Therefore

$$n(A \cap M) = n(A) + n(M) - n(A \cup M)$$
$$= 65 + 78 - 100 = 43$$

Hence 43% of people like both apples and mangoes. Also,

$$n(M) - n(A \cap M) = 78 - 43 = 35$$

Therefore, 35% of people like only mangoes.

Example 1.18

The total number of students in a school is 600. If 150 students drink apple juice, 250 students drink pineapple juice and 100 students drink both apple juice and pineapple juice, then find the number of students who drink neither apple juice nor pineapple juice.

Solution: Let

A = The set of students who drink apple juice
and P = The set of students who drink pineapple juice

We are given that $n(A) = 150$, $n(P) = 250$ and $n(A \cap P) = 100$. Then

$$n(A \cup P) = n(A) + n(P) - n(A \cap P)$$
$$= 150 + 250 - 100 = 300$$

Let S be the set of all students in the school, then S is the universal set containing A and P. We are given that $n(S) = 600$. Now,

$$n[S - (A \cup P)] = n(S) - n(A \cup P)$$
$$= 600 - 300 = 300$$

Therefore 300 students drink neither apple juice nor pineapple juice.

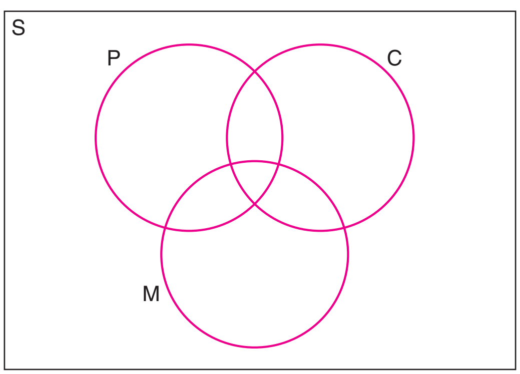
Example 1.19

In a class there are 400 students. Following is a table showing the number of students studying one or more of the subjects mentioned:

Mathematics	250
Physics	150
Chemistry	100
Mathematics and Physics	100
Mathematics and Chemistry	60
Physics and Chemistry	40
Mathematics, Physics and Chemistry	30
Only Mathematics	
Only Physics	
Only Chemistry	
None of Mathematics, Physics and Chemistry	

Fill in the empty places in the above table.

Solution: Let M, P and C stand for the set of students studying Mathematics, Physics and Chemistry. Let S be the set of all students in the class. The Venn diagram is as follows:



We are given that

$$n(S) = 400, n(M) = 250, n(P) = 150, n(C) = 100$$

Also, from the table,

$$n(M \cap P) = 100, n(M \cap C) = 60, n(P \cap C) = 40, n(M \cap P \cap C) = 30$$

We have,

$$\begin{aligned} n[M - (P \cup C)] &= n(M) - n[M \cap (P \cup C)] \\ &= n(M) - n[(M \cap P) \cup (M \cap C)] \\ &= n(M) - [n(M \cap P) + n(M \cap C) \\ &\quad - n(M \cap P \cap M \cap C)] \\ &= n(M) - n(M \cap P) - n(M \cap C) \\ &\quad + n(M \cap P \cap C) \\ &= 250 - 100 - 60 + 30 \\ &= 120 \end{aligned}$$

Therefore 120 students study only Mathematics. Also

$$\begin{aligned} n[P - (M \cup C)] &= n(P) - n[P \cap (M \cup C)] \\ &= 150 - n[(P \cap M) \cup (P \cap C)] \\ &= 150 - n(P \cap M) - n(P \cap C) \\ &\quad + n(P \cap M \cap C) \\ &= 150 - 100 - 40 + 30 \\ &= 40 \end{aligned}$$

Therefore 40 students study only Physics. Similarly,

$$\begin{aligned} n[C - (M \cup P)] &= n(C) - n[C \cap (M \cup P)] \\ &= 100 - n(C \cap M) - n(C \cap P) \\ &\quad + n(C \cap M \cap P) \\ &= 100 - 60 - 40 + 30 \\ &= 30 \end{aligned}$$

Therefore 30 students study only Chemistry. Again

$$\begin{aligned} n(M \cup P \cup C) &= n(M) + n(P) + n(C) - n(M \cap P) \\ &\quad - n(P \cap C) - n(C \cap M) + n(M \cap P \cap C) \\ &= 250 + 150 + 100 - 100 - 40 - 60 + 30 \\ &= 330 \end{aligned}$$

$$\begin{aligned} n[S - (M \cup P \cup C)] &= n(S) - n(M \cup P \cup C) \\ &= 400 - 330 = 70 \end{aligned}$$

Therefore 70 students study none of Mathematics, Physics and Chemistry.

Example 1.20

Let $X_1, X_2, \ldots, X_{30}$ be 30 sets each with five elements and $Y_1, Y_2, \ldots, Y_m$ be m sets each with 3 elements. Let

$$\bigcup_{i=1}^{30} X_i = \bigcup_{j=1}^{m} Y_j = S$$

Suppose that each element of S belongs to exactly 10 of X_i's and exactly 9 of Y_j's. Then find m.

Solution: Let $n(S) = s$. Since each element of S belongs to exactly 10 of X_i's, so

$$\sum_{i=1}^{30} n(X_i) = 10s$$

Since each X_i contains 5 elements, therefore

$$\sum_{i=1}^{30} n(X_i) = 30 \times 5 = 150$$

Therefore, $10s = 150$ and hence $s = 15$. Similarly

$$3m = \sum_{j=1}^{m} n(Y_j) = 9 \times s = 9 \times 15 = 135$$

Therefore, $m = 45$.

1.4 | Relations

Let A be the set of all straight lines in the plane and B the set of all points in the plane. For any $L \in A$ and $x \in B$, let us write $L\ \mathrm{R}\ x$ if the line L passes through the point x. This is a relation defined between elements of A and elements of B. Here $L\ \mathrm{R}\ x$ can be read as "L is related to x" and R denotes the *relation* "is passing through". Therefore $L\ \mathrm{R}\ x$ means "L is passing through x". We can also express this statement by saying that the pair of L and x is in relation R or that the ordered pair $(L, x) \in \mathrm{R}$. This pair is ordered in the sense that L and x cannot be interchanged because the first coordinate L represents a straight line and the second coordinate represents a point and because the statement "x passes through L" has no sense. Therefore, we can think of R as a set of ordered pairs (L, x) satisfying the property that L passes through x. We formalize this in the following.

DEFINITION 1.17 **Ordered Pairs** A pair of elements written in a particular order is called an *ordered pair*. It is written by listing its two elements in a particular order, separated by a comma and enclosing the pair in brackets. In the ordered pair (L, x), L is called the *first component* or the *first coordinate* and x is called the *second component* or the *second coordinate*.

The ordered pairs (3, 4) and (4, 3) are different even though they consist of same pair of elements; for example these represent different points in the Euclidean plane.

DEFINITION 1.18 **The Cartesian Product** Let A and B be any sets. The set of all ordered pairs (a, b) with $a \in A$ and $b \in B$ is called the Cartesian product of A and B and is denoted by $A \times B$; that is,

$$A \times B = \{(a, b) | a \in A \text{ and } b \in B\}$$

Examples

(1) Let $A = \{a, b, c\}$ and $B = \{1, 2\}$. Then

$$A \times B = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}$$

and $$B \times A = \{(1, a), (2, a), (1, b), (2, b), (1, c), (2, c)\}$$

(2) If $A = \{x, y, z\}$ and $B = \{a\}$, then

$$A \times B = \{(x, a), (y, a), (z, a)\}$$

and $$B \times A = \{(a, x), (a, y), (a, z)\}$$

QUICK LOOK 4

1. For any sets A and B,

$$A \times B = \phi \Leftrightarrow A = \phi \quad \text{or} \quad B = \phi$$

2. If one of A and B is an infinite set and the other is a non-empty set, then the Cartesian product $A \times B$ is an infinite set.

3. For any non-empty sets A and B,

$$A \times B = B \times A \Leftrightarrow A = B$$

DEFINITION 1.19 If $A_1, A_2, \ldots, A_n$ are sets, then their Cartesian product is defined as the set of n-tuples $(a_1, a_2, \ldots, a_n)$ such that $a_i \in A_i$ for $1 \le i \le n$. This is denoted by $A_1 \times A_2 \times \cdots \times A_n$ or $\mathop{\mathbf{X}}\limits_{i=1}^{n} A_i$ or $\prod_{i=1}^{n} A_i$. That is,

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for } 1 \le i \le n\}$$

If $A_1 = A_2 = \cdots = A_n = A$, say, then the Cartesian product $A_1 \times A_2 \times \cdots \times A_n$ is denoted by A^n; that is,

$$A^1 = A,\ A^2 = A \times A = \{(a, b) \mid a, b \in A\}$$

$$A^3 = A \times A \times A = \{(a, b, c) \mid a, b, c \in A\}$$

$$A^n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A \text{ for } 1 \le i \le n\}$$

Examples

(1) If $A = \{a, b, c\}$, then

$$A^2 = \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)\}$$

(2) If $A = \{1, 2\}$, then

$$A^3 = \{(1, 1, 1), (1, 1, 2), (1, 2, 1), (1, 2, 2), (2, 1, 1), (2, 1, 2), (2, 2, 1), (2, 2, 2)\}$$

THEOREM 1.15 For any finite sets A and B,

$$n(A \times B) = n(A) \cdot n(B)$$

PROOF Let A and B be finite sets such that $n(A) = m$ and $n(B) = n$. Then $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$ where a_i's are distinct elements of A and b_j's are distinct elements of B. In such case

$$A \times B = \bigcup_{i=1}^{m} (\{a_i\} \times B)$$

Since $\{a_i\} \times B = \{(a_i, b_j) \mid 1 \le j \le n\}$, we get that $n(\{a_i\} \times B) = n(B) = n$. Also, for any $i \ne k$, $a_i \ne a_k$ and hence

$$(\{a_i\} \times B) \cap (\{a_k\} \times B) = \phi$$

Therefore,

$$\begin{aligned} n(A \times B) &= n\left(\bigcup_{i=1}^{m} (\{a_i\} \times B)\right) \\ &= \sum_{i=1}^{m} n(\{a_i\} \times B) \\ &= \sum_{i=1}^{m} n(B) \\ &= \sum_{i=1}^{m} n \\ &= m \cdot n = n(A) \cdot n(B) \end{aligned}$$

■

COROLLARY 1.6 If $A_1, A_2, \ldots, A_m$ are finite sets, then $A_1 \times A_2 \times \cdots \times A_m$ is also finite and

$$n(A_1 \times A_2 \times \cdots \times A_m) = n(A_1) \times n(A_2) \times \cdots \times n(A_m)$$

COROLLARY 1.7 If A is a finite set and m is any positive integer, then

$$n(A^m) = [n(A)]^m$$

In particular, $n(A^2) = n(A)^2$.

QUICK LOOK 5

Let A, B, C and D be any sets. Then the following hold.

1. $(A \cup B) \times C = (A \times C) \cup (B \times C)$
2. $A \times (B \cup C) = (A \times B) \cup (A \times C)$
3. $A \times (B \cap C) = (A \times B) \cap (A \times C)$
4. $(A \cap B) \times C = (A \times C) \cap (B \times C)$
5. $(A \cup B) \times (C \cup D) =$
   $(A \times C) \cup (A \times D) \cup (B \times C) \cup (B \times D)$
6. $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$
   $= (A \times D) \cap (B \times C)$
7. $(A - B) \times C = (A \times C) - (B \times C)$
8. $A \times (B - C) = (A \times B) - (A \times C)$

Try it out Prove the equalities in Quick Look 5.

Examples

(1) If $A = \{a, b, c, d\}$ and $B = \{1, 2, 3\}$, then

$$n(A \times B) = n(A) \times n(B) = 4 \times 3 = 12$$

(2) If $A = \{a, b, c, d\}$, then

$$n(A^2) = n(A)^2 = 4^2 = 16$$

and $$n(A^3) = n(A)^3 = 4^3 = 64$$

(3) For any sets A and B, we have

$$A \times B = \bigcup_{a \in A} (\{a\} \times B) = \bigcup_{b \in B} (A \times \{b\})$$

(4) Let $S = \{(a, b) | a, b \in \mathbb{Z}^+ \text{ and } a + 2b = 7\}$. Then

$$S = \{(1, 3), (3, 2), (5, 1)\}$$

(5) Let

$A = \{1, 2, 3, 4, 5, 6\}$ and $S = \{(a, b) \mid a, b \in A$ and a divides $b\}$

Then

$$S = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6)\} \cup \{(2, 2), (2, 4), (2, 6)\} \cup \{(3, 3), (3, 6), (4, 4), (5, 5), (6, 6)\}$$

(6) If A is a finite set and $n(A) = m$, then $n[P(A \times A)] = 2^{m^2}$

(7) If A has 3 elements, then the number of subsets of $A \times A$ is $2^{3^2} = 2^9$, since $A \times A$ has 9 elements.

(8) If A has only one element, then A^n also has one element and $P(A^n)$ has two elements for any positive integer n.

(9) For any non-empty finite sets A and B,

$$n(A) = \frac{n(A \times B)}{n(B)} \quad \text{and} \quad n(B) = \frac{n(A \times B)}{n(A)}$$

Example 1.21

If A and B are sets such that $n(A \times B) = 6$ and $A \times B$ contains (1, 2), (2, 1) and (3, 2), then find the sets A, B and $A \times B$.

Solution: Since $n(A) \cdot n(B) = n(A \times B) = 6$, $n(A)$ and $n(B)$ are divisors of 6. Hence $n(A) = 1$ or 2 or 3 or 6. Since $(1, 2), (2, 1)$ and $(3, 2) \in A \times B$, $1, 2, 3 \in A$ and hence $n(A) \geq 3$. Also, $2, 1 \in B$ and hence $n(B) \geq 2$. Thus $n(A) = 3$ and $n(B) = 2$. Therefore, $A = \{1, 2, 3\}$ and $B = \{1, 2\}$, so that

$$A \times B = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 1), (3, 2)\}$$

Graphical Representation of Cartesian Product

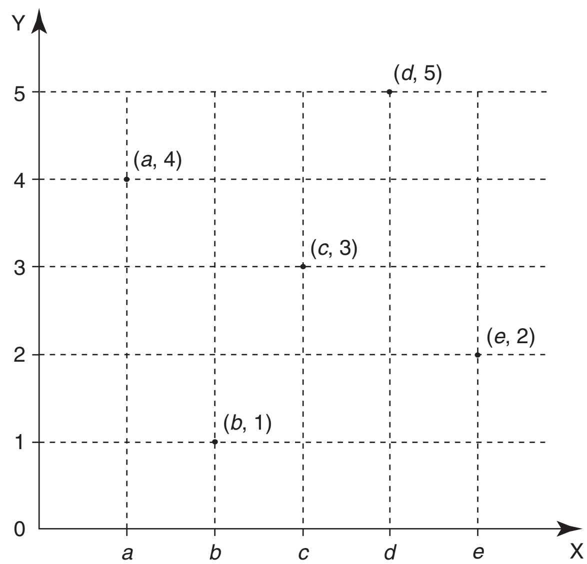


FIGURE 1.27 Graphical representation of Cartesian product.

Let A and B be non-empty sets. The Cartesian product $A \times B$ can be represented graphically by drawing two perpendicular lines OX and OY. We represent elements of A by points on OX and those of B by points on OY. Now draw a line parallel to OY through the point representing a on OX and a line parallel to OX through the point representing 4 on OY. The point of intersection of these lines represents the ordered pair $(a, 4)$ in $A \times B$. Figure 1.27 represents graphically the Cartesian product $A \times B$ where $A = \{a, b, c, d, e\}$ and $B = \{1, 2, 3, 4, 5\}$.

DEFINITION 1.20 For any sets A and B, any subset of $A \times B$ is called a *relation* from A to B.

Examples

(1) $\{(a, 2), (b, 1), (a, 4), (c, 3)\}$ is a relation from A to B, where $A = \{a, b, c, d\}$ and $B = \{1, 2, 3, 4\}$.

(2) For any sets A and B, the empty set ϕ and $A \times B$ are also relations from A to B.

DEFINITION 1.21 Let R be a relation from a set A into a set B. That is, $\mathrm{R} \subseteq A \times B$. If $(a, b) \in \mathrm{R}$, then we say that "a is R related to b" or "a is related to b with respect to R" or "a and b have relation R". It is usually denoted by $a \,\mathrm{R}\, b$.

DEFINITION 1.22 **Domain** Let R be a relation from A to B. Then the *domain* of R is defined as the set of all first components of the ordered pairs belonging to R and is denoted by Dom (R). Mathematically,

$$\mathrm{Dom(R)} = \{a \mid (a, b) \in \mathrm{R} \text{ for some } b \in B\}$$

Note that Dom(R) is a subset of A and that Dom(R) is non-empty if and only if R is non-empty.

DEFINITION 1.23 **Range** Let R be a relation from A to B. Then the *range* of R is defined as the set of all second components of the ordered pairs belonging to R and is denoted by Range(R). Mathematically,

$$\mathrm{Range(R)} = \{b \mid (a, b) \in \mathrm{R} \text{ for some } a \in A\}$$

Note that Range(R) is a subset of B and that it is non-empty if and only if R is non-empty.

Examples

(1) Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c, d, e\}$, and $R = \{(1, a), (2, c), (3, a), (2, a)\}$. Then

$$\text{Dom}(R) = \{1, 2, 3\} \quad \text{and} \quad \text{Range}(R) = \{a, c\}$$

(2) Let $A = \{2, 3, 4\}$, $B = \{2, 3, 4, 5, 6, 7, 8\}$ and $R = \{(a, b) \in A \times B \mid a \text{ divides } b\}$. Then

$$R = \{(2, 2), (2, 4), (2, 6), (2, 8), (3, 3), (3, 6), (4, 4), (4, 8)\}$$

$$\text{Dom}(R) = \{2, 3, 4\} \quad \text{and} \quad \text{Range}(R) = \{2, 4, 6, 8, 3\}$$

(3) Let $R = \{(a, b) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \mid 2a = b\}$. Then R is a relation from $\mathbb{Z}^+$ to $\mathbb{Z}^+$ and is given by

$$R = \{(a, 2a) \mid a \text{ is a positive integer}\}$$

Then

$$\text{Dom}(R) = \mathbb{Z}^+$$

and Range(R) = The set of all positive even integers

Theorem 1.16 Let A and B be non-empty finite sets with $n(A) = m$ and $n(B) = n$. Then the number of relations from A to B is 2^{mn}.

Proof It is known that the number of subsets of an n-element set is 2^n. Since the relations from A to B are precisely the subsets of $A \times B$ and since $n(A \times B) = n(A) \cdot n(B) = mn$, it follows that there are exactly 2^{mn} relations from A to B. ■

Examples

(1) Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$. Then $n(A) = 3, n(B) = 2$ and $n(A \times B) = n(A) \cdot n(B) = 3 \cdot 2 = 6$. Therefore there are exactly 64 ($=2^6$) relations from A to B.

(2) Let A and B be two finite sets and K be the number of relations from A to B. Then K is not divisible by any odd prime number, since $K = 2^{n(A) \cdot n(B)}$ and 2 is the only prime dividing 2^m for any positive integer m.

Representations of a Relation

A relation can be expressed in many forms such as:

1. **Roster form:** In this form, a relation R is represented by the set of all ordered pairs belonging to R. For example, $R = \{(1, a), (2, b), (3, a), (4, c)\}$ is a relation from the set $\{1, 2, 3, 4\}$ to the set $\{a, b, c\}$.
2. **Set-builder form:** Let $A = \{2, 3, 4, 5\}$ and $B = \{2, 4, 6, 8, 10\}$. Let $R = \{(a, b) \in A \times B \mid a \text{ divides } b\}$. Then R is a relation from A to B. This is known as the set-builder form of a relation. Note that

$$R = \{(2, 2), (2, 4), (2, 6), (2, 8), (2, 10), (3, 6), (4, 4), (4, 8), (5, 10)\}$$

3. **Arrow-diagram form:** In this form, we draw an arrow corresponding to each ordered pair (a, b) in R from the first component a to the second component b. For example, consider the relation R given in (2) above. Then R can be represented as shown in Figure 1.28. There are nine arrows corresponding to nine ordered pairs belonging to the relation R.

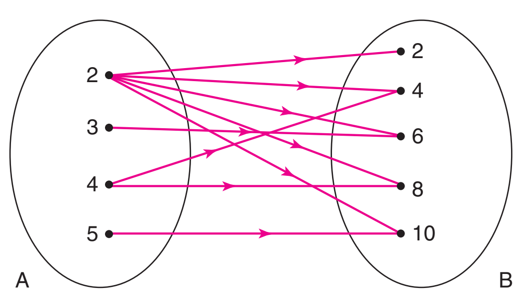


FIGURE 1.28 Representation of arrow-diagram form.

4. **Tabular form:** To represent a given relation R, sometimes it is convenient to look at it in a tabular form. Suppose R is a relation from a finite set A to a finite set B. Let

$$A = \{a_1, a_2, \ldots, a_n\} \quad \text{and} \quad B = \{b_1, b_2, \ldots, b_m\}$$

Write the elements $b_1, b_2, \ldots, b_m$ (in this order) in the top row of the table and the elements $a_1, a_2, \ldots, a_n$ (in this order) in the leftmost column. For any $1 \le i \le n$ and $1 \le j \le m$, let us define

$$r_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in \text{R} \\ 0 & \text{if } (a_i, a_j) \notin \text{R} \end{cases}$$

Write r_{ij} in the box present in the ith row written against a_i and in the jth column written against b_j. This is called the tabular form representation of the relation R.

Examples

Tabular Form

Let us consider sets $A = \{2, 3, 4, 5\}$, $B = \{2, 4, 6, 8, 10\}$, and relation R given by

$$\text{R} = \{(a, b) \in A \times B \mid a \text{ divides } b\}$$

That is

$$\text{R} = \{(2, 2), (2, 4), (2, 6), (2, 8), (2, 10), (3, 6), (4, 4), (4, 8), (5, 10)\}$$

This relation R is represented in the following tabular form.

R	2	4	6	8	10
2	1	1	1	1	1
3	0	0	1	0	0
4	0	1	0	1	0
5	0	0	0	0	1

Instead of writing 1 and 0, we can write T and F signifying whether $a_i \text{R} b_j$ is true or false.

Among all four representations of a relation, the set-builder form is most popular and convenient. The roster form, the arrow-diagram form and the tabular form can represent a relation R from A to B only when both the sets A and B are finite. The set-builder form is more general and can represent a relation even when A or B or both are infinite sets.

Examples

Let $\text{R} = \{(a, b) \in \mathbb{Z}^+ \times \mathbb{Z}^+ | a \text{ divides } b\}$. Then R is a relation from $\mathbb{Z}^+$ to $\mathbb{Z}^+$. This cannot be represented by the roster form or set-builder form or tabular form. Note that

$$\text{Dom(R)} = \mathbb{Z}^+ = \text{Range(R)}$$

DEFINITION 1.24 **Binary Relation** Any relation from a set A to itself is called a *binary relation* on A or simply a *relation* on A.

For example, the relation R given in the above example is a relation on $\mathbb{Z}^+$.

Remark: For any n-element set A, there are 2^{n^2} relations on A. For example, if $A = \{a, b, c\}$, then there are $512\ (= 2^{3^2})$ relations on A.

DEFINITION 1.25 **Composition of Relations** Let A, B and C be sets, R a relation from A to B and S a relation from B to C. Define

$$\text{S} \circ \text{R} = \{(a, c) \in A \times C \mid \text{there exists } b \in B \text{ such that } (a, b) \in \text{R and } (b, c) \in \text{S}\}$$

Then $\text{S} \circ \text{R}$ is a relation from A to C. In other words for any $a \in A$ and $c \in C$,

$$a(\text{S} \circ \text{R})c \Leftrightarrow a\text{R}b \text{ and } b\text{S}c \text{ for some } b \in B$$

$\text{S} \circ \text{R}$ is called the *composition* of R with S.

Note that, for any relations R with S, $\text{R} \circ \text{S}$ may not be defined at all even when $\text{S} \circ \text{R}$ is defined. Also even when both $\text{R} \circ \text{S}$ and $\text{S} \circ \text{R}$ are defined, they may not be equal.

Examples

(1) Let $R = \{(a, b) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \mid b = 2a\}$ and $S = \{(a, b) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \mid b = a + 2\}$. Then both R and S are relations from $\mathbb{Z}^+$ to $\mathbb{Z}^+$ and hence both $R \circ S$ and $S \circ R$ are defined. For any positive integers a and c, we have

$$a(R \circ S)c \Leftrightarrow aSb \text{ and } bRc \text{ for some } b \in \mathbb{Z}^+$$
$$\Leftrightarrow b = a + 2 \text{ and } c = 2b \text{ for some } b \in \mathbb{Z}^+$$
$$\Leftrightarrow c = 2(a + 2) = 2a + 4$$

and

$$a(S \circ R)c \Leftrightarrow aRb \text{ and } bSc \text{ for some } b \in \mathbb{Z}^+$$
$$\Leftrightarrow b = 2a \text{ and } c = b + 2 \text{ for some } b \in \mathbb{Z}^+$$
$$\Leftrightarrow c = 2a + 2$$

For example, $(3, 10) \in R \circ S$ since $(3, 5) \in S$ and $(5, 10) \in R$. Also, $(3, 8) \in S \circ R$ since $(3, 6) \in R$ and $(6, 8) \in S$.

Note that $(3, 8) \notin R \circ S$ and $(3, 10) \notin S \circ R$. Therefore $S \circ R \not\subset R \circ S$ and $R \circ S \not\subset S \circ R$.

(2) Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c, d\}$, and $C = \{x, y, z\}$. Let

$$R = \{(1, c), (2, d), (2, a), (3, d)\}$$

and $$S = \{(a, y), (b, x), (b, y), (a, z)\}$$

Then R is a relation from A to B and S is a relation from B to C.

$\text{Dom}(R) = \{1, 2, 3\}$ and $\text{Range}(R) = \{a, c, d\}$

$\text{Dom}(S) = \{a, b\}$ and $\text{Range}(S) = \{x, y, z\}$

$R \circ S$ is not defined. However $S \circ R$ is defined and

$$S \circ R = \{(2, y), (2, z)\}$$

Since $(2, a) \in R$ and $(a, y) \in S$, we have $(2, y) \in S \circ R$
Since $(2, a) \in R$ and $(a, z) \in S$, we have $(2, z) \in S \circ R$

THEOREM 1.17 Let A, B and C be sets, R a relation from A to B and S a relation from B to C. Then the following hold:

1. $S \circ R \neq \phi$ if and only if $\text{Range}(R) \cap \text{Dom}(S) \neq \phi$
2. $\text{Dom}(S \circ R) \subseteq \text{Dom}(R)$
3. $\text{Range}(S \circ R) \subseteq \text{Range}(S)$

PROOF

1. Suppose that $S \circ R \neq \phi$. Choose $(a, c) \in S \circ R$. Then there exists $b \in B$ such that $(a, c) \in R$ and $(b, c) \in S$ and hence $b \in \text{Range}(R)$ and $b \in \text{Dom}(S)$. Therefore $b \in \text{Range}(R) \cap \text{Dom}(S)$. Thus $\text{Range}(R) \cap \text{Dom}(S)$ is not empty.

   Conversely, suppose that $\text{Range}(R) \cap \text{Dom}(S) \neq \phi$. Choose $b \in \text{Range}(R) \cap \text{Dom}(S)$. Then there exist $a \in A$ and $c \in C$ such that $(a, b) \in R$ and $(b, c) \in S$ and hence $(a, c) \in S \circ R$. Thus $S \circ R$ is not empty.

2. $$a \in \text{Dom}(S \circ R) \Rightarrow (a, c) \in S \circ R \text{ for some } c \in C$$
$$\Rightarrow (a, b) \in R \quad \text{and} \quad (b, c) \in S \quad \text{for some } b \in B$$
$$\Rightarrow a \in \text{Dom}(R)$$

   Therefore $\text{Dom}(S \circ R) \subseteq \text{Dom}(R)$.

3. $$c \in \text{Range}(S \circ R) \Rightarrow (a, c) \in S \circ R \text{ for some } a \in A$$
$$\Rightarrow (a, b) \in R \quad \text{and} \quad (b, c) \in S \quad \text{for some } b \in B$$
$$\Rightarrow c \in \text{Range}(S)$$

   Therefore Range $(S \circ R) \subseteq \text{Range}(S)$. ■

Example 1.22

Find $S \circ R$, $\text{Dom}(S \circ R)$, $\text{Range}(S \circ R)$ for the following:

(1) $A = \{1, 2, 3, 4\}$, $B = \{a, b, c\}$ and $C = \{x, y, z\}$. The relations are $R = \{(2, a), (3, b), (2, b), (3, c)\}$ and $S = \{(a, y), (b, x), (b, y)\}$.

(2) The sets are the same as above. The relations are

$$R = \{(1, a), (2, b), (2, c), (4, a)\}$$

and $$S = \{(b, x), (b, y), (d, z)\}$$

Solution:

(1) From the given data, we have

$$\text{Dom(R)} = \{2, 3\} \quad \text{and} \quad \text{Range(R)} = \{a, b, c\}$$

$$\text{Dom(S)} = \{a, b\} \quad \text{and} \quad \text{Range(S)} = \{x, y\}$$

(a) $\text{S} \circ \text{R} = \{(2, y), (3, x), (3, y), (2, x)\}$

(b) $\text{Dom}(\text{S} \circ \text{R}) = \{2, 3\} = \text{Dom(R)}$

(c) $\text{Range}(\text{S} \circ \text{R}) = \{x, y\} = \text{Range(S)}$

(2) Using the given data we have

$$\text{S} \circ \text{R} = \{(2, x), (2, y)\}$$

$$\text{Dom}(\text{S} \circ \text{R}) = \{2\} \subset \{1, 2, 4\} = \text{Dom(R)}$$

$$\text{Range}(\text{S} \circ \text{R}) = \{x, y\} \subset \{x, y, z\} = \text{Range(S)}$$

THEOREM 1.18 Let A, B, C and D be non-empty sets, $\text{R} \subseteq A \times B$, $\text{S} \subseteq B \times C$ and $\text{T} \subseteq C \times D$. Then

$$(\text{T} \circ \text{S}) \circ \text{R} = \text{T} \circ (\text{S} \circ \text{R})$$

PROOF For any $a \in A$ and $d \in D$,

$$\begin{aligned}
(a, d) \in (\text{T} \circ \text{S}) \circ \text{R} &\Rightarrow (a, b) \in \text{R} \text{ and } (b, d) \in \text{T} \circ \text{S} \text{ for some } b \in B \\
&\Rightarrow (a, b) \in \text{R}, (b, c) \in \text{S} \text{ and } (c, d) \in \text{T} \text{ for some } b \in B \text{ and } c \in C \\
&\Rightarrow (a, c) \in \text{S} \circ \text{R} \text{ and } (c, d) \in \text{T}, c \in C \\
&\Rightarrow (a, d) \in \text{T} \circ (\text{S} \circ \text{R})
\end{aligned}$$

Therefore,

$$(\text{T} \circ \text{S}) \circ \text{R} \subseteq \text{T} \circ (\text{S} \circ \text{R})$$

Similarly

$$\text{T} \circ (\text{S} \circ \text{R}) \subseteq (\text{T} \circ \text{S}) \circ \text{R}$$

Thus,

$$(\text{T} \circ \text{S}) \circ \text{R} = \text{T} \circ (\text{S} \circ \text{R})$$ ■

DEFINITION 1.26 **Inverse of a Relation** Let A and B be non-empty sets and R a relation from A to B. Then the *inverse* of R is defined as the set

$$\{(b, a) \in B \times A \mid (a, b) \in R\}$$

and is denoted by R^{-1}.

Note that, if R is a relation from A to B, then R^{-1} is a relation from B to A and that $\text{R} \circ \text{R}^{-1}$ is a relation on B and $\text{R}^{-1} \circ \text{R}$ is a relation on A.

Examples

Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c, d, e\}$.

Let $\text{R} = \{(1, a), (2, b), (3, a), (4, d), (2, c), (3, e)\}$. Then

$$\text{R}^{-1} = \{(a, 1), (b, 2), (a, 3), (d, 4), (c, 2), (e, 3)\}$$

$$\text{R} \circ \text{R}^{-1} = \{(a, a), (b, b), (b, c), (a, e), (d, d), (c, b), (c, c), (e, a), (e, e)\}$$

and $\text{R}^{-1} \circ \text{R} = \{(1, 1), (2, 2), (3, 3), (4, 4)\} = \Delta_A$ (the diagonal of A).

THEOREM 1.19 Let A, B and C be non-empty sets and R a relation from A to B and S a relation from B to C. Then the following hold.

1. $(\text{S} \circ \text{R})^{-1} = \text{R}^{-1} \circ \text{S}^{-1}$
2. $(\text{R}^{-1})^{-1} = \text{R}$

PROOF **1.** $S \circ R$ is relation from A to C and therefore $(S \circ R)^{-1}$ is relation from C to A. Now consider

$$\begin{aligned}(c, a) \in (S \circ R)^{-1} &\Leftrightarrow (a, c) \in S \circ R \\ &\Leftrightarrow (a, b) \in R \text{ and } (b, c) \in S \text{ for some } b \in B \\ &\Leftrightarrow (c, b) \in S^{-1} \text{ and } (b, a) \in R^{-1} \text{ for some } b \in B \\ &\Leftrightarrow (c, a) \in R^{-1} \circ S^{-1}\end{aligned}$$

Therefore $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$.

2. It is trivial and left as an exercise for the reader. ■

1.5 | Equivalence Relations and Partitions

A partitioning of a set is dividing the set into disjoint subsets as shown in the Venn diagram in Figure 1.29. In this section we discuss a special type of relations on a set which induces a partition of the set and prove that any such partition is induced by that special type of relation. Let us begin with the following.

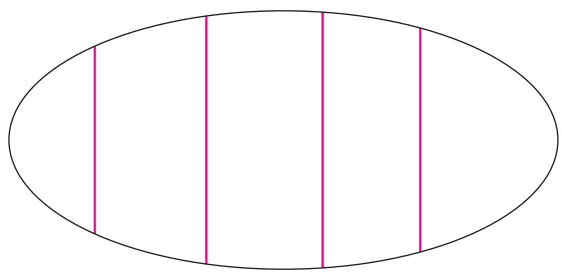


FIGURE 1.29 Partitioning of a set.

DEFINITION 1.27 Let X be a non-empty set and R a (binary) relation on X. Then,

1. R is said to be *reflexive* on X if $(x, x) \in R$ for all $x \in X$.
2. R is said to be *symmetric* if $(x, y) \in R \Rightarrow (y, x) \in R$
3. R is said to be *transitive* if $(x, y) \in R$ and $(y, z) \in R \Rightarrow (x, z) \in R$.
4. R is said to be an *equivalence relation* on X if it is a reflexive, symmetric and transitive relation on X.

Examples

(1) Let $X = \{1, 2, 3, 4\}$ and $R = \{(1, 2), (2, 1), (1, 1), (2, 2)\}$. Then R is a relation on X. R is not reflexive on X, since $3 \in X$ and $(3, 3) \notin R$. However R is symmetric and transitive. You can easily see that R is reflexive on a smaller set, namely $\{1, 2\}$. Therefore R is an equivalence relation on $\{1, 2\}$.

(2) Let $R = \{(a, b) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \mid a \text{ divides } b\}$. Then R is a reflexive and transitive relation on the set $\mathbb{Z}^+$ of positive integers. However, R is not symmetric, since $(2, 6) \in R$ and $(6, 2) \notin R$. Note that a relation R on a set S is symmetric $\Leftrightarrow R = R^{-1}$.

(3) Let $X = \{1, 2, 3, 4\}$ and $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (2, 3), (3, 2), (3, 4), (4, 3)\}$. Then R is a reflexive and symmetric relation on X. But R is not transitive, since $(2, 3) \in R$ and $(3, 4) \in R$, but $(2, 4) \notin R$.

(4) For any set X, let

$$\Delta_X = \{(x, x) | x \in X\}$$

Then Δ_X is reflexive, symmetric and transitive relation on X and hence an equivalence relation on X. Δ_X is called the diagonal on X.

(5) For any positive integer n, let

$$R_n = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid n \text{ divides } a - b\}$$

For any $a \in \mathbb{Z}$, n divides $0 = a - a$ and hence $(a, a) \in R_n$. Therefore R_n is reflexive on $\mathbb{Z}$. For any $a, b \in \mathbb{Z}$,

$$\begin{aligned}(a, b) \in R_n &\Rightarrow n \text{ divides } (a - b) \\ &\Rightarrow n \text{ divides } -(a - b) \\ &\Rightarrow n \text{ divides } (b - a) \\ &\Rightarrow (b, d) \in R_n\end{aligned}$$

Therefore R_n is symmetric. Also, for any a, b and $c \in \mathbb{Z}$,

$$\begin{aligned}(a,b) \in R_n \text{ and } (b,c) \in R_n &\Rightarrow n \text{ divides } (a-b) \text{ and } (b-c)\\ &\Rightarrow n \text{ divides } (a-b)+(b-c)\\ &\Rightarrow n \text{ divides } (a-c)\\ &\Rightarrow (a,c) \in R_n\end{aligned}$$

Therefore R_n is transitive also. Thus R_n is an equivalence relation on $\mathbb{Z}$ and is called the *congruence relation modulo n*.

(6) Let A and B be subsets of a set X such that $A \cap B = \phi$ and $A \cup B = X$. Define

$$R = \{(x, y) \in X \times X \mid \text{either } x, y \in A \text{ or } x, y \in B\}$$

Then R is an equivalence relation on X.

THEOREM 1.20 Let R be a symmetric and transitive relation on a set X. Then the following are equivalent to each other.

1. R is reflexive on X.
2. $\text{Dom}(R) = X$.
3. $\text{Range}(R) = X$.
4. R is equivalence relation on X.

PROOF Since R is already symmetric and transitive, **(1)** $\Leftrightarrow$ **(4)** is clear.
Also, since $(a, b) \in R$ if and only if $(b, a) \in R$, it follows that **(2)** $\Leftrightarrow$ **(3)**.

If R is reflexive on X, then $(x, x) \in R$ for all $x \in X$ and hence $\text{Dom}(R) = X$. Therefore **(1)** $\Leftrightarrow$ **(2)** is clear.

Finally, we shall prove **(2)** $\Rightarrow$ **(1)**. Suppose that $\text{Dom}(R) = X$. Then,

$$\begin{aligned}x \in X \Rightarrow x &\in \text{Dom}(R)\\ &\Rightarrow (x, y) \in R \text{ for some } y \in X\\ &\Rightarrow (x, y) \in R \text{ and } (y, x) \in R \text{ (since R is symmetric)}\\ &\Rightarrow (x, x) \in R \text{ (since R is transitive)}\end{aligned}$$

Therefore $(x, x) \in R$ for all $x \in X$. Thus R is reflexive on X. ■

DEFINITION 1.28 **Partition** Let X be a non-empty set. A class of non-empty subsets of X is called a *partition* of X if the members of the class are pairwise disjoint and their union is X. In other words, a class of sets $\{A_i\}_{i \in I}$ is called a partition of X if the following are satisfied:

1. For each $i \in I$, A_i is a non-empty subset of X
2. $A_i \cap A_j = \phi$ for all $i \neq j \in I$
3. $\bigcup_{i \in I} A_i = X$

Examples

(1) For any set X, the class $\{\{x\}\}_{x \in X}$ is a partition of X; that is, the class of all singleton subsets of X is a partition of X.

(2) Let E = the set of all even integers and O = the set of all odd integers. Then the class $\{E, O\}$ is a partition of $\mathbb{Z}$.

(3) For any non-empty proper subset A of a set X, the class $\{A, X - A\}$ is a partition of X. Note that $X - A$ is not empty since A is a proper subset of X.

DEFINITION 1.29 Let R be an equivalence relation on a set X and $x \in X$. Then define

$$R(x) = \{y \in x \mid (x, y) \in R\}$$

$R(x)$ is a subset of X and is called the *equivalence class of x* with respect to R or the R-equivalence *class of x* or simply the R-*class of x*.

Examples

(1) Let $X = \{1, 2, 3, 4\}$ and $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (2, 3), (3, 2)\}$. Then R is an equivalence relation on X and the R-classes are as follows:

$$R(1) = \{x \in X \mid (1, x) \in R\} = \{1\}$$
$$R(2) = \{x \in X \mid (2, x) \in R\} = \{2, 3\}$$
$$R(3) = \{x \in X \mid (3, x) \in R\} = \{2, 3\}$$
$$R(4) = \{x \in X \mid (4, x) \in R\} = \{4\}$$

(2) Let n be a positive integer and

$$R_n = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid n \text{ divides } a - b\}$$

Then R_n is an equivalence relation on the set $\mathbb{Z}$ of integers. For any $a \in \mathbb{Z}$, the R_n-class of "a" denoted by $R_n(a)$ is given by

$$\begin{aligned} R_n(a) &= \{y \in X \mid (a, y) \in R_n\} \\ &= \{y \in X \mid n \text{ divides } a - y\} \\ &= \{y \in X \mid a - y = nx \text{ for some } x \in \mathbb{Z}\} \\ &= \{a + nx \mid x \in \mathbb{Z}\} \end{aligned}$$

We can prove that $R_n(0), R_n(1), \ldots, R_n(n-1)$ are all the distinct R_n-classes in $\mathbb{Z}$. If $a \geq n$ or $a < 0$, we can write by the division algorithm that

$$a = qn + r$$

where $q, r \in \mathbb{Z}$ and $0 \leq r < n$. Hence $R_n(a) = R_n(r)$, $0 \leq r < n$.

THEOREM 1.21 Let R be an equivalence relation on a set X and $a, b \in X$. Then the following are equivalent to each other:

1. $(a, b) \in R$
2. $R(a) = R(b)$
3. $R(a) \cap R(b) \neq \phi$

PROOF **(1) ⇒ (2):** Suppose that $(a, b) \in R$. Then $(b, a) \in R$ (since R is symmetric) and

$$\begin{aligned} x \in R(a) &\Rightarrow (a, x) \in R \\ &\Rightarrow (b, a) \in R \text{ and } (a, x) \in R \\ &\Rightarrow (b, x) \in R \quad \text{(since R is transitive)} \\ &\Rightarrow x \in R(b) \end{aligned}$$

Therefore $R(a) \subseteq R(b)$. Similarly $R(b) \subseteq R(a)$. Thus $R(a) = R(b)$.

(2) ⇒ (3) is trivial, since $a \in R(a)$ and if $R(a) = R(b)$, then $a \in R(a) \cap R(b)$.

(3) ⇒ (1): Suppose that $R(a) \cap R(b) \neq \phi$. Choose an element $c \in R(a) \cap R(b)$. Then $(a, c) \in R$ and $(b, c) \in R$ and hence $(a, c) \in R$ and $(c, b) \in R$. Since R is transitive, we get that $(a, b) \in R$. ■

THEOREM 1.22 Let R be an equivalence relation on a set X. Then the class of all distinct R-classes forms a partition of X; that is,

1. $R(a)$ is a non-empty subset of X for each $a \in X$.
2. Any two distinct R-classes are disjoint.
3. The union of all R-classes is the whole set X.

PROOF

1. By definition of the R-class $R(a)$, we have

$$R(a) = \{x \in X \mid (a, x) \in R\}$$

Therefore $R(a)$ is a subset of X. Since $(a, a) \in R$ we have $a \in R(a)$. Thus $R(a)$ is a non-empty subset of X for each $a \in X$.

2. This is a consequence of **(2) ⇔ (3)** of Theorem 1.21.
3. Since $a \in R(a)$ for all $a \in X$, we have

$$\bigcup_{a \in X} R(a) = X$$

■

Examples

Let $X = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $R = \{(x, y) \in X \times X \mid$ both x and y are either even or odd$\}$. Then

$$R(1) = \{1, 3, 5, 7\} = R(3) = R(5) = R(7)$$

and $$R(2) = \{2, 4, 6, 8\} = R(4) = R(6) = R(8)$$

Therefore, there are only two distinct R-classes, namely $R(1) = \{1, 3, 5, 7\}$ and $R(2) = \{2, 4, 6, 8\}$ and these two form a partition of X.

In Theorem 1.22, we have obtained a partition from a given equivalence relation on set a X. Infact, for any given partition of X, we can define an equivalence relation on X which induces the given partition. This is proved in the following.

THEOREM 1.23 Let X be a non-empty set and $\{A_i\}_{i\in I}$ a partition of X. Define

$$R = \{(x, y) \in X \times X \mid \text{both } x \text{ and } y \text{ belong to same } A_i, i \in I\}$$

Then R is an equivalence relation whose R-classes are precisely A_i's.

PROOF We are given that $\{A_i\}_{i\in I}$ is a partition of X, that is,

1. Each A_i is a non-empty subset of X.
2. $A_i \cap A_j = \phi$ for all $i \neq j \in I$.
3. $\bigcup_{i\in I} A_i = X$.

For any $x \in X$, there exists only one $i \in I$ such that $x \in A_i$ and hence $(x, x) \in R$. This means that R is reflexive on X; clearly R is symmetric. Also, $(x, y) \in R$ and $(y, z) \in R \Rightarrow x, y \in A_i$ and $y, z \in A_j$ for some $i, j \in I$. This implies

$$\begin{aligned} &A_i \cap A_j \neq \phi \text{ and hence } i = j \text{ and } A_i = A_j \\ &\Rightarrow x, z \in A_i, i \in I \\ &\Rightarrow (x, z) \in R \end{aligned}$$

Thus R is transitive also. Therefore R is an equivalence relation on X. For any $i \in I$ and $x \in A_i$, we have

$$y \in A_i \Leftrightarrow (x, y) \in R \Leftrightarrow y \in R(x)$$

and have $A_i = R(x)$. This shows that A_i's are all the R-classes in X. ■

Theorems 1.22 and 1.23 imply that we can get a partition of X from an equivalence relation on X and conversely we can get an equivalence relation from a partition of X and that these processes are inverses to each other.

Examples

For any $i = 0, 1$ or 2, let

$$A_i = \{a \in \mathbb{Z}^+ \mid \text{on dividing } a \text{ with 3, the remainder is } i\}$$

That is,

$$\begin{aligned} A_0 &= \{3, 6, 9, 12, \ldots\} = \{3n \mid n \in \mathbb{Z}^+\} \\ A_1 &= \{1, 4, 7, 10, \ldots\} = \{3n + 1 \mid 0 \leq n \in \mathbb{Z}^+\} \\ A_2 &= \{2, 5, 8, 11, \ldots\} = \{3n + 2 \mid 0 \leq n \in \mathbb{Z}^+\} \end{aligned}$$

Then $\{A_0, A_1, A_2\}$ is a partition of $\mathbb{Z}^+$. The equivalence relation corresponding to this partition is

$$\begin{aligned} R &= \{(a, b) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \mid a, b \in A_0 \text{ or } a, b \in A_1 \text{ or } a, b \in A_2\} \\ &= \{(a, b) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \mid \text{The remainders are same when } a \text{ and } b \text{ are divided by } 3\} \\ &= \{(a, b) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \mid 3 \text{ divides } a - b\} \end{aligned}$$

In this case, $R(1) = A_1$, $R(2) = A_2$ and $R(3) = A_0$ and these three are the only R-classes in $\mathbb{Z}^+$.

THEOREM 1.24 Let R and S be two equivalence relations on a non-empty set X. Then $R \cap S$ is also an equivalence relation on X and, for any $x \in X$,

$$(R \cap S)(x) = R(x) \cap S(x)$$

PROOF For any $x \in X$, $(x, x) \in R$ and $(x, x) \in S$ (since R and S are reflexive on X). Hence $(x, x) \in R \cap S$. Therefore $R \cap S$ is reflexive on X. Also,

$$\begin{aligned}(x, y) \in R \cap S &\Rightarrow (x, y) \in R \quad \text{and} \quad (x, y) \in S \\ &\Rightarrow (y, x) \in R \quad \text{and} \quad (y, x) \in S \\ &\Rightarrow (y, x) \in R \cap S\end{aligned}$$

Therefore $R \cap S$ is symmetric. Further

$$\begin{aligned}(x, y), (y, z) \in R \cap S &\Rightarrow (x, y), (y, z) \in R \quad \text{and} \quad (x, y), (y, z) \in S \\ &\Rightarrow (x, z) \in R \quad \text{and} \quad (x, z) \in S \\ &\Rightarrow (x, z) \in R \cap S\end{aligned}$$

Therefore $R \cap S$ is an equivalence relation. For any $x \in X$, we have

$$\begin{aligned}(R \cap S)(x) &= \{y \in X \mid (x, y) \in R \cap S\} \\ &= \{y \in X \mid (x, y) \in R\} \cap \{y \in X \mid (x, y) \in S\} \\ &= R(x) \cap S(x)\end{aligned}$$

■

We have proved in Theorem 1.24 that the intersection of equivalence relations on a given set X is again an equivalence relation. This result cannot be extended to the composition of equivalence relations. In this direction, we have the following theorem that gives us several equivalent conditions for the composition of equivalence relations to again become an equivalence relation.

THEOREM 1.25 Let R and S be equivalence relations on a set X. Then the following are equivalent to each other.

1. $R \circ S$ is an equivalence relation on X.
2. $R \circ S$ is symmetric.
3. $R \circ S$ is transitive.
4. $R \circ S = S \circ R$.

PROOF **(1) ⇒ (2)** is clear.

(2) ⇒ (3): Suppose that $R \circ S$ is symmetric. Then

$$R \circ S = (R \circ S)^{-1} = S^{-1} \circ R^{-1} = S \circ R$$

and

$$\begin{aligned}(R \circ S) \circ (R \circ S) &= R \circ (S \circ R) \circ S \\ &= R \circ (R \circ S) \circ S \\ &= (R \circ R) \circ (S \circ S) \\ &= (R \circ S)\end{aligned}$$

Since R and S are reflexive, we get that $R \circ \Delta_X = R = \Delta_X \circ R$ and $S \circ \Delta_X = S = \Delta_X \circ S$. Also, since R and S are transitive, $R \circ R \subseteq R = R \circ \Delta_X \subseteq R \circ R^{-1}$ so that $R \circ R = R$. Similarly, $S \circ S = S$. Therefore, $R \circ S$ is transitive.

(3) ⇒ (4): Suppose that $R \circ S$ is transitive. Then $(R \circ S) \circ (R \circ S) = R \circ S$. Now, consider

$$S \circ R = (\Delta_X \circ S) \circ (R \circ \Delta_X) \subseteq (R \circ S) \circ (R \circ S) = R \circ S$$

and

$$R \circ S = R^{-1} \circ S^{-1} = (S \circ R)^{-1} \subseteq (R \circ S)^{-1} = S^{-1} \circ R^{-1} = S \circ R$$

Therefore

$$R \circ S = S \circ R$$

(4) ⇒ (1): Suppose that $R \circ S = S \circ R$. Then

$$(R \circ S)^{-1} = S^{-1} \circ R^{-1} = S \circ R = R \circ S$$

Hence $R \circ S$ is symmetric and transitive also. Further $\Delta_X = \Delta_X \circ \Delta_X \subseteq R \circ S$ and therefore $R \circ S$ is reflexive on X. Thus, $R \circ S$ is an equivalence relation on X. ■

1.6 | Functions

Functions are a special kind of relations from one set to another set. The concept of a function is an important tool in any area of logical thinking, not only in science and technology but also in social sciences. The word "function" is derived from a Latin word meaning operation. For example, when we multiply a given real number x by 2, we are performing an operation on the number x to get another number $2x$. A function may be viewed as a rule which provides new element from some given element. Function is also called a *map* or a *mapping*. In this section, we discuss various types of functions and their properties. The following is a formal definition of a function.

DEFINITION 1.30 **Function** A relation R from a set A to a set B is called a *function* (or a *mapping* or a *map*) from A *into* B if the following condition is satisfied:

For each element a in A there exists one and only one element b in B such that $(a, b) \in R$.

That is, $R \subseteq A \times B$ is called a function from A into B if the following hold:

1. For each $a \in A$, there exists $b \in B$ such that $(a, b) \in R$.
2. If $(a, b) \in R$ and $(a, c) \in R$, then $b = c$.

ALTERNATE DEFINITION A relation R from A to B is a function from A into B if $\text{Dom}(R) = A$ and whenever the first components of two ordered pairs in R are equal, then the second components are also equal.

Examples

(1) Let $R = \{(x, 2x) \mid x \in \mathbb{Z}\}$. Then R is a function from $\mathbb{Z}$ into $\mathbb{Z}$.

(2) Let $R = \{(x, |x|) \mid x \in \mathbb{R}\}$. Then R is a function from the real number system $\mathbb{R}$ into itself.

(3) Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c\}$. Let $R = \{(1, a), (2, a), (3, b), (4, b)\}$. Then R is a function from A into B.

(4) Let A and B be as in (3) above and $R = \{(1, a), (2, b), (3, c), (3, a), (4, a)\}$. Then R is not a function from A into B, since we have two ordered pairs $(3, c)$ and $(3, a)$ in R whose first components are equal and the second components are different. Also, if $S = \{(1, a), (2, b), (4, c)\}$, then S is not a function of A into B, since $\text{Dom}(S) \neq A$.

Notation

1. If R is a function from A into B and $a \in A$, then the unique element b in B such that $(a, b) \in R$ is denoted by $R(a)$.
2. Usually functions will be denoted by lower case letters $f, g, h, \ldots$.
3. If f is a function from A into B, then we denote this by $f : A \to B$.
4. If $f : A \to B$ is a function and $a \in A$, then there exists a unique element b in A such that $(a, b) \in f$. This unique element is denoted by $f(a)$. We write $f(a) = b$ to say that $(a, b) \in f$ or afb. Some authors also write $(a)f = b$ or simply $af = b$ to say that $(a, b) \in f$. However in this chapter we prefer to use $f(a) = b$.

DEFINITION 1.31 Let $f : A \to B$ be a function. Then A is called the *domain* of f and is denoted by $\text{Dom}(f)$. B is called the co-domain of f and is denoted by $\text{codom}(f)$. The range of f is also called the *image* of f or *the image of A under f* and is denoted by $\text{Im}(f)$. That is,

$$\text{Im}(f) = \{f(a) \mid a \in A\}$$

Note that $\text{Im}(f)$ is a subset of B and may not be equal to B. If $f(a) = b$, then b is called the *image of a under f* and a is called a *pre-image of b*. Note that for any $a \in A$, the image of a under f is unique. But, for $b \in B$, there may be several pre-images of b or there may not be any pre-image of b at all. To describe a function $f: A \to B$ it is enough if we prescribe the image $f(a)$ of each $a \in A$ under f.

Examples

(1) Define a function $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$ for all $x \in \mathbb{R}$. That is, $f = \{(x, x^2) \mid x \in \mathbb{R}\}$. Here x^2 is the image of any $x \in \mathbb{R}$. Note that x^2 is always non-negative for any $x \in \mathbb{R}$ and hence a negative real number has no pre-image under f. For example, there is no $x \in \mathbb{R}$ such that $f(x) = -1$. Here both the domain and co-domain of the function are $\mathbb{R}$ and the image of f (or range of f) is equal to the set of non-negative real numbers.

(2) Define $f: \mathbb{Z} \to \mathbb{R}$ by $f(x) = x/2$ for all $x \in \mathbb{Z}$. Then the domain of f is $\mathbb{Z}$ and the co-domain of f is $\mathbb{R}$. Also

$$\text{Im}(f) = \{f(x) \mid x \in \mathbb{Z}\} = \left\{\frac{x}{2} \mid x \in \mathbb{Z}\right\}$$

Here note that every integer n has a pre-image, namely $2n$, since $f(2n) = n$. The real number $1/3$ has no pre-image.

Quite often a function is given by an equation of type $f(x) = y$ without specifically mentioning the domain and co-domain. We can identify the domain and co-domain by looking at the validity of the equation. The following examples illustrate these.

Example 1.23

Let f be the function defined by

$$f(x) = \frac{x^2 + 2x + 1}{x^2 - 8x + 12}$$

Find out the domain of f.

Solution: We are given that

$$f(x) = \frac{x^2 + 2x + 1}{x^2 - 8x + 12}$$

The expression of the right-hand side has meaning for all real numbers except when $x = 6$ or $x = 2$. Therefore, the domain of f is the set at all real number other than 6 and 2, that is,

$$\text{Dom}(f) = \mathbb{R} - \{2, 6\}$$

Example 1.24

Consider a function defined by

$$f = \left\{\left(x, \frac{x^2}{1+x^2}\right) \middle| \, x \in \mathbb{R}\right\}$$

Then f is a function from $\mathbb{R}$ into $\mathbb{R}$. Find the range of f.

Solution: We have

$$f(x) = \frac{x^2}{1+x^2} \text{ for all } x \in \mathbb{R}$$

Suppose

$$y = f(x) = \frac{x^2}{1+x^2}$$

Then

$$y + yx^2 = x^2 \quad \text{or} \quad x^2(1-y) = y$$

Therefore

$$x^2 = \frac{y}{1-y} \quad \text{or} \quad x = \pm\sqrt{\frac{y}{1-y}}$$

provided $y/(1-y) \geq 0$; that is, $0 \leq y < 1$. Thus the range of f is $[0, 1)$.

DEFINITION 1.32 Let $f: A \to B$ and $g: B \to C$ be functions. Then the *composition* of f with g is defined as the function $g \circ f: A \to C$ given by

$$(g \circ f)(a) = g(f(a)) \text{ for all } a \in A$$

Note that $g \circ f$ is defined only when the range of f is contained in the domain of g. If $f : A \to B$ is a function and $g : D \to C$ is another function such that Range$(f) \subseteq D = \text{Dom}(g)$, then $g \circ f$ can be defined as a function from A into C. When we regard functions as relations, then the composition of functions is same as that of the relations as given in Definition 1.25. That is,

$$\begin{aligned}(a, c) \in g \circ f &\Leftrightarrow (a, b) \in f \quad \text{and} \quad (b, c) \in g \quad \text{for some } b \in B\\ &\Leftrightarrow f(a) = b \quad \text{and} \quad g(b) = c\\ &\Leftrightarrow g(f(a)) = c\\ &\Leftrightarrow (g \circ f)(a) = c\end{aligned}$$

Example 1.25

Let $f: \mathbb{Z} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \frac{x+2}{3} \quad \text{for all } x \in \mathbb{Z}$$

and $$g(x) = \frac{x^2 - 1}{x^2 + 1} \quad \text{for all } x \in \mathbb{R}$$

Find $(g \circ f)(x)$.

Solution: We have

$$(g \circ f)(x) = g(f(x)) = g\left(\frac{x+2}{3}\right)$$

$$= \frac{[(x+2)/3]^2 - 1}{[(x+2)/3]^2 + 1}$$

$$= \frac{(x+2)^2 - 9}{(x+2)^2 + 9}$$

$$= \frac{x^2 + 4x - 5}{x^2 + 4x + 13}$$

Example 1.26

Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c\}$ and $C = \{x, y, z\}$. Let

$$f = \{(1, a), (2, c), (3, b), (4, a)\}$$

and $$g = \{(a, y), (b, z), (c, x)\}$$

Find $g \circ f$.

Solution: We have $f : A \to B$ and $g : B \to C$ are functions. Then $g \circ f : A \to C$ is given by

$$g \circ f = \{(1, y), (2, x), (3, z), (4, y)\}$$

Try it out

THEOREM 1.26 Let $f : A \to B$ and $g : B \to C$ be functions. Then Dom$(g \circ f)$ = Dom(f) and codom$(g \circ f)$ = codom(g).

Two functions f and g are said to be equal if their domains are equal and $f(x) = g(x)$ for all elements x in Dom (f). For any functions f and g, even when both $g \circ f$ and $f \circ g$ are defined, $g \circ f$ may be different from $f \circ g$, as seen in the following example.

Example 1.27

Let $f: \mathbb{Z} \to \mathbb{Z}$ and $g : \mathbb{Z} \to \mathbb{Z}$ be defined by

$$f(x) = x^2 \quad \text{and} \quad g(x) = x + 2 \quad \text{for all } x \in \mathbb{Z}$$

Show that $g \circ f \neq f \circ g$.

Solution: We have

$$(g \circ f)(x) = g(f(x)) = g(x^2) = x^2 + 2$$

$$\begin{aligned}(f \circ g)(x) &= f(g(x))\\ &= f(x+2) = (x+2)^2\\ &= x^2 + 4x + 4\end{aligned}$$

Therefore $g \circ f \neq f \circ g$.

The following is an easy verification and is a direct consequence of Theorem 1.18.

Try it out

THEOREM 1.27 Let $f: A \to B$, $g: B \to C$ and $h: C \to D$ be functions. Then

$$h \circ (g \circ f) = (h \circ g) \circ f$$

In the following we discuss certain special types of functions. If $f: A \to B$ is a function, a_1 and a_2 are elements of A and b_1 and b_2 are elements of B such that $f(a_1) = b_1$ and $f(a_2) = b_2$ and if $a_1 = a_2$, then necessarily $b_1 = b_2$. In other words, two elements of B are equal if their pre-images are equal. It is quite possible that two distinct elements of A may have equal images under f. A function $f: A \to B$ is called an injection if distinct elements of A have distinct images under f. The following is a formal definition.

DEFINITION 1.32 **Injection** A function $f: A \to B$ is called an *injection* or "*one-one function*" if $f(a_1) \neq f(a_2)$ for any $a_1 \neq a_2$ in A; in other words,

$$f(a_1) = f(a_2) \Rightarrow a_1 = a_2$$

for any $a_1, a_2 \in A$.

Examples

(1) Let $f: \mathbb{Z} \to \mathbb{Z}$ be defined by

$$f(x) = x + 2 \quad \text{for all } x \in \mathbb{Z}$$

Then f is an injection, since, for any $x, y, \in \mathbb{Z}$,

$$f(x) = f(y) \Rightarrow x + 2 = y + 2 \Rightarrow x = y$$

(2) Let $f: \mathbb{Z} \to \mathbb{Z}$ be defined by

$$f(x) = x^2 \quad \text{for all } x \in \mathbb{Z}$$

Then f is not an injection, since two distinct elements have the same image; for example, $1 \neq -1$ but $f(1) = 1^2 = (-1)^2 = f(-1)$.

THEOREM 1.28 Let $f: A \to B$ and $g: B \to C$ be functions. Then the following hold.

1. If f and g are injections, then so is $g \circ f$.
2. If $g \circ f$ is an injection, then f is an injection.

PROOF
1. Suppose that both f and g are injections. For any $a_1, a_2 \in A$, we have

$$\begin{aligned}
&(g \circ f)(a_1) = (g \circ f)(a_2) \\
&\Rightarrow g(f(a_1)) = g(f(a_2)) \\
&\Rightarrow f(a_1) = f(a_2) \quad \text{(since } g \text{ is an injection)} \\
&\Rightarrow a_1 = a_2 \quad \text{(since } f \text{ is an injection)}
\end{aligned}$$

Therefore, $g \circ f$ is an injection.

2. Suppose that $g \circ f$ is an injection. Then, for any $a_1, a_2 \in A$, we have

$$\begin{aligned}
f(a_1) = f(a_2) &\Rightarrow g(f(a_1)) = g(f(a_2)) \quad (\because g \text{ is a function}) \\
&\Rightarrow (g \circ f)(a_1) = (g \circ f)(a_2) \\
&\Rightarrow a_1 = a_2 \quad \text{(since } g \circ f \text{ is an injection)}
\end{aligned}$$

Therefore f is an injection. ■

Note that $g \circ f$ can be an injection without g being an injection. An example of this case is given in the following.

Example

Define

$$f : \mathbb{Z}^+ \to \mathbb{Z} \text{ by } f(x) = x + 2 \text{ for all } x \in \mathbb{Z}^+$$

and $g : \mathbb{Z} \to \mathbb{Z}$ by $g(x) = x^2$ for all $x \in \mathbb{Z}$

Then $g \circ f : \mathbb{Z}^+ \to \mathbb{Z}$ is given by

$$(g \circ f)(x) = g(f(x)) = g(x + 2) = (x + 2)^2 \text{ for all } x \in \mathbb{Z}^+$$

Now, for any $x, y \in \mathbb{Z}^+$,

$$(g \circ f)(x) = (g \circ f)(y)$$

$$\Rightarrow (x + 2)^2 = (y + 2)^2$$

$$\Rightarrow x + 2 = y + 2 \quad \text{(since } x \text{ and } y \text{ are positive)}$$

$$\Rightarrow x = y$$

Therefore $g \circ f$ is an injection. However, g is not an injection, since

$$g(2) = 2^2 = (-2)^2 = g(-2)$$

Next we discuss functions under which every element in the codomain is the image of some element in the domain.

DEFINITION 1.33 **Surjection** A function $f : A \to B$ is called a *surjection* or "*onto function*" if the range of f is equal to the co-domain B; that is, for each $b \in B$, $b = f(a)$ for some $a \in A$.

Examples

(1) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = 2x + 1 \quad \text{for all } x \in \mathbb{R}$$

Then, for any element y in the co-domain $\mathbb{R}$, we have $(y - 1)/2$ is in the domain $\mathbb{R}$ and

$$f\left(\frac{y-1}{2}\right) = \frac{2(y-1)}{2} + 1 = y$$

Therefore f is a surjection. Note that f is an injection also, since

$$f(x) = f(y) \Rightarrow 2x + 1 = 2y + 1 \Rightarrow x = y$$

(2) Let $\mathbb{N}$ be the set of all non-negative integers. Define $f : \mathbb{Z} \to \mathbb{N}$ by $f(x) = |x|$ for all $x \in \mathbb{Z}$. Then f is a surjection, since $f(x) = x$ for all $x \in \mathbb{N}$ and $\mathbb{N} \subseteq \mathbb{Z}$. However, f is not an injection since

$$f(-1) = |-1| = 1 = f(1) \quad \text{and} \quad -1 \neq 1$$

(3) Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2 + 1$ for all $x \in \mathbb{R}$. Then f is neither an injection nor a surjection. It is not an injection, since

$$f(-1) = (-1)^2 + 1 = 2 = 1^2 + 1 = f(1) \quad \text{and} \quad -1 \neq 1$$

f is not a surjection, since we cannot find an element x in $\mathbb{R}$ such that $x^2 + 1 = 0$; that is $f(x) = 0$.

(4) Define $f : \mathbb{Z} \to \mathbb{R}$ by $f(x) = x + 2$ for all $x \in \mathbb{Z}$. Then f is an injection and it is not a surjection, since we cannot find an integer x such that $f(x) = 1/2$. Note that $f(x) = x + 2$ is always an integer for any integer x.

THEOREM 1.29 Let $f : A \to B$ and $g : B \to C$ be functions. Then the following hold:

1. If f and g are surjections, then so is $g \circ f$.
2. If $g \circ f$ is a surjection, then g is a surjection.

PROOF 1. Suppose that f and g are surjections. Also $g \circ f$ is a function from A into C. The domain of $g \circ f$ is A and the co-domain of $g \circ f$ is C. Now,

$$c \in C \Rightarrow c = g(b) \text{ for some } b \in B \quad \text{(since } g \text{ is a surjection)}$$

$$\Rightarrow f(a) = b \text{ and } g(b) = c \text{ for some } a \in A \text{ and } b \in B \quad \text{(since } f \text{ is a surjection)}$$

$$\Rightarrow a \in A \text{ and } (g \circ f)(a) = g(f(a)) = g(b) = c$$

$$\Rightarrow (g \circ f)(a) = c \quad \text{for some } a \in A$$

Thus $g \circ f$ is a surjection.

2. Suppose $g \circ f$ is a surjection. To prove that $g : B \to C$ is a surjection, let $c \in C$. Since $g \circ f : A \to C$ is a surjection, there exists $a \in A$ such that $(g \circ f)(a) = c$. Then $f(a) \in B$ and

$$g(f(a)) = (g \circ f)(a) = c$$

Thus g is a surjection. ■

Note that $g \circ f$ can be a surjection without f being a surjection. This is substantiated in the following.

Example

Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = [2x]$ for $x \in \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{Z}$ by $g(x) = [x]$ for all $x \in \mathbb{R}$, where $[x]$ is the integral part of x (i.e., $[x]$ is the largest integer $\leq x$). Then $g \circ f : \mathbb{R} \to \mathbb{Z}$ is given by

$$(g \circ f)(x) = g(f(x)) = [[2x]] = [2x]$$

In this case $g \circ f$ is a surjection, since, for any $n \in \mathbb{Z}, n/2 \in \mathbb{R}$ and

$$(g \circ f)\left(\frac{n}{2}\right) = \left[2 \cdot \frac{n}{2}\right] = [n] = n$$

However f is not a surjection, since $f(x)$ is always an integer and we cannot find $x \in \mathbb{R}$ such that $f(x) = 1/2$.

It is a convention that, when $f : A \to B$ is a surjection, we often denote this by saying "f is a function of A *onto* B" or f is a surjection of A *onto* B. We use the word *onto* only in the case of surjections. Whenever we want to mention that $f : A \to B$ is a surjection, we say that f is a surjection (or surjective function or onto function) of A onto B.

DEFINITION 1.34 **Bijection** A function $f : A \to B$ is said to be a *bijection* or a *one-one* and *onto function* or a *one-to-one function* if f is both injective and surjective.

Examples

(1) For any set X, define $I : X \to X$ by $I(x) = x$ for all $x \in X$. Then clearly I is an injection and a surjection, and hence a bijection. This is called the *identity function on* X or *identity map on* X. To specify the set X also, we denote the identity function I on X by I_X.

(2) Define $f : \mathbb{Z} \to \mathbb{Z}$ by $f(x) = x + 3$ for all $x \in \mathbb{Z}$. Then f is a bijection of $\mathbb{Z}$ onto $\mathbb{Z}$ (the term "onto" is used, since any bijection is necessarily a surjection).

(3) For any real numbers a and b with $a \neq 0$, define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = ax + b \text{ for all } x \in \mathbb{R}$$

Then f is an injection, since

$$\begin{aligned} f(x) = f(y) &\Rightarrow ax + b = ay + b \Rightarrow ax = ay \\ &\Rightarrow x = y \quad (\text{since } a \neq 0) \end{aligned}$$

Also, f is surjective, since, for any $y \in \mathbb{R}$,

$$\frac{y-b}{a} \in \mathbb{R} \quad \text{and} \quad f\left(\frac{y-b}{a}\right) = a\left(\frac{y-b}{a}\right) + b = y$$

Thus, f is a bijection of $\mathbb{R}$ onto itself.

(4) Let E be the set of all even integers and $\mathbb{Z}$ the set of all integers. Define $f : E \to \mathbb{Z}$ by

$$f(x) = \begin{cases} 2y & \text{if } x = 4y \\ y & \text{if } x = 2y \text{ and } y \text{ is odd} \end{cases}$$

Then f is a bijection. One can verify that

$$\begin{array}{ll} f(0) = 0 & f(-2) = -1 \\ f(2) = 1 & f(-4) = -2 \\ f(4) = 2 & \\ f(6) = 3 & f(-n) = -f(n) \\ f(8) = 4 & \end{array}$$

Try it out

THEOREM 1.30 Let $f : A \to B$ be any function. Then

$$I_B \circ f = f = f \circ I_A$$

THEOREM 1.31 If $f: A \to B$ and $g: B \to C$ are bijections, then $g \circ f: A \to C$ is a bijection.

PROOF This is an immediate consequence of Theorems 1.28 [part (1)] and 1.29 [part (1)], since a bijection is both an injection as well as a surjection. ■

In the following, we give a characterization property for bijections.

THEOREM 1.32 Let $f: A \to B$ be a mapping. Then f is a bijection if and only if there exists a function $g: B \to A$ such that

$$g \circ f = I_A \quad \text{and} \quad f \circ g = I_B$$

that is, $g(f(a)) = a$ for all $a \in A$ and $f(g(b)) = b$ for all $b \in B$.

PROOF If there is a function $g: B \to A$ such that

$$g \circ f = I_A \quad \text{and} \quad f \circ g = I_B$$

then, by Theorem 1.28 [part (2)], f is an injection (since $g \circ f = I_A$ which is an injection. Also, by Theorem 1.29 [part (2)], f is a surjection (since $f \circ g = I_B$ which is a surjection). Thus f is a bijection. Conversely suppose that f is a bijection. Define $g: B \to A$ as follows:

$$g(b) = \text{The pre-image of } b \text{ under } f$$

That is, if $f(a) = b$, then $g(b)$ is defined as a. First observe that every element $b \in B$ has a pre-image $a \in A$ under f (since f is a surjection). Also, this pre-image is unique (since f is an injection). Therefore g is properly defined as a function from B into A. Now, for any $a \in A$ and $b \in B$, we have

$$(g \circ f)(a) = g(f(a)) = a$$

since a is the pre-image of $f(a)$ and

$$(f \circ g)(b) = f(g(b)) = b$$

since $g(b) = a$ if $f(a) = b$. Thus $g \circ f = I_A$ and $f \circ g = I_B$. ■

DEFINITION 1.35 **Inverse of a Bijection** Let $f: A \to B$ and $g: B \to A$ be functions such that $g \circ f = I_A$ and $f \circ g = I_B$. Then both f and g are bijections (by the above theorem). Also, g is unique such that $g \circ f = I_A$ and $f \circ g = I_B$, since, for any $a \in A$ and $b \in B$, we have

$$f(a) = b \Leftrightarrow g(f(a)) = g(b) \Leftrightarrow a = g(b)$$

The function g is called the *inverse function* of f and f is called the *inverse function* of g. Both f and g are interrelated by the property

$$f(a) = b \Leftrightarrow a = g(b)$$

for all $a \in A$ and $b \in B$. The inverse function of f is denoted by f^{-1}. When we look at f as a relation, then f^{-1} is precisely the inverse relation as defined in Definition 1.26.

To confirm that f is a bijection, the existence of g satisfying both the properties $g \circ f = I_A$ and $f \circ g = I_B$ are necessary. Just $g \circ f = I_A$ may not imply that f is a bijection. In this context, we have the following two results.

THEOREM 1.33 Let $f: A \to B$ be a function. Then f is an injection if and only if there exists a function $g: B \to A$ such that $g \circ f = I_A$.

PROOF If $g: B \to A$ is a function such that $g \circ f = I_A$, then by Theorem 1.28 [part (2)], f is an injection, Conversely suppose that f is an injection. Choose an arbitrary element $a_0 \in A$ and define $g: B \to A$ as follows:

$$g(b) = \begin{cases} a & \text{if } b = f(a) \text{ for some } a \in A \\ a_0 & \text{if } b \notin \text{Range}(f) \end{cases}$$

Recall that $\text{Range}(f) = \{f(a) \mid a \in A\} \subseteq B$. Since f is an injection, there can be at most one $a \in A$ for any $b \in B$ such that $f(a) = b$. Therefore, g is a well-defined function from B into A. Also, for any $a \in A$,

$$(g \circ f)(a) = g(f(a)) = a$$

and hence $g \circ f = I_A$. ■

THEOREM 1.34 Let $f : A \to B$ be a function. Then f is a surjection if and only if there exists a function $g : B \to A$ such that $f \circ g = I_B$.

PROOF If there is a function $g : B \to A$ such that $f \circ g = I_B$, then, by Theorem 1.29 [part (2)], f is a surjection.

Conversely, suppose that f is a surjection. Then each element b in B has a pre-image a in A [i.e., a is an element in A such that $f(a) = b$]. Now, for each $b \in B$, choose one element a_b in A such that $f(a_b) = b$. Define $g : B \to A$ by

$$g(b) = a_b \text{ for each } b \in B$$

Then g is a function from B into A and, for any $b \in B$, we have

$$(f \circ g)(b) = f(g(b)) = f(a_b) = b$$

Therefore $f \circ g = I_B$. ■

DEFINITION 1.36 **Real-Valued Function** If $f: A \to B$ is a function and $a \in A$ then the image $f(a)$ is also called a *value of f at a*. If the value of f at each $a \in A$ is a real number, then f is called a *real-valued function on A*; that is, any function from a set A into a subset of the real number system $\mathbb{R}$ is called a real-valued function on A.

If $f: A \to B$ is a function and $B \subseteq C$, then f can be treated as a function from A into C as well. Therefore, a real-valued function on A is just a function from A into $\mathbb{R}$.

QUICK LOOK 6

Let f and g be real valued functions on a set A. Then we define the real-valued functions $f + g$, $-f$, $f - g$ and $f \cdot g$ on A as follows:

1. $(f + g)(a) = f(a) + g(a)$
2. $(-f)(a) = -f(a)$
3. $(f - g)(a) = f(a) - g(a)$
4. $(f \cdot g)(a) = f(a)f(b)$

Note that the operation symbols are those in the real number system $\mathbb{R}$. Also, if $g(a) \neq 0$ for all $a \in A$, then the function f/g is defined as follows:

5. $(f/g)(a) = f(a)/g(a)$ for all $a \in A$

Examples

(1) Let f be a polynomial over $\mathbb{R}$, that is

$$f = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

where $a_0, a_1, a_2, \ldots, a_n$ are all real numbers. For any $a \in \mathbb{R}$, let us define

$$f(a) = a_0 + a_1a + a_2a^2 + \cdots + a_na^n$$

Then $f: \mathbb{R} \to \mathbb{R}$ is a real-valued function on $\mathbb{R}$ and is called a *polynomial function*.

(2) Define $f: \mathbb{R} \to \mathbb{R}$ by

$$f(a) = e^a \quad \text{for all } a \in \mathbb{R}$$

Then f is a real-valued function on $\mathbb{R}$.

(3) Define $f : [0, 2\pi] \to \mathbb{R}$ by

$$f(a) = \sin a \quad \text{for all } a \in \mathbb{R}$$

Then f is a real-valued function defined on $[0, 2\pi]$ and is denoted by sin.

(4) Define $f : \mathbb{R}^+ \to \mathbb{R}$ by

$$f(a) = \sqrt{a} \quad \text{for all } a \in \mathbb{R}^+$$

This is a real-valued function defined on $\mathbb{R}^+$. Here $\sqrt{a}$ stands for the positive square root of a.

We have earlier made use of the notation $[x]$ to denote the largest integer $\leq x$ and called it the integral part of x. Now, we shall formally define this concept before going on to prove certain important properties.

DEFINITION 1.37 For any real number x, the largest integer less than or equal to x is called the *integral part of* x and in denoted by $[x]$. The real number $x-[x]$ is called the *fractional part of* x and is denoted by $\{x\}$.

Note that, for any real number x, $[x]$ is an integer and $\{x\}$ is a real number such that

$$x=[x]+\{x\} \quad \text{and} \quad 0 \leq \{x\} < 1$$

Also, this expression of x is unique in the sense that, if n is an integer and a is a real number such that $x=n+a$ and $0 \leq a < 1$, then $n=[x]$ and $a=\{x\}$.

Examples

(1) $\left[\frac{5}{6}\right]=0$ and $\left\{\frac{5}{6}\right\}=\frac{5}{6}$

(2) For any $0 \leq a < 1, [a]=0$ and $\{a\}=a$

(3) $\left[-\frac{1}{4}\right]=-1$ and $\left\{\frac{-1}{4}\right\}=\frac{3}{4}$

(4) $\left[\frac{-11}{10}\right]=-2$ and $\left\{\frac{-11}{10}\right\}=\frac{9}{10}$

THEOREM 1.35 The following hold for any real number x.

1. $[x] \leq x < [x]+1$
2. $x-1 < [x] \leq x$
3. $0 \leq \{x\}=x-[x] < 1$
4. $[x]=\sum_{1 \leq i \leq x} i$, if $x>0$
5. $[x]=x \Leftrightarrow x \in \mathbb{Z} \Leftrightarrow \{x\}=0$
6. $\{x\}=x$ if and only if $[x]=0$
7. $[x]+[-x]=\begin{cases} 0 & \text{if } x \text{ is an integer} \\ -1 & \text{if } x \text{ is not an integer} \end{cases}$

PROOF **(1)** through **(6)** are all straight-forward verifications using the definition that $[x]$ is the largest integer n such that $n \leq x$ and that $x-[x]=\{x\}$.

To prove **(7)**, let $[x]=n$. Then $n \leq x < n+1$ and therefore

$$-n-1 < -x \leq -n$$

If x is an integer, then so is $-x$ and hence $[x]+[-x]=x+(-x)=0$. If x is not an integer, then $-x$ is also not an integer and therefore

$$-n-1 < -x \; -n$$

So $[-x]=-n-1$ and hence $[x]+[-x]=n+(-n-1)=-1$. ■

Examples

(1) $\left[-\frac{9}{5}\right]+\left[\frac{9}{5}\right]=-2+1=-1$

(2) $[-3]+[3]=-3+3=0$

(3) $\left[\frac{6}{5}\right]+\left[\frac{-6}{5}\right]=1+(-2)=-1$

(4) $\left[\frac{-7}{8}\right]+\left[\frac{7}{8}\right]=-1+0=-1$

THEOREM 1.36 The following hold for any real numbers x and y:

1. $[x+y]=\begin{cases}[x]+[y] & \text{if } \{x\}+\{y\}<1\\ [x]+[y]+1 & \text{if } \{x\}+\{y\}\geq 1\end{cases}$
2. $[x+y]\geq[x]+[y]$ and equality holds if and only if $\{x\}+\{y\}<1$
3. If x or y is an integer, then $[x+y]=[x]+[y]$

PROOF 1. Let $x=n+r$ and $y=m+s$, where n and m are integers, $0\leq r<1$ and $0\leq s<1$. Then $[x]=n$, $\{x\}=r$, $[y]=m$ and $\{y\}=s$. Now,

$$x+y=[x]+[y]+(\{x\}+\{y\})$$

and

$$0\leq\{x\}+\{y\}<2$$

Therefore

$$[x+y]=\begin{cases}[x]+[y] & \text{if } \{x\}+\{y\}<1\\ [x]+[y]+1 & \text{if } \{x\}+\{y\}\geq 1\end{cases}$$

2. This is a consequence of (1).
3. This is a consequence of (2) and the fact that x is an integer if and only if $\{x\}=0$. ■

Examples

(1) $\left[\frac{8}{5}\right]+\left[\frac{9}{5}\right]=1+1=2$

and $\left[\frac{8}{5}+\frac{9}{5}\right]=\left[\frac{17}{5}\right]=3=\left[\frac{8}{5}\right]+\left[\frac{9}{5}\right]+1$

Note that

$$\left\{\frac{8}{5}\right\}+\left\{\frac{9}{5}\right\}=\frac{3}{5}+\frac{4}{5}=\frac{7}{5}>1$$

(2) $\left[\frac{7}{4}+\frac{6}{5}\right]=\left[\frac{51}{20}\right]=2=1+1=\left[\frac{7}{4}\right]+\left[\frac{6}{5}\right]$

Note that

$$\left\{\frac{7}{4}\right\}+\left\{\frac{6}{5}\right\}=\frac{3}{4}+\frac{1}{5}=\frac{19}{20}<1$$

(3) $\left[\frac{8}{3}+5\right]=\left[\frac{23}{3}\right]=7=2+5=\left[\frac{8}{3}\right]+[5]$

(4) $\left[\frac{7}{6}+\frac{17}{6}\right]=\left[\frac{24}{6}\right]=4=1+2+1=\left[\frac{7}{6}\right]+\left[\frac{17}{6}\right]+1$

Note that

$$\left\{\frac{7}{6}\right\}+\left\{\frac{17}{6}\right\}=\frac{1}{6}+\frac{5}{6}=1$$

THEOREM 1.37 The following hold for any real number x and any non-zero integer m:

1. $\left[\frac{x}{m}\right]=\left[\frac{[x]}{m}\right]$
2. If n and k are positive integers and $k>1$, then

$$\left[\frac{n}{k}\right]+\left[\frac{n+1}{k}\right]\leq\left[\frac{2n}{k}\right]$$

PROOF 1. Let $[x]=n$. Then $x=n+r, 0\leq r<1$ (where $r=\{x\}$). Let $m>0$. By division algorithm, we have

$$n=qm+s,\quad q,s\in\mathbb{Z}\quad\text{and}\quad 0\leq s<m$$

Alternately

$$\frac{n}{m}=q+\frac{s}{m},\quad 0\leq\frac{s}{m}<1$$

Therefore,

$$\left[\frac{[x]}{m}\right]=\left[\frac{n}{m}\right]=q$$

Also,

$$\frac{x}{m}=\frac{n+r}{m}=\frac{n}{m}+\frac{r}{m}=q+\frac{s+r}{m},\quad 0\le s+r<s+1\le m$$

and therefore

$$\left[\frac{x}{m}\right]=q=\left[\frac{[x]}{m}\right]$$

Similar technique proves this when $m<0$ also.

2. Let n and k be positive integers and $k>1$. Let

$$\left[\frac{n}{k}\right]=m$$

Then

$$\frac{n}{k}=m+r,\quad 0\le r<1$$

Therefore

$$\frac{n+1}{k}=m+r+\frac{1}{k}\quad\text{and}\quad\frac{2n}{k}=2m+2r$$

Now,

$$\left[\frac{n}{k}\right]+\left[\frac{n+1}{k}\right]=\begin{cases}2m & \text{if } r+(1/k)<1\\ 2m+1 & \text{if } r+(1/k)\ge 1\end{cases}$$

and

$$\left[\frac{2n}{k}\right]=\begin{cases}2m & \text{if } 2r<1\\ 2m+1 & \text{if } 2r\ge 1\end{cases}$$

Note that, if $r+(1/k)\ge 1$, then $2r\ge 2-(2/k)\ge 1\,(\because k\ge 2)$. Thus

$$\left[\frac{n}{k}\right]+\left[\frac{n+1}{k}\right]\le\left[\frac{2n}{k}\right]$$

■

Examples

(1) $\left[\frac{16}{25}\right]+\left[\frac{17}{25}\right]=0+0=0<1=\left[\frac{2\times 16}{25}\right]$

(2) $\left[\frac{7}{8}\right]+\left[\frac{8}{8}\right]=0+1=1=\left[\frac{2\times 7}{8}\right]$

(3) $\left[\frac{13}{15}\right]+\left[\frac{14}{15}\right]=0+0=0<1=\left[\frac{2\times 13}{15}\right]$

(4) $\left[\frac{-21}{29}\right]+\left[\frac{-21+1}{29}\right]=(-1)+(-1)=-2=\left[\frac{-21\times 2}{29}\right]$

DEFINITION 1.38 Let A be a subset of $\mathbb{R}$ and $f: A\to\mathbb{R}$ be a function. A positive real number p is called a *period* of f if $f(x)=f(x+p)$ whenever x and $x+p\in A$. A function with a period is called a *periodic function*.

Note that, if p is a period of a function $f:\mathbb{R}\to\mathbb{R}$, then np is also a period of f for any positive integer n, since for any $x\in\mathbb{R}$,

$$f(x)=f(x+p)=f(x+2p)=\cdots$$

Examples

(1) Define the function $f: \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \{x\}, \text{ the fractional part of } x$$

Note that any real number x can be uniquely expressed as $x = n + r$, where n is an integer and $0 \le r < 1$ and this r is the fractional part of x denoted by $\{x\}$ and this n is the integral part of x denoted by $[x]$. If $x = n + r$, then $x + 1 = (n + 1) + r$ and hence

$$f(x) = \{x\} = \{x + 1\} = f(x + 1)$$

for all $x \in \mathbb{R}$. Thus, 1 is a period of f and hence every positive integer n is a period of f. Therefore, f is a periodic function.

(2) We will be learning later in Vol. II that functions like $\sin x$, $\cos x$, $\operatorname{cosec} x$, etc. are all periodic functions and 2π is a period of all these.

(3) The function $f: \mathbb{R} \to \mathbb{R}$, defined by $f(x) = c$, for all $x \in \mathbb{R}$, where c is a given constant, is a periodic function. Infact, every positive real number is a period of this.

(4) The function $f: \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = [x] \quad \text{(the integral part of } x\text{)}$$

is not a periodic function. Note that

$$[x + n] = [x] + n$$

for all $x \in \mathbb{R}$ and for all integers n.

1.7 | Graph of a Function

A function f from a set A into a set B is a relation from A to B; that is, $f \subseteq A \times B$ and hence it can be represented as a subset of the Cartesian product $A \times B$ graphically. In particular, when the function is a real-valued function defined on the real number system or a subset of $\mathbb{R}$, we can plot the point $(a, f(a))$ on the coordinate plane by treating the x-axis as the domain and the y-axis as the co-domain of the function. This type of representation facilitates a better insight into understanding various properties of the function. First, let us have the formal definition of the graph in the following.

DEFINITION 1.39 **Graph of a Function** Let $f: A \to B$ be a function. Then the *graph* of f is defined as the set of all ordered pairs whose first coordinate is an element a of A and the second coordinate is the image of a under f. This is denoted by Graph (f). That is,

$$\text{Graph } (f) = \{(a, f(a)) \mid a \in A\}$$

Note that the graph of a function $f: A \to B$ is a subset of the Cartesian product $A \times B$. For each $a \in A$, there is exactly one ordered pair in Graph (f) with a as the first coordinate. In the following, we shall provide graphs of certain important functions and draw diagrams of these.

Example 1.28

Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x$ for all $x \in \mathbb{R}$. (Recall that f is called the identity function on $\mathbb{R}$ and is denoted by I_R.) What is the graph of f? Sketch the same.

Solution: The graph of f is

$$\{(a, f(a)) \mid a \in \mathbb{R}\} = \{(a, a) \mid a \in \mathbb{R}\}$$

This is known as the diagonal relation on $\mathbb{R}$. As shown in Figure 1.30, it is a straight line passing through the origin, contained in the first and third quadrants and bisecting the right angle $X\hat{O}Y$.

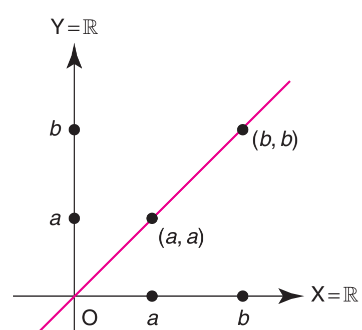


FIGURE 1.30 Example 1.28.

Example 1.29

For any given real numbers m and c, let us define the function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = mx + c \quad \text{for all } x \in \mathbb{R}$$

Sketch the graph of the same.

Solution: The graph of f is

$$\{(x, mx + c) \mid x \in \mathbb{R}\}$$

As shown in Figure 1.31, this is a straight line whose slope is m and the intercept on the y-axis is c. If we take $m = 1$ and $c = 0$, we get the identity function given in Example 1.28.

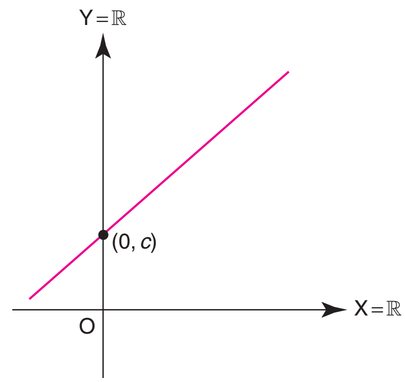


FIGURE 1.31 Example 1.29.

Example 1.30

Sketch the graph for $m = 0$ in Example 1.29.

Solution: If $m = 0$ in Example 1.29, then we get

$$f(x) = c \quad \text{for all } x \in \mathbb{R}$$

This is called the *constant function* with image c. The graph of f is a straight line parallel to the x-axis (Figure 1.32).

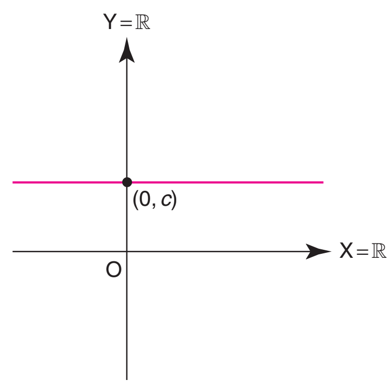


FIGURE 1.32 Example 1.30.

Example 1.31

Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = |x| \quad \text{for all } x \in \mathbb{R}$$

Sketch the graph of f.

Solution: The given function is

$$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

The graph of f is

$$\{(x, x) \mid x \geq 0\} \cup \{(x, -x) \mid x < 0\}$$

This is the combination of two straight lines: one passing through the origin, bisecting $X\hat{O}Y$ and contained in the first quadrant and the second passing through the origin, bisecting $X'\hat{O}Y$ and contained in the second quadrant (Figure 1.33).

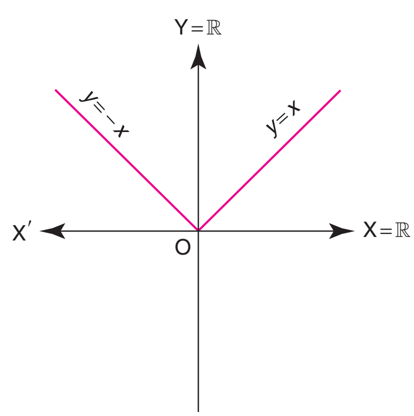


FIGURE 1.33 Example 1.31.

Example 1.32

Let $f:\mathbb{R}\to\mathbb{R}$ be defined by $f(x)=[x]$ for all $x\in\mathbb{R}$, where $[x]$ is the largest integer $\le x$. For example

$$f\left(1\frac{1}{2}\right)=1,\qquad f(-2.5)=-3,\qquad f(2.5)=2$$
$$f(2)=2,\qquad f(-4.2)=-5,\qquad f(5.01)=5$$
$$f(3.9)=3,\qquad f(-8.9)=-9,\qquad f(-6.01)=-7$$

Sketch the graph of f.

Solution: The graph of f is $\bigcup_{n\in\mathbb{Z}}([n,n+1)\times\{n\})$ and is given in Figure 1.34. This function is called the *step function*. The graph of f restricted to an interval $[n,n+1)$, with n an integer, is a line segment parallel to the x-axis.

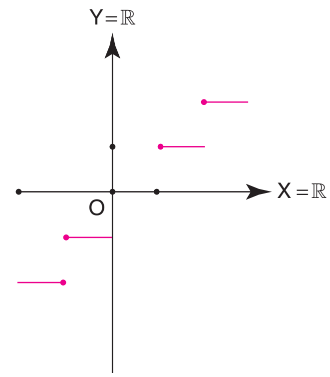


FIGURE 1.34 Example 1.32.

Example 1.33

Let $f:\mathbb{R}\to\mathbb{R}$ be defined by

$$f(x)=\begin{cases}0 & \text{if } x=0\\ \dfrac{|x|}{x} & \text{if } x\neq 0\end{cases}$$

Sketch the graph of this function.

Solution: We have

$$f(x)=\begin{cases}0 & \text{if } x=0\\ \dfrac{|x|}{x} & \text{if } x\neq 0\end{cases}$$

Then

$$f(x)=\begin{cases}-1 & \text{if } x<0\\ 0 & \text{if } x=0\\ 1 & \text{if } x>0\end{cases}$$

This function is called the *signum function*. The graph of this f is in three parts: one is the line $y=1$ which is parallel to x-axis and contained in the first quadrant; the second is the origin (0, 0) and the third is the line $y=-1$ which is parallel to the x-axis and contained in the third quadrant (Figure 1.35).

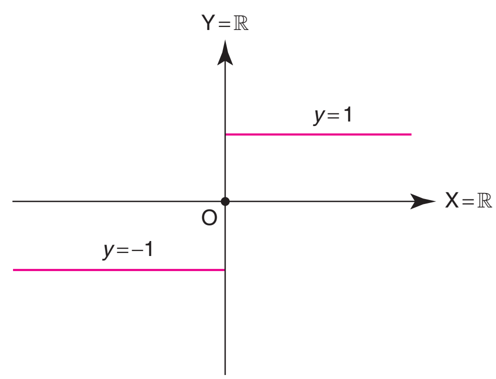


FIGURE 1.35 Example 1.33.

Example 1.34

Let $f:\mathbb{R}\to\mathbb{R}$ be defined by

$$f(x)=\begin{cases}1-x & \text{if } x<0\\ 1 & \text{if } x=0\\ 1+x & \text{if } x>0\end{cases}$$

Sketch the graph for this function.

Solution: Note that $f(x)=1+|x|$ for all $x\in\mathbb{R}$. The graph of f is given by

$$\{(x,1+x)\,|\,x>0\}\cup\{(0,1)\}\cup\{(x,1-x)\,|\,x<0\}$$

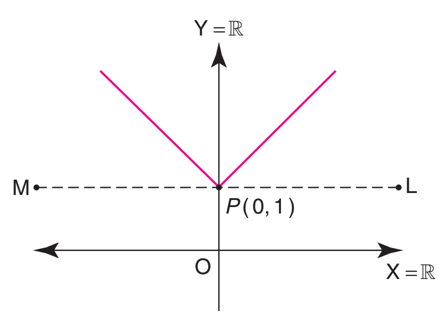


FIGURE 1.36 Example 1.34.

This is in three parts: one is the straight line bisecting $L\hat{P}Y$ and contained in the first quadrant, the second is the point $P = (0, 1)$ and the third is the straight line bisecting $M\hat{P}Y$ and contained in the second quadrant (Figure 1.36).

DEFINITION 1.40 Let A be a subset of $\mathbb{R}$ and $f: A \to \mathbb{R}$ be a function. Then we say that f is *increasing* if $f(x) \le f(y)$ whenever $x \le y$. f is said to be *decreasing* if $f(x) \ge f(y)$ whenever $x \le y$.

Example 1.35

Let $1 < a \in \mathbb{R}$ and define $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = a^x$ for all $x \in \mathbb{R}$. Sketch the graph of f.

Solution: Since $a > 1$, f is an increasing function. The graph of f is a curve which goes upward when x increases [i.e., $f(x)$ increases when x increases] and goes downwards when x decreases [i.e., $f(x)$ decreases when x decreases]. Also, since $a > 1$, a is positive and hence a^x is positive for all x. This implies that the graph of $f(x) = a^x$ is contained in the first and second quadrants (Figure 1.37).

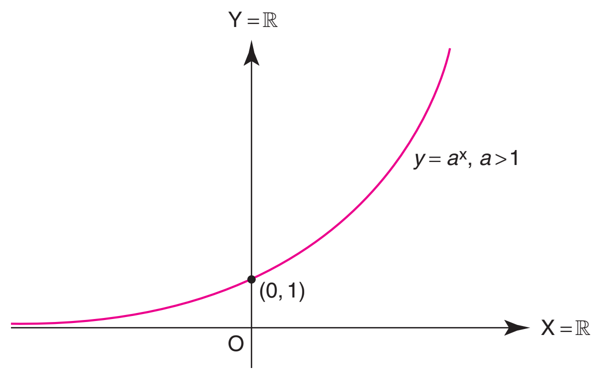


FIGURE 1.37 Example 1.35.

Example 1.36

Let $0 < a < 1$ and define $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = a^x$ for all $x \in \mathbb{R}$. Sketch the graph of f. Here, $f(x)$ decreases as x increases (since $0 < a < 1$) and hence f is a decreasing function. The graph of f is the curve shown in Figure 1.38. The curve cuts the y-axis at (0, 1). Also, since $a > 0$, $a^x > 0$ for all $x \in \mathbb{R}$. Therefore, the graph of f is contained in the first and second quadrants only.

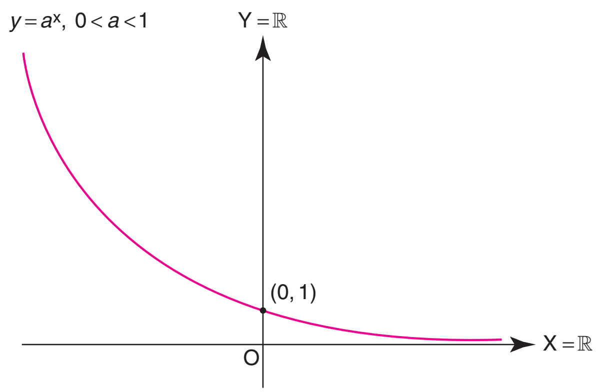


FIGURE 1.38 Example 1.36.

Example 1.37

Let $f: \mathbb{R} \to \mathbb{R}$ be a periodic function with a period p. What would the graph of this function look like?

Solution: In this case, the graph of f between the lines $x = 0$ and $x = p$ is similar to that between the lines $x = p$ and $x = 2p$. For example, consider the function $f: \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \{x\}, \text{ the fractional part of } x$$

This is a periodic function with 1 as a period. The graph of this function is as shown in Figure 1.39. Note that $0 \le f(x) < 1$ for all real numbers x.

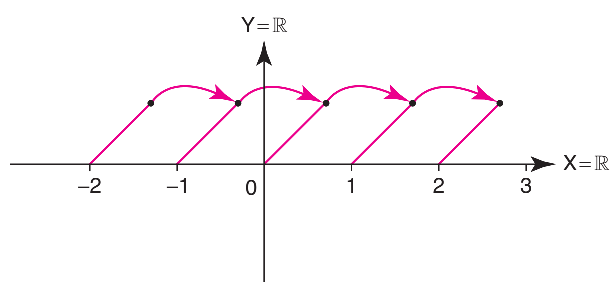


FIGURE 1.39 Example 1.37.

1.8 | Even Functions and Odd Functions

If we consider the function $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=x^2$, then we have $f(x) = f(-x)$. Functions satisfying this property are called *even functions*. If f is a real-valued function such that $f(x)=-f(x)$ for all x, then f is called an odd function. In this section we discuss certain elementary properties of even and odd functions. We shall begin with a formal definition in the following.

Even Functions

DEFINITION 1.41 **Symmetric Set** A subset X of the real number system $\mathbb{R}$ is said to be a *symmetric set* if

$$x \in X \Leftrightarrow -x \in X$$

Examples

(1) The interval $[-1,1]$ is a symmetric set, since $-1\le x\le 1$ if and only if $-1\le -x\le 1$.

(2) The interval $[0,1]$ is not symmetric.

(3) The set $\mathbb{Z}$ of integers, the set $\mathbb{Q}$ of rational numbers and the whole set $\mathbb{R}$ are all symmetric sets.

(4) The sets $\{0\}, \{-1,1\}, \{-1,0,1\}$ are symmetric.

(5) $[-2,-1]\cup[1,2]$ is a symmetric set.

DEFINITION 1.42 **Even Function** Let X be a symmetric set and $f:X\to\mathbb{R}$ a function. Then f is said to be an *even function* if

$$f(-x)=f(x) \quad \text{for all } x\in X$$

Examples

(1) If $f:\mathbb{R}\to\mathbb{R}$ is the function defined by $f(x)=x^2$ for all $x\in\mathbb{R}$, then f is an even function, since, for any $x\in\mathbb{R}$,

$$f(-x)=(-x)^2=x^2=f(x)$$

(2) The function $f:\mathbb{R}\to\mathbb{R}$, defined by $f(x)=|x|$ for all $x\in\mathbb{R}$, is even, since

$$f(-x)=|-x|=|x|=f(x) \quad \text{for all } x\in\mathbb{R}$$

(3) Any constant function $f:\mathbb{R}\to\mathbb{R}$ is even, that is, for any $c\in\mathbb{R}$, the function $f:\mathbb{R}\to\mathbb{R}$, defined by $f(x)=c$ for all $x\in\mathbb{R}$, is even.

(4) The function $f:[-\pi,\pi]\to\mathbb{R}$, defined by $f(x)=\cos x$ for all $-\pi\le x\le\pi$, is an even function, since $\cos(-x)=\cos x$.

Graphs of Even Functions

The graph of an even function is symmetric about the y-axis, in the sense that, when y-axis is assumed as plane mirror, the graph in the left part is the image of the right part. Equivalently, if the graph is rotated through 180° about the y-axis, we get the appearance of the graph as original. Figure 1.40 shows the graphs of the even functions given in the example above.

Odd Functions

DEFINITION 1.41 **Odd Function** Let X be a symmetric set. A function $f:X\to\mathbb{R}$ is said to be an *odd function* if

$$f(-x)=-f(x) \quad \text{for all } x\in X$$

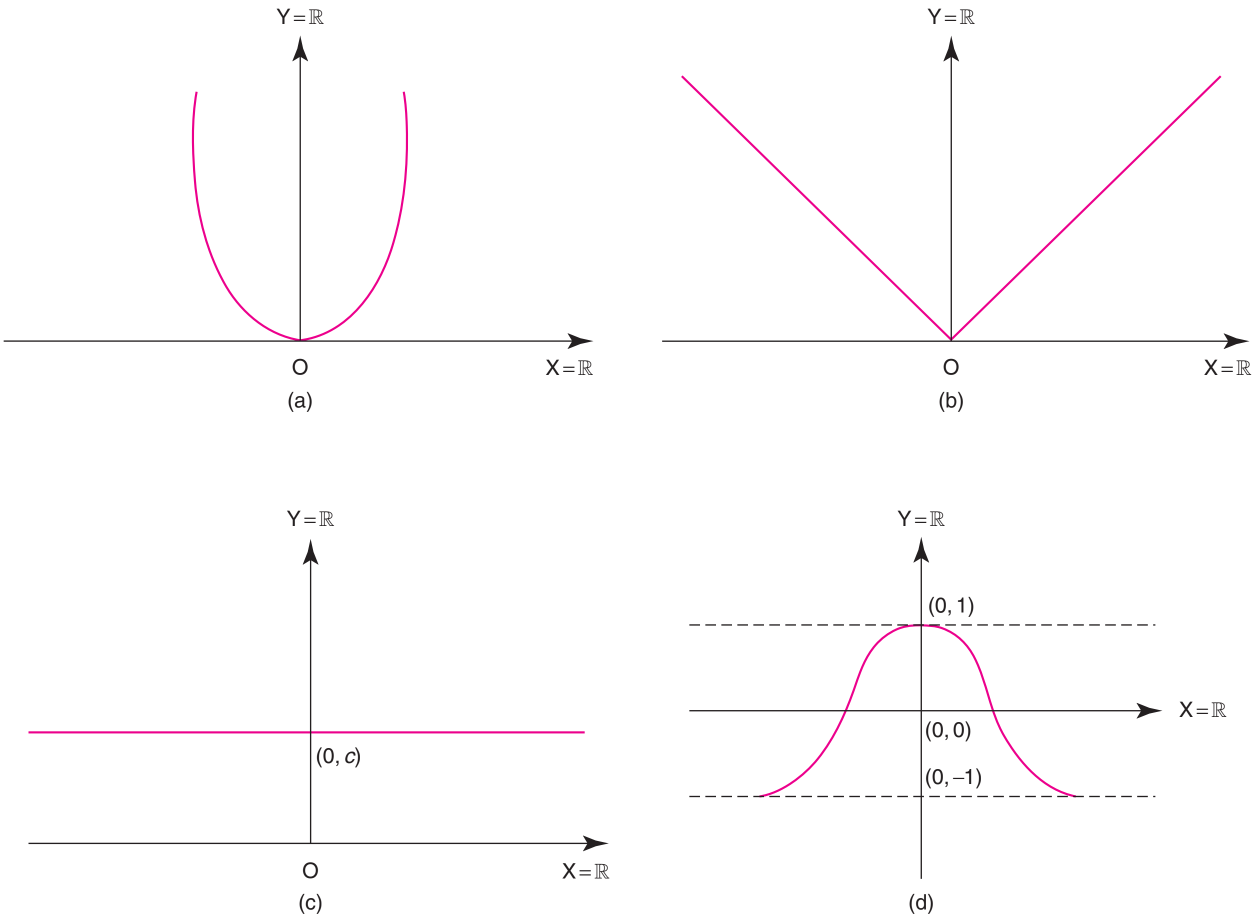


FIGURE 1.40 Graphs of the functions: (a) $f(x) = x^2$; (b) $f(x) = |x|$; (c) $f(x) = c$; (d) $f(x) = \cos x$.

Examples

(1) The identity function $f: \mathbb{R} \to \mathbb{R}$, defined by $f(x) = x$ for all $x \in \mathbb{R}$, is an odd function, since $f(-x) = -x = -f(x)$ for all $x \in \mathbb{R}$.

(2) In general, for any integer n, the function $f: \mathbb{R} \to \mathbb{R}$, defined by $f(x) = x^{2n+1}$, is an odd function, since $f(-x) = (-x)^{2n+1} = -x^{2n+1} = -f(x)$ for all $x \in \mathbb{R}$.

(3) Define $f: [-\pi, \pi] \to \mathbb{R}$ by $f(x) = \sin x$ for all $-\pi \le x \le \pi$. Then f is an odd function, since $f(-x) = \sin(-x) = -\sin x = -f(x)$ for all $x \in [-\pi, \pi]$.

(4) Define f: $(-\pi/2, \pi/2) \to \mathbb{R}$ by $f(x) = \tan x$ for all $-\pi/2 < x < \pi/2$. Then f is an odd function, since $\tan(-x) = -\tan x$ for all $x \in (-\pi/2, \pi/2)$.

Note: If f is an odd function defined on a symmetric set S containing 0, then necessarily $f(0) = 0$, for $f(0) = f(-0) = -f(0)$. Hence $2f(0) = 0$, so that $f(0) = 0$.

Graphs of Odd Functions

The graph of an odd function is symmetric about the origin. If the graph is rotated through 180°, either clockwise or anticlockwise, about the origin, the resulting figure gives the same appearance as original. Figure 1.41 gives the graphs of the odd functions given in the above example.

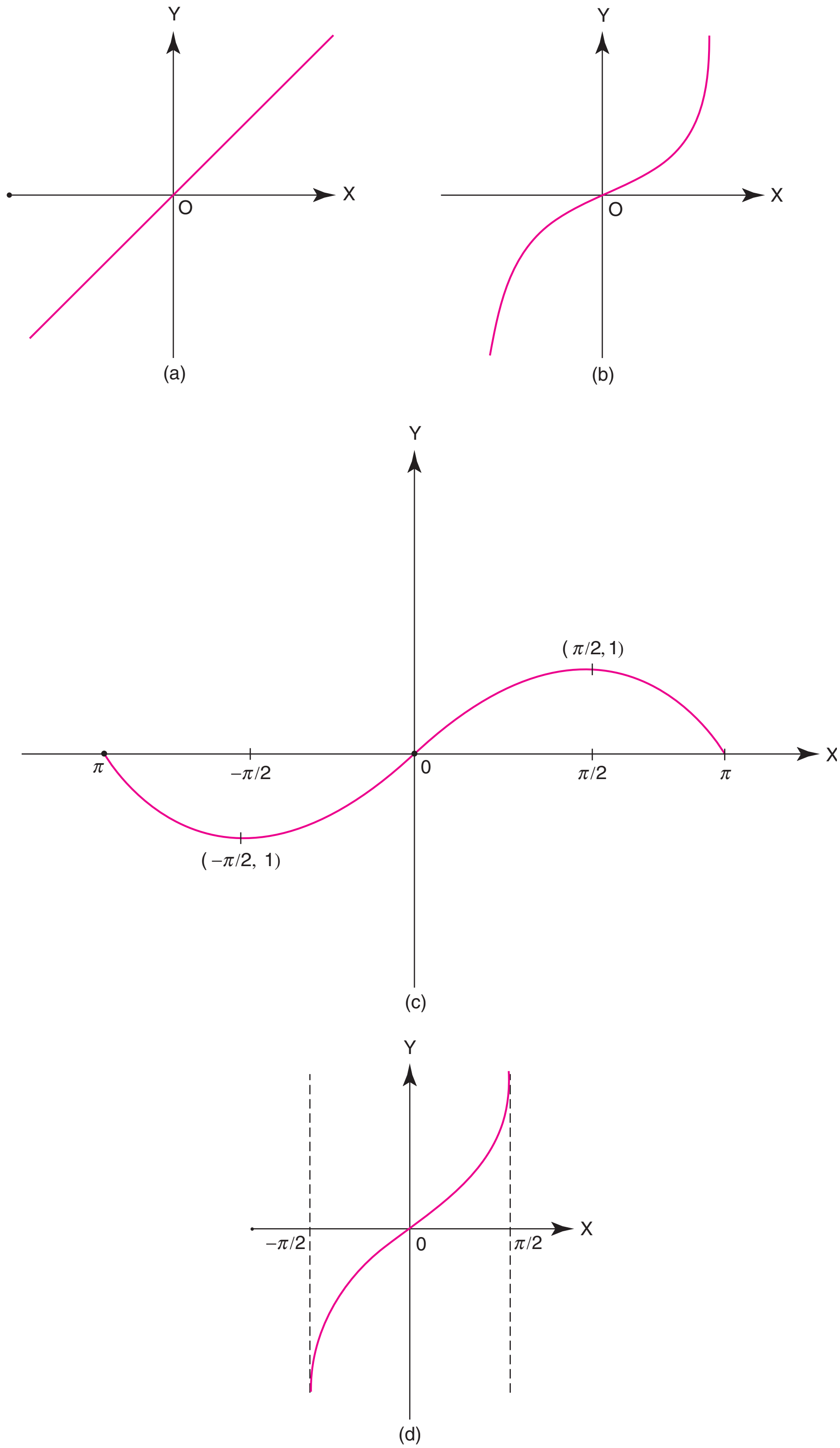


FIGURE 1.41 Graphs of the functions: (a) $f(x) = x$; (b) $f(x) = x^3$; (c) $f(x) = \sin x$; (d) $f(x) = \tan x$.

Remark: Unlike in integers, a function can be neither even nor odd. For example, consider the function $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=x^2+x+1$ for all $x\in\mathbb{R}$. Then $f(-1)=1$ and $f(1)=3$ and hence

$$f(-1)\neq f(1) \quad \text{and} \quad f(-1)\neq -f(1)$$

Therefore f is neither even nor odd. Next, note that a function f is both even and odd if and only if $f(x)=0$ for all x.

Examples

(1) Define $f:\mathbb{R}\to\mathbb{R}$ by $f(x)=e^x+e^{-x}$ for all $x\in\mathbb{R}$. Then $f(-x)=e^{-x}+e^{-(-x)}=e^x+e^{-x}=f(x)$ for all $x\in\mathbb{R}$ and therefore f is an even function.

(2) Define $f:[-1,1]\to\mathbb{R}$ by $f(x)=\sqrt[4]{1-x^3}-\sqrt[4]{1+x^3}$ for all $-1\le x\le 1$. Then

$$f(-x)=\sqrt[4]{1-(-x)^3}-\sqrt[4]{1+(-x)^3}$$
$$=\sqrt[4]{1+x^3}-\sqrt[4]{1-x^3}$$
$$=-(\sqrt[4]{1-x^3}-\sqrt[4]{1+x^3})$$
$$=-f(x)$$

for all $x\in[-1,1]$. Therefore f is an odd function.

Theorem 1.38 Let X be a symmetric set and f and g functions of X into $\mathbb{R}$. Then, the product fg is an even function if both f and g are even or both f and g are odd.

Proof Suppose that both f and g are even functions. Then, for any $x\in X$, we have

$$(fg)(x)=f(x)g(x)=f(-x)g(-x)=(fg)(-x)$$

and hence fg is an even function. One the other hand, suppose that both f and g are odd functions. Then, for any $x\in X$, we have

$$(fg)(-x)=f(-x)g(-x)=(-f(x))(-g(x))=f(x)g(x)=(fg)(x)$$

and therefore fg is an even function. ■

Theorem 1.39 For any real-valued functions f and g defined on a symmetric set X, the product fg is an odd function if one of f and g is odd and the other is even.

Proof Note that $fg = gf$, since $rs = sr$ for any real numbers r and s. Without loss of generality, we can suppose that f is even and g is odd. Then, for any $x\in X$, we have

$$(fg)(-x)=f(-x)g(-x)=f(x)(-g(x))=-(f(x)g(x))=-(fg)(x)$$

Therefore fg is an odd function. ■

Theorem 1.40 Let f be a real-valued function on a symmetric set X. Then the following hold:

1. f is even if and only if af is even for any $0\neq a\in\mathbb{R}$.

2. f is odd if and only if af is odd for any $0\neq a\in\mathbb{R}$.

3. f is even (odd) if and only if $-f$ is even (odd).

Proof **1.** Let us recall that for any $a\in\mathbb{R}$ the function af is defined by $(af)(x)=af(x)$ for all $x\in X$. Suppose that f is even. Then, for any $a\in\mathbb{R}$ and $x\in X$,

$$(af)(-x)=af(-x)=af(x)=(af)(x)$$

and hence af is even. Conversely, suppose that $0\neq a\in\mathbb{R}$ such that af is even. Then, for any $x\in X$, we have

$$af(-x)=(af)(-x)=(af)(x)=af(x)$$

Now, since $a\neq 0$, $f(-x)=f(x)$. Therefore, f is even.

2. It can be proved similarly.

3. It is a simple consequence of **(1)** and **(2)**; take $a = 1$ in **(1)** and **(2)**. ■

Theorem 1.41 If f and g are even (odd), then so is $f \pm g$.

Proof Suppose that f and g are even. Then, for any $x \in X$, we have

$$(f+g)(-x) = f(-x) + g(-x) = f(x) + g(x) = (f+g)(x)$$

Therefore $f+g$ is even. This together with the above theorem implies that $f-g$ is also even. Similarly, we can prove that, if f and g odd, then so is $f \pm g$. ■

Theorem 1.42 Any function can be expressed as a sum of an even function and an odd function.

Proof Let $f: X \to \mathbb{R}$ be a function whose domain X is a symmetric set. Define $g: X \to \mathbb{R}$ and $h: X \to \mathbb{R}$ by

$$g(x) = \frac{f(x) + f(-x)}{2} \quad \text{and} \quad h(x) = \frac{f(x) - f(-x)}{2}$$

for all $x \in X$. Then

$$g(-x) = \frac{f(-x) + f(-(-x))}{2} = \frac{f(x) + f(-x)}{2} = g(x)$$

and

$$h(-x) = \frac{f(-x) = f(-(-x))}{2} = \frac{f(-x) - f(x)}{2} = -h(x)$$

for all $x \in X$. Therefore, g is an even function and h is an odd function. Also, for any $x \in X$,

$$g(x) + h(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2} = f(x)$$

and hence $f = g + h$. ■

Note: The above representation of f is unique in the sense that if $g + h = f = \alpha + \beta$, where g and α are even and h and β are odd, then $g = \alpha$ and $h = \beta$; for, in this case $g - \alpha = \beta - h$, which is both even and odd. Therefore, $g - \alpha = 0 = \beta - h$ or $g = \alpha$ and $h = \beta$.

The unique functions g and h given in the proof of Theorem 1.42 are called the *even extension of f* and *odd extension of f*, respectively.

Examples

(1) Let $f: \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = x^2 + 2x + 1 = (x+1)^2$$

Note that f is neither even nor odd, since

$$f(-1) = (-1)^2 + 2(-1) + 1 = 0$$

and
$$f(1) = (1)^2 + 2(1) + 1 = 4$$

Therefore $f(-1) \neq f(1)$ and $f(-1) \neq -f(1)$. However, consider the functions g and h defined by

$$g(x) = x^2 + 1 \quad \text{and} \quad h(x) = 2x$$

Then g is even, h is odd and $f = g + h$. Note that

$$\frac{f(x) + f(-x)}{2} = g(x) \quad \text{and} \quad \frac{f(x) - f(-x)}{2} = h(x)$$

(2) Consider the function $f: \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = e^x \text{ for all } x \in \mathbb{R}$$

Then $f = g + h$, where

$$g(x) = \frac{e^x + e^{-x}}{2} \quad \text{and} \quad h(x) = \frac{e^x - e^{-x}}{2}$$

Note that g is even and h is odd.

Example 1.38

Determine the even and odd extensions of the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = e^{-x}$.

Solution: The even extension of f is given by

$$g(x) = \frac{f(x) + f(-x)}{2} = \frac{e^{-x} + e^{x}}{2}$$

and the odd extension of f is given by

$$h(x) = \frac{f(x) - f(-x)}{2} = \frac{e^{-x} - e^{x}}{2}$$

WORKED-OUT PROBLEMS

Single Correct Choice Type Questions

1. If A is the set of positive divisors of 20, B is the set of all prime numbers less than 15 and C is the set of all positive even integers less than 11, then $(A \cap B) \cup C$ is

(A) $\{2, 3, 5, 7, 8, 10\}$ (B) $\{2, 4, 5, 7, 8, 10\}$
(C) $\{2, 4, 5, 6, 7, 8, 10\}$ (D) $\{2, 4, 5, 6, 8, 10\}$

Solution: It is given that

$$A = \{1, 2, 4, 5, 10, 20\}$$
$$B = \{2, 3, 5, 7, 11, 13\}$$
$$C = \{2, 4, 6, 8, 10\}$$

Therefore

$$A \cap B = \{2, 5\} \quad \text{and} \quad (A \cap B) \cup C = \{2, 4, 5, 6, 8, 10\}$$

Answer: (D)

2. Which of the following sets is empty?

(A) $\{x \in \mathbb{R} \mid x^2 = 9 \text{ and } 2x = 6\}$
(B) $\{x \in \mathbb{R} \mid x^2 = 9 \text{ and } 2x = 4\}$
(C) $\{x \in \mathbb{R} \mid x + 4 = 4\}$
(D) $\{x \in \mathbb{R} \mid 2x + 1 = 3\}$

Solution: We have $x^2 = 9$ only if $x = \pm 3$. For this value of x the equation $2x = 4$ is not satisfied. Sets in (A), (B), and (D) are non-empty.

Answer: (B)

3. For each positive integer n, let

$$A_n = \text{The set of all positive multiples of } n$$

Then $A_6 \cap A_{10}$ is

(A) A_{10} (B) A_{20} (C) A_{30} (D) A_{60}

Solution: Given that $A_n = \{a \in \mathbb{Z}^+ \mid n \text{ divides } a\}$. Now

$$\begin{aligned} a \in A_n \cap A_m &\Leftrightarrow \text{Both } n \text{ and } m \text{ divide } a \\ &\Leftrightarrow \text{The LCM of } \{n, m\} \text{ divides } a \\ &\Leftrightarrow a \in A_r, \text{ where } r = \text{LCM } \{n, m\} \end{aligned}$$

Therefore,

$$A_6 \cap A_{10} = A_{30}$$

since LCM $\{6, 10\} = 30$.

Answer: (C)

4. Let $A = \{a, b, c, d\}$ and $B = \{a, b, c\}$. Then the number of sets X contained in A and not contained in B is

(A) 8 (B) 6 (C) 16 (D) 12

Solution: If $X \subseteq A$ and $X \not\subseteq B$, then necessarily $d \in X \subseteq A$ and hence $X = Y \cup \{d\}$, where Y is any subset of B. The number of subsets of B is 2^3 and therefore the required number is 8.

Answer: (A)

5. Let A, B and C be three sets and X be the set of all elements which belong to exactly two of the sets A, B and C. Then X is equal to

(A) $(A \cap B) \cup (B \cap C) \cup (C \cap A)$
(B) $A \,\Delta\, (B \,\Delta\, C)$
(C) $(A \cup B) \cap (B \cup C) \cap (C \cup A)$
(D) $(A \cup B \cup C) - [A \,\Delta\, (B \,\Delta\, C)]$

Solution: We have

$$x \in X \Leftrightarrow x \in A \cap B \quad \text{and} \quad x \notin C$$

or $$x \in B \cap C \quad \text{and} \quad x \notin A$$

or $$x \in C \cap A \quad \text{and} \quad x \notin B$$

Therefore

$$\begin{aligned} X &= [(A \cap B) - C] \cup [(B \cap C) - A] \cup [(C \cap A) - B] \\ &= (A \cup B \cup C) - [A \,\Delta\, (B \,\Delta\, C)] \end{aligned}$$

since

$$\begin{aligned} A \,\Delta (B \,\Delta\, C) = (A^c \cap B^c \cap C) \cup (A \cap B^c \cap C^c) \cup \\ (A^c \cap B \cap C^c) \cup (A \cap B \cap C) \end{aligned}$$

Answer: (D)

6. Let $\wp(x)$ denote the power set of X. If $A = \{a, b, c, d, e\}$ and $B = \{a, c, d, x, y\}$, then $\wp(A \cap B) =$

(A) $\{\phi, \{a, c\}, \{c, d\}, \{a, c, d\}, \{a\}, \{c\}, \{d\}\}$

(B) $\{\phi, \{a\}, \{c\}, \{a, c\}, \{c, d\}, \{a, d\}, \{a, c, d\}\}$

(C) $\wp(A \cup B)$

(D) $\wp(A) \cap \wp(B)$

Solution: We have $X \subseteq A \cap B \Leftrightarrow X \subseteq A$ and $X \subseteq B$.

Answer: (D)

7. Let A and B be finite sets with $n(A) = m$ and $n(B) = n$. If the number of elements in $\wp(A)$ is 56 more than those in $\wp(B)$, then

(A) $m = 6, n = 4$

(B) $m = 6, n = 3$

(C) $m = 7, n = 4$

(D) $m = 5, n = 3$

Solution: It is given that

$$n(\wp(A)) = 2^m = 56 + n(\wp(B)) = 56 + 2^n$$

Now $2^m - 2^n = 56$ and $m > n$. Hence we get

$$2^n(2^{m-n} - 1) = 56 = 8 \times 7 = 2^3(2^3 - 1)$$

Therefore $n = 3$ and $m - n = 3$ and hence $m = 6$ and $n = 3$.

Answer: (B)

8. If A and B are two subsets of a universal set X, then $A^c - B^c =$

(A) $A - B$ (B) $(A - B)^c$

(C) $B \cap A^c$ (D) $(B - A)^c$

Solution: We have

$$A^c - B^c = A^c \cap (B^c)^c = A^c \cap B = B \cap A^c$$

Answer: (C)

9. If $A = \{1, 2, 3, 4\}$, $B = \{1, 2, 5, 6\}$, $C = \{2, 7, 8, 9\}$ and $D = \{2, 4, 8, 9\}$, then $(A \,\Delta\, B)\,\Delta\,(C \,\Delta\, D) =$

(A) $\{3, 4, 5, 6, 7\}$ (B) $\{3, 4, 5, 7\}$

(C) $\{3, 5, 7, 8\}$ (D) $\{3, 5, 6, 7\}$

Solution: We have

$$A \,\Delta\, B = (A - B) \cup (B - A) = \{3, 4, 5, 6\}$$

$$C \,\Delta\, D = (C - D) \cup (D - C) = \{7, 4\}$$

and $(A \,\Delta\, B)\,\Delta\,(C \,\Delta\, D) = (A \,\Delta\, B) - (C \,\Delta\, D)$

$$\cup[(C \,\Delta\, D) - (A \,\Delta\, B)]$$

$$= \{3, 5, 6\} \cup \{7\} = \{3, 5, 6, 7\}$$

Answer: (D)

10. Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $A = \{2, 3, 4, 5, 6\}$. Then the number of subsets B of S such that $A \,\Delta\, B = \{5\}$ is

(A) 1 (B) 2 (C) 3 (D) 0

Solution: For any subsets X, Y and Z of S, we have

$$X \,\Delta\, X = \phi,\ X \,\Delta\, \phi = X$$

and
$$X \,\Delta\, Y = Z \Leftrightarrow Y = X \,\Delta\, Z$$

Now,

$$A \,\Delta\, B = \{5\} \Leftrightarrow B = A \,\Delta\, \{5\} = \{2, 3, 4, 6\}$$

Answer: (A)

11. Let A, B and C be finite sets such that $A \cap B \cap C = \phi$ and each one of the sets $A \,\Delta\, B$, $B \,\Delta\, C$ and $C \,\Delta\, A$ has 100 elements. The number of elements in $A \cup B \cup C$ is

(A) 250 (B) 200 (C) 150 (D) 300

Solution: Let $n(X)$ denote the number of elements in X. Then,

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B)$$
$$- n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$$
$$= \sum n(A) - \sum n(A \cap B)$$

(since $A \cap B \cap C = \phi$)

Now,

$$A \,\Delta\, B = (A - B) \cup (B - A) = (A \cup B) - (A \cap B)$$

Therefore

$$n(A \,\Delta\, B) = n(A \cup B) - n(A \cap B)$$
$$= n(A) + n(B) - 2n(A \cap B)$$

and

$$300 = \sum n(A \,\Delta\, B) = \sum[n(A) + n(B) - 2n(A \cap B)]$$
$$= 2\left[\sum n(A) - \sum n(A \cap B)\right]$$

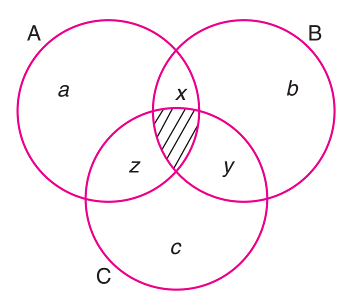
Therefore

$$n(A \cup B \cup C) = \sum n(A) - \sum n(A \cap B) = 300/2 = 150$$

Answer: (C)

Alternate Method

Draw the Venn diagram as follows:



The shaded part is $A \cap B \cap C$ which is given to be empty. Let a, b, c denote $n[A-(B \cup C)]$, $n[B-(C \cup A)]$, $n[C-(A \cup B)]$ respectively. Let x, y, z denote the number of elements in $(A \cap B)-C$, $(B \cap C)-A$, $(C \cap A)-B$ respectively. Then

$$n(A \cup B \cup C) = a+b+c+x+y+z$$

We are given that

$$100 = n(A \,\Delta\, B) = (a+z)+(b+y)$$

$$100 = n(B \,\Delta\, C) = (b+x)+(c+z)$$

and $$100 = n(C \Delta A) = (c+y)+(a+x)$$

Adding the above three, we get that

$$300 = 2(a+b+c)+2(x+y+z) = 2n(A \cup B \cup C)$$

and hence $n(A \cup B \cup C) = 150$.

Answer: (C)

12. Let n be a positive integer and

$$R = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid a-b = nm \text{ for some } 0 \neq m \in \mathbb{Z}\}$$

Then R is

(A) Reflexive on $\mathbb{Z}$
(B) Symmetric
(C) Transitive
(D) Equivalence relation on $\mathbb{Z}$

Solution: R is not reflexive, since $(2, 2) \notin R$. R is symmetric, since

$$(a, b) \in R \Rightarrow a-b = nm \text{ for some } 0 \neq m \in \mathbb{Z}$$

$$\Rightarrow b-a = n(-m) \quad \text{and} \quad 0 \neq -m \in \mathbb{Z}$$

$$\Rightarrow (b, a) \in R$$

R is not transitive, since $(2, n+2) \in R$ and $(n+2, 2) \in R$, but $(2, 2) \notin R$.

Answer: (B)

13. Let $P_0 = 1$ and P_n be the number of partitions on a finite set with n elements. For $n \geq 1$, a recursion formula for P_n is given by

(A) $P_n = P_{n-1} + P_{n-2}$ for $n \geq 2$

(B) $P_n = \sum_{r=1}^{n-1} \binom{n-1}{r} P_r$

(C) $P_{n+1} = \sum_{r=0}^{n} \binom{n}{r} P_r$

(D) $P_{n+1} = P_n + nP_{n-1}$

Solution: We are given that $P_0 = 1$. If X is a set with only one element, then clearly $P_1 = 1$. Now, let X be a set with $n+1$ elements, $n > 0$. If A is a non-empty subset of X with K-elements (such sets are $\binom{n+1}{K}$ in number), then the number of partitions of the set $X-A$ is $P_{(n+1)-K}$. For each $\phi \neq A \subseteq X$ and for each partition of $X-A$, we get a partition of X. Conversely, any partition of X corresponds to a non-empty subset A of X and a partition of $X-A$. Therefore

$$P_{(n+1)} = \sum_{K=1}^{n+1} \binom{n+1}{K} P_{(n+1)-K} = \sum_{r=0}^{n} \binom{n}{r} P_r$$

Answer: (C)

Note: If n is a positive integer and $0 \leq r \leq n$ is an integer, then $\binom{n}{r}$ denotes the number of selections of n distinct objects taken r at a time (see Chapter 6).

14. The number of equivalence relations on a five element set is

(A) 32 (B) 42 (C) 50 (D) 52

Solution: Note that equivalence relations and partitions are same in number. By Problem 13, we have

$$P_5 = \sum_{r=0}^{4} \binom{4}{r} P_r = \binom{4}{0}P_0 + \binom{4}{1}P_1 + \binom{4}{2}P_2 + \binom{4}{3}P_3 + \binom{4}{4}P_4$$

Now,

$$P_0 = 1,\ P_1 = 1,\ P_2 = \binom{1}{0}P_0 + \binom{1}{1}P_1 = 1+1 = 2$$

$$P_3 = \binom{2}{0}P_0 + \binom{2}{1}P_1 + \binom{2}{2}P_2 = 1 + 2\cdot 1 + 1\cdot 2 = 5$$

$$P_4 = \binom{3}{0}P_0 + \binom{3}{1}P_1 + \binom{3}{2}P_2 + \binom{3}{3}P_3$$

$$= 1\cdot 1 + 3\cdot 1 + 3\cdot 2 + 1\cdot 5 = 15$$

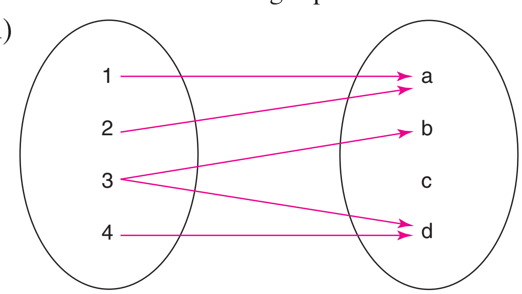
$$P_5 = \binom{4}{0}P_0 + \binom{4}{1}P_1 + \binom{4}{2}P_2 + \binom{4}{3}P_3 + \binom{4}{4}P_4$$

$$= 1\cdot 1 + 4\cdot 1 + 6\cdot 2 + 4\cdot 5 + 1\cdot 15 = 52$$

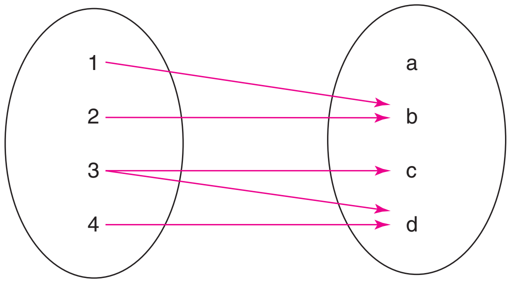
Answer: (D)

15. Which one of the following represents a function?

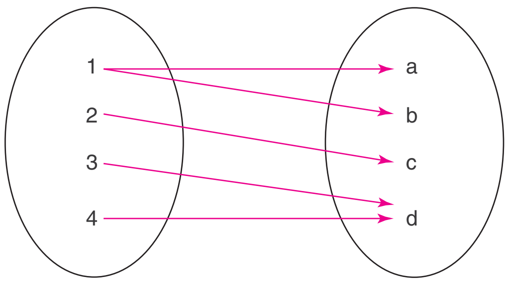
(A)



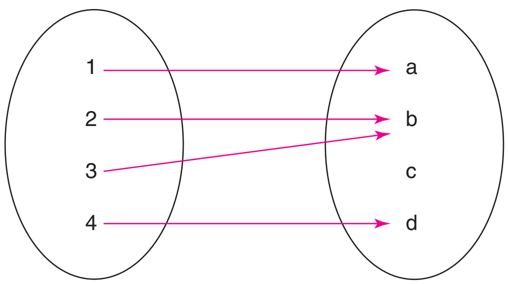
(B)



(C)



(D)



Solution: In (A), $3 \to$ b and $3 \to$ d. It does not represent a function, since one element in the domain cannot be sent to two elements in the codomain. Similarly (B) and (C) do not represent functions. But (D) represents a function f, where $f(1) = $ a, $f(2) = $ b, $f(3) = $ b and $f(4) = $ d.

Answer: (D)

16. Let A be the set of all men living in a town. Which one of the following relations is a function from A to A?

(A) $\{(a, b) \in A \times A \mid b$ is the son of $a\}$
(B) $\{(a, b) \in A \times A \mid b$ is the father of $a\}$
(C) $\{(a, b) \in A \times A \mid a$ and b are same$\}$
(D) $\{(a, b) \in A \times A \mid a$ is the grandfather of $b\}$

Solution: Here (B) is not a function, since for any $a \in A$, there should be exactly one b such that b is the father of a. Then again there should be $c \in A$ such that c is the father of b and so on. This chain breaks at some stage, where there is man a whose father is not in that town. Therefore, not every element in A has an image. In (A) and (D) an element can have more than one images and hence they do not represent a function. However, (C) is a function; in fact, it is the identity function on A.

Answer: (C)

17. Let $f: \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} x^2 - 4x + 3 & \text{if } x < 2 \\ x - 3 & \text{if } x \geq 2 \end{cases}$$

Then number of real numbers x for which $f(x) = 3$ is

(A) 1 (B) 2 (C) 3 (D) 4

Solution: We have

$$\begin{aligned} x < 2 \text{ and } f(x) = 3 &\Rightarrow x^2 - 4x + 3 = 3 \\ &\Rightarrow x(x-4) = 0 \\ &\Rightarrow x = 0 \quad (\text{since } x < 2) \end{aligned}$$

Also $x \geq 2$ and $f(x) = 3 \Rightarrow x - 3 = 3 \Rightarrow x = 6$. Therefore, only $x = 0$ or 6 satisfy $f(x) = 3$.

Answer: (B)

18. Let

$$f(x) = \frac{ax}{x+1} \quad \text{for} \quad x \neq -1$$

Then the value of a such that $(f \circ f)(x) = x$ for all $x \neq -1$ is

(A) -1 (B) $\sqrt{2}$ (C) $-\sqrt{2}$ (D) 1

Solution: We have

$$x = (f \circ f)(x) = f\left(\frac{ax}{x+1}\right) = \frac{a[ax/(x+1)]}{[ax/(x+1)] + 1}$$

Therefore

$$x = \frac{a^2 x}{ax + x + 1} \quad \text{for all } x \neq -1$$

$$(a+1)x^2 + (1 - a^2)x = 0 \quad \text{for all } x \neq -1$$

This is a quadratic equation which is satisfied by more than two values of x (infact, for all $x \neq -1$). Therefore, the coefficients of x^2 and x must be both zero. Hence

$$a + 1 = 0 \quad \text{and} \quad 1 - a^2 = 0$$

and so

$$a = -1$$

Answer: (A)

19. If $f(x)$ is a polynomial function satisfying the relation

$$f(x) + f\left(\frac{1}{x}\right) = f(x) f\left(\frac{1}{x}\right) \quad \text{for all } x \neq 0$$

and $f(4) = 65$, then $f(2) =$

(A) 7 (B) 4 (C) 9 (D) 6

Solution: Since $f(4) = 65$, $f(x)$ must be a non-zero polynomial. Let

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n, \quad a_n \neq 0$$

Suppose that

$$f(x)+f\left(\frac{1}{x}\right)=f(x)f\left(\frac{1}{x}\right) \quad \text{for all } x \neq 0$$

Then

$$\sum_{r=0}^{n} a_r x^r + \sum_{r=0}^{n} \frac{a_r}{x^r} = \left(\sum_{r=0}^{n} a_r x^r\right)\left(\sum_{r=0}^{n} \frac{a_r}{x^r}\right)$$

Multiplying throughout by x^n, we get that

$$\sum_{r=0}^{n} a_r x^{n+r} + \sum_{r=0}^{n} a_r x^{n-r} = \left(\sum_{r=0}^{n} a_r x^r\right)\left(\sum_{r=0}^{n} a_r x^{n-r}\right)$$

That is,

$$(a_0x^n + a_1x^{n+1} + \cdots + a_nx^{2n}) + (a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n)$$
$$= (a_0 + a_1x + \cdots + a_nx^n)(a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n)$$

Equating the corresponding coefficients of powers of x, we have

$$a_n = a_0a_n,\ a_{n-1} = a_0a_{n-1} + a_1a_n$$
$$a_{n-2} = a_2a_n + a_1a_{n-1} + a_{n-2}a_0$$
$$2a_0 = a_0^2 + a_n^2$$
$$a_n = a_0a_n \Rightarrow a_0 = 1 \text{ (since } a_n \neq 0)$$
$$a_{n-1} = a_0a_{n-1} + a_1a_n \Rightarrow a_1a_n = 0 \Rightarrow a_1 = 0$$
$$a_{n-2} = a_2a_n + a_1a_{n-1} + a_{n-2}a_0 \Rightarrow a_{n-2} = a_2a_n + a_{n-2} \Rightarrow a_2 = 0$$

Continuing this process, we get that $a_{n-1} = 0$ and $2 = 1 + a_n^2$. Hence $a_n = \pm 1$. Therefore

$$f(x) = 1 \pm x^n$$

Since we are given that $f(4) = 65$ we have

$$65 = 1 \pm 4^n$$

Therefore $f(x)$ cannot be $1 - x^n$. Thus, $f(x) = 1 + x^n$ and $65 = 1 + 4^n$ and hence $n = 3$. So $f(x) = 1 + x^3$ and $f(2) = 9$.

Answer: (C)

20. Let $f(x) = 9^x/(9^x + 3)$ for all $x \in \mathbb{R}$. Then the value of $\sum_{r=1}^{2008} f(r/2009)$ is

(A) 1004 (B) 1005 (C) 1004.5 (D) 1005.5

Solution: Consider

$$f(x)+f(1-x) = \frac{9^x}{9^x+3} + \frac{9^{1-x}}{9^{1-x}+3}$$
$$= \frac{9^x}{9^x+3} + \frac{9}{9+3\cdot 9^x}$$
$$= \frac{9^x}{9^x+3} + \frac{9}{3(3+9^x)}$$
$$= \frac{3\cdot 9^x + 9}{3(3+9^x)} = 1$$

Therefore,

$$\sum_{r=1}^{2008} f\left(\frac{r}{2009}\right) = \left[f\left(\frac{1}{2009}\right) + f\left(\frac{2008}{2009}\right)\right] + \cdots$$
$$= \sum_{r=1}^{1004}\left[f\left(\frac{r}{2009}\right) + \left(\frac{2009-r}{2009}\right)\right]$$
$$= \sum_{r=1}^{1004} f\left(\frac{r}{2009}\right) + f\left(1 - \frac{r}{2009}\right)$$
$$= \sum_{r=1}^{1004} 1 = 1004$$

Answer: (A)

Note: If a is any positive integer and $f(x) = a^{2x}/(a^{2x} + a)$, then

$$f(x) + f(1-x) = 1$$

21. Let $[x]$ and $\{x\}$ denote the integral part and fractional part of x, respectively. Then the number of solutions of the equation $4\{x\} = x + [x]$ is

(A) 1 (B) 2 (C) 0 (D) infinite

Solution: Let $4\{x\} = x + [x] = 2[x] + \{x\}$. Therefore $3\{x\} = 2[x]$. Since $0 \le \{x\} < 1$, we have $0 \le 3\{x\} < 3$ and therefore $0 \le 2\{x\} < 3$. Since $2[x]$ is even integer,

$$[x] = 0 \text{ or } 1 \quad \text{and} \quad \{x\} = 0 \text{ or } \frac{2}{3}$$

Therefore

$$x = 0 \quad \text{or} \quad [x] = 1$$
$$x = 0 \quad \text{or} \quad 4\left(\frac{2}{3}\right) = x + 1$$
$$x = 0 \quad \text{or} \quad \frac{5}{3}$$

Answer: (B)

22. If the function $f: \mathbb{R} \to \mathbb{R}$ satisfies the relation $f(x) + f(x+4) = f(x+2) + f(x+6)$ for all $x \in \mathbb{R}$, then a period of f is

(A) 3 (B) 7 (C) 5 (D) 8

Solution: The given relation is

$$f(x) + f(x+4) = f(x+2) + f(x+6) \tag{1.3}$$

Replacing x with $x-2$, we get that

$$f(x-2)+f(x+2)=f(x)+f(x+4) \tag{1.4}$$

From Eqs. (1.3) and (1.4) we get

$$f(x-2)=f(x+6) \text{ for all } x\in\mathbb{R}$$

or $$f(x)=f(x+8) \text{ for all } x\in\mathbb{R}$$

Answer: (D)

23. Let $A=\mathbb{R}\times\mathbb{R}$, $\mathbb{R}$ the real number system and

$$\text{R}=\{((x,y),(a,b))\in A\times A \mid \text{either } x<a \text{ or } x=a \text{ and } y>b\}$$

Then which one of the following is true, if $((x,y),(a,b))\in\text{R}$ and $((a,b),(p,q))\in\text{R}$?

(A) $((x,y),(p,q))\in\text{R}$ (B) $((x,y),(q,p))\in\text{R}$
(C) $((x,y),(y,q))\in\text{R}$ (D) $((y,x),(p,q))\in\text{R}$

Solution: Suppose that $((x,y),(a,b)),((a,b),(p,q))\in\text{R}$. Then

$$\text{either } x<a \text{ or } x=a \text{ and } y>b$$

and $$\text{either } a<p \text{ or } a=p \text{ and } b>q$$

If $x<a$ and $a<p$, then $x<p$ and hence $((x,y),(p,q))\in\text{R}$. Same is the case when $x<a$ and $a=p$ and also when $x=a$ and $a<p$. If $x=a$, $y>b$, $a=p$ and $b>q$, then $x=p$ and $y>b>q$. Therefore $((x,y),(p,q))\in\text{R}$.

Answer: (A)

24. Let a be a positive real number and $f:\mathbb{R}\to\mathbb{R}$ a function such that

$$f(x+a)=1+\sqrt{2f(x)-f^2(x)} \text{ for all } x\in\mathbb{R}$$

Then a period of f is

(A) $2a$ (B) $3a$ (C) $4a$ (D) $5a$

Solution: Given $f(x+a)=1+\sqrt{2f(x)-f^2(x)}$ for all $x\in\mathbb{R}$. Replacing x with $x-a$ we get

$$f(x)=1+\sqrt{2f(x-a)-(f(x-a))^2} \text{ and } 1\le f(x)\le 2$$

Therefore

$$[f(x+a)-1]^2=2f(x)-[f(x)]^2 \tag{1.5}$$

Replacing x with $x+a$, we get

$$\begin{aligned}[f(x+2a)-1]^2 &= 2f(x+a)-[f(x+a)]^2\\ &= -[f(x+a)-1]^2+1\\ &= -[2f(x)-\{f(x)\}^2]+1 \quad \text{[by Eq. (1.5)]}\\ &= [f(x)-1]^2\end{aligned}$$

Therefore,

$$f(x+2a)-1=f(x)-1 \quad [\text{since } f(x+a), f(x)\ge 1]$$

$$f(x+2a)=f(x) \text{ for all } x\in\mathbb{R}$$

Thus $2a$ is a period of f.

Answer: (A)

25. The range of the function f defined by

$$f(x)=\frac{e^x-e^{|x|}}{e^x+e^{|x|}}$$

is

(A) $[0,1]$ (B) $(-1,0]$ (C) $(0,1)$ (D) $[-1,0]$

Solution: Here $f(x)$ is defined for all real x, since $e^x+e^{|x|}\ne 0$ for all $x\in\mathbb{R}$. Also

$$f(x)=\begin{cases}0 & \text{for } x\ge 0\\ \dfrac{e^x-e^{-x}}{e^x+e^{-x}}=\dfrac{e^{2x}-1}{e^{2x}+1} & \text{for } x<0\end{cases}$$

Therefore

$$f(x)=1-\frac{2}{e^{2x}+1} \quad \text{for all } x<0$$

For $x<0$,

$$y=f(x)\Leftrightarrow 0\ge y=1-\frac{2}{e^{2x}+1}>-1$$

From this it follows that the range of f is $(-1,0]$.

Answer: (B)

Multiple Correct Choice Type Questions

1. Let A and B be two sets. If X is any set such that $A\cap X=B\cap X$ and $A\cup X=B\cup X$, then

(A) $B\subseteq A$ (B) $A\subseteq B$ (C) $A=B$ (D) $A\,\Delta\, B=\phi$

Solution: We have

$$\begin{aligned}A &= (A\cup X)\cap A\\ &= (B\cup X)\cap A\\ &= (B\cap A)\cup(X\cap A)\\ &= (B\cap A)\cup(X\cap B)\\ &= B\cap(A\cup X)\\ &= B\cap(B\cup X)=B\end{aligned}$$

Therefore $A=B$ and hence all are correct answers.

Answers: (A), (B), (C) and (D)

2. S is a set and the Cartesian product $S \times S$ has 9 elements of which two elements are $(-2, 1)$ and $(1, 2)$. Then

(A) $(2, -2) \in S \times S$ (B) $(-2, -2) \in S \times S$

(C) $(-2, 2) \notin S \times S$ (D) $S = \{-2, 1, 2\}$

Solution: $S \times S$ has $9 = 3^2$ elements and hence S must have 3 elements. Since $(-2, 1)$ and $(1, 2) \in S \times S$, we have $-2, 1, 2 \in S$ and therefore $S = \{-2, 1, 2\}$. Therefore $(2, -2) \in S \times S$ and $(-2, -2) \in S \times S$.

Answers: (A), (B) and (D)

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying the following:

(a) $f(-x) = -f(x)$

(b) $f(x+1) = f(x) + 1$

(c) $f\left(\dfrac{1}{x}\right) = \dfrac{f(x)}{x^2}$ for all $x \neq 0$

Then

(A) $f(x) = x$ for all $x \in \mathbb{R}$

(B) $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$

(C) $f(xy) = f(x)f(y)$ for all $x, y \in \mathbb{R}$

(D) $f\left(\dfrac{x}{y}\right) = \dfrac{f(x)}{f(y)}$ for all $x, y \in \mathbb{R}$ with $y \neq 0$

Solution: We shall prove that $f(x) = x$ for all $x \in \mathbb{R}$ and hence (A), (B), (C) and (D) are all true. By (a), f is an odd function and hence $f(0) = 0$.

$$0 = f(0) = f(-1+1) = f(-1) + 1 \quad \text{[by (b)]}$$

Therefore $f(-1) = -1$. For any $x \neq 0$ and -1, we have

$$f\left(\frac{1}{x+1}\right) = \frac{f(x+1)}{(x+1)^2} = \frac{f(x)+1}{(x+1)^2} \tag{1.6}$$

Also,

$$\begin{aligned}
f\left(\frac{1}{x+1}\right) &= f\left(\frac{-x}{x+1} + 1\right) \\
&= f\left(\frac{-x}{x+1}\right) + 1 \\
&= -f\left(\frac{x}{x+1}\right) + 1 \\
&= -f\left(\frac{1}{(x+1)/x}\right) + 1 \\
&= \frac{-f[(x+1)/x]}{[(x+1)/x]^2} + 1 \\
&= \frac{-f[1+(1/x)]}{[(x+1)/x]^2} + 1 \\
&= \frac{-x^2[f(1/x)+1]}{(x+1)^2} + 1 \\
&= \frac{-x^2[\{f(x)/x^2\}+1]}{(x+1)^2} + 1 \\
&= \frac{-[f(x)+x^2]}{(x+1)^2} + 1
\end{aligned}$$

Therefore

$$f\left(\frac{1}{x+1}\right) = \frac{(x+1)^2 - x^2 - f(x)}{(x+1)^2} \tag{1.7}$$

From Eqs. (1.6) and (1.7), we get

$$f(x) + 1 = (x+1)^2 - x^2 - f(x) = 2x + 1 - f(x)$$

Therefore, $2f(x) = 2x$ and hence $f(x) = x$ for all $x \in \mathbb{R}$.

Answers: (A), (B), (C) and (D)

4. If $a \neq b \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ is a function such that

$$af(x) + bf\left(\frac{1}{x}\right) = x - 1 \text{ for all } 0 \neq x \in \mathbb{R}$$

Then

(A) $f(2) = \dfrac{2a+b}{2(a^2-b^2)}$ (B) $f(1) = 0$

(C) $f(-1) = -2/(a+b)$ (D) $f(-1) = 2(a-b)$

Solution: We are given that

$$af(x) + bf\left(\frac{1}{x}\right) = x - 1 \tag{1.8}$$

Replacing x with $1/x$, we get

$$bf(x) + af\left(\frac{1}{x}\right) = \frac{1}{x} - 1 \tag{1.9}$$

From Eqs. (1.8) and (1.9), we have

$$(a^2 - b^2)f(x) = a(x-1) - b\left(\frac{1}{x} - 1\right)$$

Therefore

$$f(2) = \frac{a + b/2}{a^2 - b^2} = \frac{2a+b}{2(a^2-b^2)}$$

$$f(1) = 0$$

and

$$f(-1) = \frac{-2a+2b}{a^2-b^2} = \frac{-2}{a+b}$$

Answers: (A), (B) and (C)

5. Let $P(x)$ be a polynomial function of degree n such that

$$P(k) = \frac{k}{k+1}$$

for $k = 0, 1, 2, \ldots, n$. Then $P(n+1)$ is equal to

(A) -1 if n is even (B) 1 if n is odd

(C) $\dfrac{n}{n+2}$ if n is even (D) $\dfrac{n}{n+2}$ if n is odd

Solution: Consider the polynomial

$$Q(x) \equiv P(x)(x+1) - x$$

Then $Q(x)$ is a polynomial of degree $n+1$ and $0, 1, 2, \ldots, n$ are the roots of the equation $Q(x) = 0$. Therefore

$$Q(x) = Ax(x-1)(x-2)\cdots(x-n)$$

where A is a non-zero constant. Substituting $x = -1$, we get that

$$1 = Q(-1) = A(-1)^{n+1}(n+1)!$$

Therefore

$$P(x) \equiv \frac{1}{x+1}\left[x + \frac{(-1)^{n+1}x(x-1)(x-2)\cdots(x-n)}{(n+1)!}\right]$$

$$P(n+1) = \frac{1}{n+2}[(n+1) + (-1)^{n+1}]$$

$$= \begin{cases} 1 & \text{if } n \text{ is odd} \\ \dfrac{n}{n+2} & \text{if } n \text{ is even} \end{cases}$$

Answers: (B) and (C)

Matrix-Match Type Questions

1. If $A = \{1, 2, 4, 5\}$, $B = \{2, 3, 4, 5\}$ and $C = \{4, 5, 6, 7\}$, then match the items in Column I with those in Column II.

Column I	*Column II*
(A) $(A - B) \cup C$	(p) $\{1, 2, 3\}$
(B) $(A - B) \cup (B - C)$	(q) $\{1, 3, 4, 5, 6, 7\}$
(C) $(A \cup B) - C$	(r) $\{1, 4, 5, 6, 7\}$
(D) $(A \,\Delta\, B) \,\Delta\, C$	(s) $\{1, 2, 3, 4\}$

Solution: This can be solved by simple checking.

Answer: (A)→(r), (B)→(p), (C)→(p), (D)→(q)

2. Let A, B and C be subsets of a finite universal set X. Let $n(P)$ denote the number of elements in a set P. Then match the items in Column I with those in Column II.

Column I	*Column II*
(A) $n(A - B)$	(p) $n(X) - n(A \cap B)$
(B) $n(A \,\Delta\, B)$	(q) $n(C) - n(C \cap B)$
(C) $n(A^c \cup B^c)$	(r) $n(A) - n(A \cap B)$
(D) $n(C \cap B^c)$	(s) $n(A) + n(B) - 2n(A \cap B)$

Solution: This can be solved by simple checking.

Answer: (A)→(r), (B)→(s), (C)→(p), (D)→(q)

3. Let $P : [0, \infty) \to \mathbb{Z}^+$ be defined as

$$P(x) = \begin{cases} 13 & \text{if } 0 \le x < 1 \\ 13 + 15n & \text{if } n \le x < n+1, n \in \mathbb{Z}^+ \end{cases}$$

Then match the items in Column I with those in Column II.

Column I	*Column II*
(A) $P(3 \cdot 01)$	(p) 68
(B) $P(4 \cdot 9)$	(q) 63
(C) $P(3 \cdot 999)$	(r) 73
(D) $P([4 \cdot 99])$	(s) 58

Solution: Given that $P(x) = 13 + 15[x]$ for all $x \ge 0$, where $[x]$ is the integral part of x. Then

$$P(3 \cdot 01) = 13 + 15 \times 3 = 58$$

Remaining parts can be solved similarly.

Answer: (A)→(s), (B)→(r), (C)→(s), (D)→(r)

Note: Functions of this type are called *Postage-stamp functions*.

Comprehension-Type Questions

1. In a group of 25 students aged between 16 years and 18 years, it was found that 15 play cricket, 12 play tennis, 11 play football, 5 play both cricket and foot ball, 9 play both cricket and tennis, 4 play tennis and football and 3 play all the three games. Based on this, answer the following questions.

(i) The number of students in the group who play only football is

(A) 2 (B) 3 (C) 4 (D) 5

(ii) The number of students in the group who play only cricket is

(A) 1 (B) 2 (C) 3 (D) 4

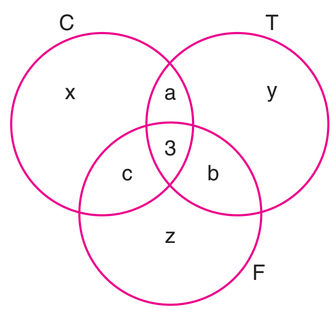
(iii) The number of students in the group who play only tennis is

(A) 1 (B) 2 (C) 3 (D) 4

(iv) The number of students who do not play any of the three games is

(A) 1 (B) 2 (C) 3 (D) 4

Solution: Let C, T and F denote the sets of students in the group who play cricket, tennis and football, respectively. Consider the Venn diagram.



We are given that

$$n(C) = x + a + c + 3 = 15$$
$$n(T) = y + b + a + 3 = 12$$
$$n(F) = z + c + b + 3 = 11$$

Then

$$n(C \cap T) = a + 3 = 9$$
$$n(T \cap F) = b + 3 = 4$$
$$n(C \cap F) = c + 3 = 5$$
$$n(C \cap T \cap F) = 3$$

and by solving these, we get $a = 6, b = 1, c = 2, x = 4, y = 2$ and $z = 5$. The number of students who do not play any of these games is $25 - (a + b + c + x + y + z + 3) = 2$.

Answer: (i) → (D); (ii) → (D); (iii) → (B); (iv) → (B)

2. Let $f: \mathbb{R} - \{0, 1\} \to \mathbb{R}$ be a function satisfying the relation

$$f(x) + f\left(\frac{x-1}{x}\right) = x$$

for all $x \in \mathbb{R} - \{0, 1\}$. Based on this, answer the following questions.

(i) $f(x)$ is equal to

(A) $\frac{1}{2}\left[x + \frac{1}{1-x} - \frac{x-1}{x}\right]$

(B) $\frac{1}{2}\left[x - \frac{1}{1-x} + \frac{x-1}{x}\right]$

(C) $\frac{1}{2}\left[x - \frac{1}{1-x} - \frac{x-1}{x}\right]$

(D) $\frac{1}{2}\left[x + \frac{1}{1-x} + \frac{x-1}{x}\right]$

(ii) $f(-1)$ is equal to

(A) 3/4 (B) –3/4 (C) 5/4 (D) –5/4

(iii) $f(1/2)$ is equal to

(A) 5/4 (B) –7/4 (C) 7/4 (D) 9/4

Solution: Given that

$$f(x) + f\left(\frac{x-1}{x}\right) = x \tag{1.10}$$

for all $x \neq 0, 1$. Replacing x with $(x-1)/x$ both sides, we get that

$$f\left(\frac{x-1}{x}\right) + f\left(\frac{[(x-1)/x] - 1}{(x-1)/x}\right) = \frac{x-1}{x}$$

That is,

$$f\left(\frac{x-1}{x}\right) + f\left(\frac{1}{1-x}\right) = \frac{x-1}{x} \tag{1.11}$$

Again replacing x with $(x-1)/x$ in this, we get

$$f\left(\frac{1}{1-x}\right) + f(x) = \frac{1}{1-x} \tag{1.12}$$

Then by taking Eq. (1.10) + Eq. (1.12) – Eq. (1.11), we get that

$$2f(x) = x + \frac{1}{1-x} - \frac{x-1}{x}$$

or
$$f(x) = \frac{1}{2}\left[x + \frac{1}{1-x} - \frac{x-1}{x}\right] \tag{1.13}$$

Substituting the values $x = -1$ and 1/2 in Eq. (1.13) we get the solution for (ii) and (iii).

Answer: (i) → (A); (ii) → (D); (iii) → (C)

Assertion–Reasoning Type Questions

In the following question, a Statement I is given and a corresponding Statement II is given just below it. Mark the correct answer as:

(A) Both I and II are true and II is a correct reason for I

(B) Both I and II are true and II is not a correct reason for I

(C) I is true, but II is false

(D) I is false, but II is true

1. **Statement I:** If $A = \{1, 2, 3, 4\}$ and $B = \{2, 3, 5, 6, 7\}$, then $n((A \times B) \cap (B \times A)) = 4$.

   Statement II: If two sets A and B have n elements in common, then the sets $A \times B$ and $B \times A$ have n^2 elements in common.

Solution: Note that, for any sets A, B, C and D,

$$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$$

and hence

$$(A \times B) \cap (B \times A) = (A \cap B) \times (B \cap A)$$
$$= (A \cap B) \times (A \cap B)$$

Therefore

$$n((A \times B) \cap (B \times A)) = [n(A \cap B)]^2$$

Answer: (A)

Integer Answer Type Questions

1. Let $f: \mathbb{R} \to \mathbb{R}$ be a function such that $f(1) = f(0) = 0$ and $|f(x) - f(y)| < |x - y|$ for all $x \neq y$ in $[0, 1]$. If $2|f(x) - f(y)| < K$ for all $x, y \in [0,1]$, then K can be ______.

Solution: Let $0 < x < y < 1$. Then

$$|f(x) - f(y)| \leq |f(x)| + |f(y)|$$
$$= |f(x) - f(0)| + |f(y) - f(1)|$$
$$< |x - 0| + |y - 1|$$
$$= x + 1 - y \tag{1.14}$$

Also,

$$|f(x) - f(y)| < |x - y| = y - x \tag{1.15}$$

By adding Eqs. (1.14) and (1.15), we have

$$2|f(x) - f(y)| < 1$$

Answer: 1

2. Let $f: \mathbb{R} \to \mathbb{R}$ be a function such that $f(x + y) = f(x) + f(y) - xy - 1$ for all $x, y \in \mathbb{R}$ and $f(1) = 1$. Then the number of positive integers n such that $f(n) = n$ is ______.

Solution: By taking $x = 0 = y$, we get that $f(0) = 1$. By hypothesis, $f(1) = 1$. For any integer $n > 1$,

$$f(n) = f[(n-1) + 1] = f(n-1) + f(1) - (n-1)1 - 1$$
$$= f(n-1) - (n-1) < f(n-1)$$

Therefore, we have $1 = f(1) > f(2) > f(3) > \cdots$ and hence $f(n) \neq n$ for all $n > 1$. Thus 1 is the only positive integer n such that $f(n) = n$.

Answer: 1

3. Let $f: \mathbb{R} \to \mathbb{R}$ be a function such that $f(2 + x) = f(2 - x)$ and $f(7 + x) = f(7 - x)$ for all real numbers x. If $f(0) = 0$ and there are atleast m number of integer solutions for $f(x) = 0$ in the interval $[-2010, 2010]$, then m can be ______.

Solution: For all $x \in \mathbb{R}$, we have

$$f(2 + x) = f(2 - x) = f[7 - (5 + x)]$$
$$= f[7 + (5 + x)] = f(12 + x)$$

By replacing x with $x - 2$ we get that

$$f(x) = f(x + 10) \text{ for all } x \in \mathbb{R} \tag{1.16}$$

Now,

$$0 = f(0) = f(2 - 2) = f(2 + 2) = f(4) \tag{1.17}$$

From Eqs. (1.16) and (1.17), we have $f(4 + 10n) = 0$ for all integers n. Also, since $f(0) = 0$, we have $f(10n) = 0$ for all integers n. There are 403 integers of the form $10n$ and 402 integers of the form $10n + 4$ in the interval $[-2010, 2010]$. Therefore, there are atleast 805 integers n in $[-2010, 2010]$ for which $f(n) = 0$.

Answer: 805

SUMMARY

1.1 ***Set:*** Any collection of well-defined objects.

1.2 ***Elements:*** Objects belonging to a set.

1.3 ***Empty set:*** Set having no elements and is denoted Ø.

1.4 ***Equal sets:*** Two sets A and B are said to be equal, if they contain same elements or every element of A belong to B and vice-versa.

1.5 ***Finite set:*** A set having definite number of elements is called *finite* set. A set which is not a finite set is called *infinite* set.

1.6 ***Family or class of sets:*** A set whose numbers are family of sets or class of sets. Family of sets or class of sets are denoted by script letters $\mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{D}, P$ etc.

1.7 ***Indexed family of sets:*** A family C of sets is called indexed family if there exists a set I such that for each element $i \in I$, there exists unique member $A \in C$ associated with i. In this case the set I is called index set, C is called indexed family sets and we write $C = \{A_i : i \in I\}$.

1.8 ***Intervals:*** Let a, b be real numbers and $a < b$. Then

$$(a, b) = \{x \in \mathbb{R} | a < x < b\}$$

$$[a, b) = \{x \in \mathbb{R} | a \le x < b\}$$

$$(a, b] = \{x \in \mathbb{R} | a < x \le b\}$$

$$[a, b] = \{x \in \mathbb{R} | a \le x \le b\}$$

$$(-\infty, +\infty) \quad \text{or} \quad (-\infty, \infty) \text{ is } \mathbb{R}$$

1.9 ***Subset and superset:*** A set A is called a subset of a set B, if every element of A is also an element of B. In this case we write $A \subseteq B$. If A is a subset of B, then B is called superset of A. If A is not a subset of B, then we write $A \not\subseteq B$.

1.10 ***Proper subset:*** Set A is called a proper subset of a set B if A is a subset of B and is not equal to B.

1.11 ***Powerset:*** If X is a set, then the collection of all subsets of X is called the powerset of X and is denoted by $P(X)$.

1.12 ***Cardinality of a set:*** If X is a finite set having n elements, then n called cardinality of X and is denoted by $|X|$ or $n(X)$.

1.13 If $|X| = n$, then $|P(X)| = 2^n$.

1.14 ***Intersection of sets:*** For any two sets A and B, the intersection of A and B is the set of all elements belonging to both A and B and is denoted by

$$A \cap B = \{x | x \in A \text{ and } x \in B\}$$

1.15 ***Theorem:*** The following hold for any sets A, B and C.

(1) $A \subseteq B \Leftrightarrow A = A \cap B$

(2) $A \cap A = A$

(3) $A \cap B = B \cap A$ (Commutative law)

(4) $(A \cap B) \cap C = A \cap (B \cap C)$ (Associative law)

(5) $A \cap Ø = Ø$, where Ø is the empty set.

(6) For any set X, $X \subseteq A \cap B \Leftrightarrow X \subseteq A$ and $X \subseteq B$.

(7) In view of (4) we write $A \cap B \cap C$ for $A \cap (B \cap C)$.

(8) For any sets $A_1, A_2, \ldots, A_n$ we write $\bigcap_{i=1}^{n} A_i$ for $A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_n$.

1.16 ***Disjoint sets:*** Two sets A and B are said to be disjoint sets if $A \cap B = Ø$.

1.17 ***Union of sets:*** For any two sets A and B, their union is defined to be the set of all elements belonging to either A or to B and this set is denoted by $A \cup B$. That is $A \cup B = \{x | x \in A \text{ or } x \in B\}$.

1.18 ***Theorem:*** For any sets A, B and C the following hold.

(1) $A \cap B \subseteq A \cup B$

(2) For any set X, $A \cup B \subseteq X \Leftrightarrow A \subseteq X$ and $B \subseteq X$

(3) $A \cup A = A$

(4) $A \cup B = B \cup A$ (Commutative law)

(5) $(A \cup B) \cup C = A \cup (B \cup C)$ and we write $A \cup B \cup C$ for $(A \cup B) \cup C$

(6) $A \cap B = A \Leftrightarrow A \cup B = B$

(7) $A \cup Ø = A$

(8) $A \cap (A \cup B) = A$

(9) $A \cup (A \cap B) = A$

1.19 ***Theorem (Distributive laws):*** If A, B and C are three sets, then

(1) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(2) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

1.20 ***Theorem:*** For any sets A, B and C, $A \cap B = A \cap C$ and $A \cup B = A \cup C \Rightarrow B = C$.

1.21 If $\{A_i\}_{i\in I}$ is an indexed family of sets then $\bigcup_{i\in I} A_i$ is the set of all elements x where x belongs to atleast one A_i.

1.22 ***Set difference:*** For any two sets A and B, $A - B = \{x \in A \mid x \notin B\} = A - (A \cap B)$

1.23 ***De Morgan's laws:*** If A, B and C are any sets, then

(1) $A - (B \cup C) = (A - B) \cap (A - C)$

(2) $A - (B \cap C) = (A - B) \cup (A - C)$

1.24 ***Theorem:*** Let A, B and C be sets. Then

(1) $B \subseteq C \Rightarrow A - C \subseteq A - B$

(2) $A \subseteq B \Rightarrow A - C \subseteq B - C$

(3) $(A \cup B) - C = (A - C) \cup (B - C)$

(4) $(A \cap B) - C = (A - C) \cap (B - C)$

(5) $(A - B) - C = A - (B \cup C) = (A - B) \cap (A - C)$

(6) $A - (B - C) = (A - B) \cup (A \cap C)$

1.25 ***Universal set:*** If $\{A_i\}_{i\in I}$ is a class of sets, then the set $X = \bigcup_{i\in I} A_i$ is called universal set. In fact the set X whose subsets are under our consideration is called universal set.

Caution: Do not be mistaken that universal set means the set which contains all objects in the universe. Do not be carried away with word universal. In fact, the fundamental axiom of set theory is:

Given any set, there is always an element which does not belong to the given set.

1.26 ***Complement of a set:*** If X is an universal set and $A \subseteq X$ then the set $X - A$ is called complement of A and is denoted by A' or A^c.

1.27 ***Relative complement:*** If X is an universal set and A, B are subsets of X, then $A - B = A \cap B'$ is called relative complement of B in A.

1.28 ***De Morgan's laws (General form):*** If A and B are two sets, then

(1) $(A \cup B)' = A' \cap B'$

(2) $(A \cap B)' = A' \cup B'$

1.29 ***Symmetric difference:*** For any two sets A and B, the set $(A - B) \cup (B - A)$ is called symmetric difference of A and B and is denoted by $A \Delta B$. Since $A - B = A \cap B'$ and $B - A = B \cap A'$, $A \Delta B = (A \cap B') \cup (B \cap A')$.

1.30 ***Theorem:*** The following hold for any sets A, B and C.

(1) $A \Delta B = B \Delta A$ (Commutative law)

(2) $(A \Delta B) \Delta C = A \Delta (B \Delta C)$ (Associative law)

(3) $A \Delta \emptyset = A$

(4) $A \Delta A = \emptyset$

1.31 ***Theorem:*** If A and B are disjoint sets, then

(1) $n(A \cup B) = n(A) + n(B)$

(2) If $A_1, A_2, \ldots, A_m$ are pairwise disjoint sets, then

$$n\left(\bigcup_{i=1}^{m} A_i\right) = n(A_1) + n(A_2) + \cdots + n(A_m)$$

Recall that for any finite set P, $n(P)$ denotes the number of elements in P.

1.32 ***Theorem:*** For any finite sets A and B, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

1.33 ***Theorem:*** For any finite sets A, B and C,

$$\begin{aligned} n(A \cup B \cup C) = {} & n(A) + n(B) + n(C) - n(A \cap B) \\ & - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C) \end{aligned}$$

1.34 ***Theorem:*** If A, B and C are finite sets, then the number of elements belonging to exactly two of the sets is

$$n(A \cap B) + n(B \cap C) + n(C \cap A) - 3n(A \cap B \cap C)$$

1.35 ***Theorem:***

(1) If A, B and C are finite sets, then the number of elements belonging to exactly one of the sets is

$$\begin{aligned} & n(A) + n(B) + n(C) - 2n(A \cap B) - 2n(B \cap C) \\ & - 2n(C \cap A) + n(A \cap B \cap C) \end{aligned}$$

(2) If A and B are finite sets, then the number of elements belonging to exactly one of the sets equals

$$\begin{aligned} n(A \Delta B) &= n(A) + n(B) - 2n(A \cap B) \\ &= n(A \cup B) - n(A \cap B) \end{aligned}$$

Relations

1.36 ***Ordered pair:*** A pair of elements written in a particular order is called an ordered pair and is written by listing its two elements in a particular order, separated by a comma and enclosing the pair in brackets. In the ordered pair (x, y), x is the first element called first component and y is the second element called second component. Also x is called first coordinate and y is called second coordinate.

1.37 ***Cartesian product:*** If A and B are sets, then the set of all ordered pairs (a, b) with $a \in A$ and $b \in B$ is called the Cartesian product of A and B and is denoted by $A \times B$ (read as A cross B). That is

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$$

1.38 Let A, B be any sets and $\emptyset$ is the empty set. Then

(1) $A \times B = \emptyset \Leftrightarrow A = \emptyset$ or $B = \emptyset$.

(2) If one of A and B is an infinite set and the other is a non-empty set, then $A \times B$ is an infinite set.

(3) $A \times B = B \times A \Leftrightarrow A = B$.

1.39 ***Cartesian product of n sets (n is a finite positive integer greater than or equal to 2):*** Let $A_1, A_2, A_3, \ldots, A_n$ be n sets. Then their Cartesian product is defined to be the set of all n-tuples $(a_1, a_2, \ldots, a_n)$ such that $a_i \in A_i$ for $i = 1, 2, 3, \ldots, n$ and is denoted by

$$A_1 \times A_2 \times A_3 \times \cdots \times A_n \quad \text{or} \quad \mathop{\mathbf{X}}_{i=1}^{n} A_i \quad \text{or} \quad \prod_{i=1}^{n} A_i$$

That is,

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for } 1 \le i \le n\}$$

The Cartesian product of a set A with itself n times is denoted by A^n.

1.40 ***Theorem:*** If A and B are finite sets, then $n(A \times B) = n(A) \cdot n(B)$. In general, if $A_1, A_2, \ldots, A_m$ are infinite sets, then $n(A_1 \times A_2 \times \cdots \times A_m) = n(A_1) \times n(A_2) \times \cdots \times n(A_m)$. In particular, $n(A^m) = (n(A))^m$ where A is a finite set.

1.41 ***Theorem:*** Let A, B, C and D be any sets. Then

(1) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(2) $(A \cup B) \times C = (A \times C) \cup (B \times C)$

(3) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(4) $(A \cap B) \times C = (A \times C) \cap (B \times C)$

(5) $(A \cup B) \times (C \cup D) = (A \times C) \cup (A \times D) \cup (B \times C) \cup (B \times D)$

(6) $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D) = (A \times D) \cap (B \times C)$

(7) $(A - B) \times C = (A \times C) - (B \times C)$

(8) $A \times (B - C) = (A \times B) - (A \times C)$

1.42 ***Relation:*** For any two sets A and B, any subset of $A \times B$ is called a relation from A to B.

1.43 ***Symbol aRb:*** Let R be a relation from a set A to a set B ($\mathrm{R} \subseteq A \times B$). If $(a, b) \in \mathrm{R}$, then a is said to be R related to b or a is said to be related to b and we write $a\mathrm{R}b$.

1.44 ***Domain:*** Let R be a relation from a set A to a set B. Then the set of all first components of the ordered pairs belonging to R is called the domain of R and is denoted by Dom(R).

1.45 ***Range:*** If R is a relation from a set A to a set B, then the set of all second components of the ordered pairs belonging to R is called range of R and is denoted by Range(R).

1.46 ***Theorem:*** If A and B are finite non-empty sets such that $n(A) = m$ and $n(B) = n$, then the number of relations from A to B is 2^{mn} which include the empty set and the whole set $A \times B$.

1.47 ***Relation on a set:*** If A is a set, then any subset of $A \times A$ is called a binary relation on A or simply a relation on A.

1.48 ***Composition of relations:*** Let A, B and C be sets, R is a relation from A to B and S is a relation from B to C. Then, the composition of R and S denoted by $\mathrm{S} \circ \mathrm{R}$ defined to be

$$\mathrm{S} \circ \mathrm{R} = \{(a, c) \in A \times C \mid \text{there exist } b \in B \text{ such that } (a, b) \in \mathrm{R} \text{ and } (b, c) \in \mathrm{S}\}$$

1.49 ***Theorem:*** Let A, B and C be sets, R a relation from A to B and S a relation from B to C. Then the following hold:

(1) $\mathrm{S} \circ \mathrm{R} \ne \emptyset$ if and only if $\mathrm{Range(R)} \cap \mathrm{Dom(S)} \ne \emptyset$

(2) $\mathrm{Dom}(\mathrm{S} \circ \mathrm{R}) = \mathrm{Dom(R)}$

(3) $\mathrm{Range}(\mathrm{S} \circ \mathrm{R}) \subseteq \mathrm{Range(S)}$

1.50 ***Theorem:*** Let A, B, C and D be non-empty sets, $R \subseteq A \times B$, $S \subseteq B \times C$ and $T \subseteq C \times D$. Then

$$(\mathrm{T} \circ \mathrm{S}) \circ \mathrm{R} = \mathrm{T} \circ (\mathrm{S} \circ \mathrm{R}) \text{ (Associative law)}$$

1.51 ***Inverse relation:*** Let A and B be non-empty sets and R a relation from A to B. Then the inverse of R is defined as the set $\{(b, a) \in B \times A \mid (a, b) \in \mathrm{R}\}$ and is denoted by R^{-1}.

1.52 ***Theorem:*** Let A, B and C be non-empty sets, R a relation from A to B and S a relation from B to C. Then the following hold:

(1) $(\mathrm{S} \circ \mathrm{R})^{-1} = \mathrm{R}^{-1} \circ \mathrm{S}^{-1}$

(2) $(\mathrm{R}^{-1})^{-1} = \mathrm{R}$

Types of Relations

1.53 ***Reflexive relation:*** Let X be a non-empty set and R relation from X to X. Then R is said to be reflexive on X if $(x, x) \in \mathrm{R}$ for all $x \in X$.

1.54 ***Symmetric relation:*** A relation R on a non-empty set X is called symmetric if $(x, y) \in \mathrm{R} \Rightarrow (y, x) \in \mathrm{R}$.

1.55 ***Transitive relation:*** A relation R on a non-empty set X is called transitive if $(x, y) \in R$ and $(y, z) \in R \Rightarrow (x, z) \in R$.

1.56 ***Equivalence relation:*** A relation R on a non-empty set X is called an equivalence relation if it is reflexive, symmetric and transitive.

1.57 ***Partition of a set:*** Let X be a non-empty set. A class of subsets of X is called a partition of X if they are pairwise disjoint and their union is X.

1.58 ***Equivalence class:*** Let X be a non-empty set and R an equivalence relation on X. If $x \in X$, then the set $\{y \in X \mid (x, y) \in R\}$ is called the equivalence class of x with respect to R or the R-equivalence of x or simply the R-class of x and is denoted by $R(x)$.

1.59 ***Theorem:*** Let R be an equivalence relation on a set X and $a, b \in X$. Then the following statements are equivalent:

(1) $(a, b) \in R$

(2) $R(a) = R(b)$

(3) $R(a) \cap R(b) \neq \emptyset$

1.60 ***Theorem:*** Let R be an equivalence relation on X. Then the class of all R-classes form a partition of X.

1.61 ***Theorem:*** Let X be a non-empty and $\{A_i\}_{i \in I}$ a partition of X. Then

$$R = \{(x, y) \in X \times X \mid \text{both } x \text{ and } y \text{ belong to the same } A_i, i \in I\}$$

is an equivalence relation on X, whose R-classes are precisely A_i's.

1.62 ***Theorem:*** Let R and S be equivalence relations on a non-empty X. Then $R \cap S$ is also an equivalence relation on X and for any $x \in X$, $(R \cap S)(x) = R(x) \cap S(x)$.

1.63 ***Theorem:*** Let R and S be equivalence relations on a set X. Then the following statements are equivalent.

(1) $R \circ S$ is an equivalence relation on X

(2) $R \circ S$ is symmetric

(3) $R \circ S$ is transitive

(4) $R \circ S = S \circ R$

Functions

1.64 ***Function:*** A relation f from a set A to a set B is called a function from A into B or simply A to B, if for each $a \in A$, there exists unique $b \in B$ such that $(a, b) \in f$. That is $f \subseteq A \times B$ is called a function from A to B, if

(1) Dom $(f) = A$

(2) $(a, b) \in f$ and $(a, c) \in f \Rightarrow b = c$

If f is a function from A to B, then we write $f: A \to B$ is a function and for $(a, b) \in f$, we write $b = f(a)$ and b is called f-image of a and a is called f-preimage of b.

1.65 ***Domain, codomain and range:*** Let $f: A \to B$ be a function. Then A is called domain, B is called codomain and Range of f denoted by $f(A) = \{f(a) \mid a \in A\}$. $f(A)$ is also called the image set of A under the function f.

1.66 ***Composition of functions:*** Let $f: A \to B$ and $g: B \to C$ be functions. Then the composition of f with g denoted by $g \circ f$ is defined as $g \circ f: A \to C$ given by

$$(g \circ f)(a) = g(f(a)) \quad \text{for all } a \in A$$

1.67 ***Theorem:*** Let $f: A \to B$, $g: B \to C$ and $h: C \to D$ be functions. Then

$$(h \circ g) \circ f = h \circ (g \circ f)$$

1.68 ***One-one function or injection:*** A function $f: A \to B$ is called "one-one function" if $f(a_1) \neq f(a_2)$ for any $a_1 \neq a_2$ in A.

1.69 ***Theorem:*** If $f: A \to B$ and $g: B \to C$ be functions. Then the following hold:

(1) If f and g are injections, then so is $g \circ f$.

(2) If $g \circ f$ is an injection, then f is an injection.

1.70 ***Onto function or surjection:*** A function $f: A \to B$ is called "onto function" if the range of f is equal to the codomain B. That is, to each $b \in B$, there exists $a \in A$ such that $f(a) = b$.

1.71 ***Theorem:*** Let $f: A \to B$ and $g: B \to C$ be functions. Then, the following hold:

(1) If f and g are surjections, then so is $g \circ f$.

(2) If $g \circ f$ is a surjection, then g is a surjection.

1.72 ***Bijection or one-one and onto function:*** A function $f: A \to B$ is called "bijection" if f is both an injection and a surjection.

1.73 ***Theorem:*** If $f: A \to B$ and $g: B \to C$ are bijections, then $g \circ f: A \to C$ is a bijection.

1.74 ***Identity function:*** A function $f: A \to A$ is called an identity function if $f(x) = x$ for all $x \in A$ and is denoted by I_A.

1.75 ***Theorem:*** If $f: A \to B$ is a function, then $I_B \circ f = f = f \circ I_A$.

QUICK LOOK

Identity function is always a bijection.

1.76 ***Theorem:*** Let $f: A \to B$ be a function. Then, f is a bijection if and only if there exists a function $g: B \to A$ such that

$$g \circ f = I_A \quad \text{and} \quad f \circ g = I_B$$

That is

$$g(f(a)) = a \quad \text{for all } a \in A$$

and $$f(g(b)) = b \quad \text{for all } b \in B$$

1.77 ***Inverse of a bijective function:*** Let $f: A \to B$ and $g: B \to A$ be functions such that $g \circ f = I_A$ and $f \circ g = I_B$. Then f and g are bijections. Also g is unique such that $g \circ f = I_A$ and $f \circ g = I_B$. g is called the inverse of f and f is called the inverse of g. The inverse function of f is denoted by f^{-1}.

QUICK LOOK

If $f: A \to B$ is a bijection, then $f^{-1}: B \to A$ is also a bijection and $f^{-1}(b) = a \Leftrightarrow f(a) = b$ for $b \in B$.

1.78 ***Real-valued function:*** If the range of a function is a subset of the real number set $\mathbb{R}$, then the function is called a real-valued function.

1.79 ***Operations among real-valued functions:*** Let f and g be real-valued functions defined on a set A. Then we define the real-valued functions $f + g$, $-f$, $f - g$ and $f \cdot g$ on the set A as follows:

(1) $(f + g)(a) = f(a) + g(a)$

(2) $(-f)(a) = -f(a)$

(3) $(f - g)(a) = f(a) - g(a)$

(4) $(f \cdot g)(a) = f(a)\, g(a)$

(5) If $g(a) \neq 0$ for all $a \in A$, then

$$\left(\frac{f}{g}\right)(a) = \frac{f(a)}{g(a)}$$

(6) If n is a positive integer, then $f^n(a) = (f(a))^n$.

1.80 ***Integral part and fractional part:*** If x is a real number, then the largest integer less than or equal to x is called the integral part of x and is denoted by $[x]$. $x - [x]$ is called the fractional part of x and will be denoted by $\{x\}$.

QUICK LOOK

$0 \leq \{x\} < 1$ for any real number x.

1.81 ***Theorem:*** The following hold for any real numbers x and y.

(1) $$[x + y] = \begin{cases} [x] + [y] & \text{if } \{x\} + \{y\} < 1 \\ [x] + [y] + 1 & \text{if } \{x\} + \{y\} \geq 1 \end{cases}$$

(2) $[x + y] \geq [x] + [y]$ and equality holds if and only if $\{x\} + \{y\} < 1$.

(3) If x or y is an integer, then $[x + y] = [x] + [y]$.

(4) $\left[\dfrac{x}{m}\right] = \left[\dfrac{[x]}{m}\right]$ for any real number x and non-zero-integer m.

(5) If n and k are positive integers and $k > 1$, then

$$\left[\frac{n}{k}\right] + \left[\frac{n+1}{k}\right] \leq \left[\frac{2n}{k}\right]$$

1.82 ***Periodic function:*** Let A be a subset of $\mathbb{R}$ and $f: A \to \mathbb{R}$ a function. A positive real number p is called a period of f if $f(x + p) = f(x)$ whenever x and $x + p$ belong to A. A function with a period is called periodic function. Among the periods of f, the least one (if it exists) is called the *least period*.

1.83 ***Step function (greatest integer function):*** Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = [x]$ for all $x \in \mathbb{R}$ where $[x]$ is the largest integer less than or equal to x. This function f is called step function.

1.84 ***Signum function:*** Let $f: \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

is called Signum function and is written as sign(x).

1.85 ***Increasing and decreasing functions:*** Let A be a subset of $\mathbb{R}$ and $f: A \to \mathbb{R}$ a function. Then, we say that f is an increasing function if $f(x) \leq f(y)$ whenever $x \leq y$. f is said to be decreasing function if $f(x) \geq f(y)$ whenever $x \leq y$.

1.86 ***Symmetric set:*** A subset X of $\mathbb{R}$ is called a symmetric set if $x \in X \Leftrightarrow -x \in X$.

1.87 ***Even function:*** Let X be a symmetric set and $f: X \to \mathbb{R}$ a function. Then f is said to be even function if $f(-x) = f(x)$ for all $x \in X$.

1.88 ***Odd function:*** Let X be a symmetric set and $f: X \to \mathbb{R}$ a function. Then f is said to be odd function if $f(-x) = -f(x)$ for all $x \in X$.

QUICK LOOK

If f is an odd function on a symmetric set X and 0 belongs to X, then $f(0)$ is necessarily 0.

1.89 ***Theorem:*** Let X be a symmetric set and f, g be functions from X to $\mathbb{R}$. Then, the following hold:

(1) $f \cdot g$ is even if either both f and g are even or both are odd.

(2) $f \cdot g$ is odd if one of them is odd and the other is even.

1.90 ***Theorem:*** Let f be a real valued function defined on a symmetric set X. Then the following hold:

(1) f is even if and only if af is even for any non-zero $a \in \mathbb{R}$.

(2) f is odd if and only if af is odd for any non-zero $a \in \mathbb{R}$.

(3) f is even (odd) if and only if $-f$ is even (odd).

1.91 ***Theorem:*** If f and g are even (odd) functions then so is $f \pm g$.

1.92 ***Theorem:*** Every real-valued function can be uniquely expressed as a sum of an even function and an odd function. The representation is

$$f(x) = \frac{1}{2}[f(x) + f(-x)] + \frac{1}{2}[f(x) - f(-x)]$$

1.93 ***Number of partions of a finite set:*** Let $P_0 = 1$ and P_n be the number of partions on a finite set with n elements. Then for $n \geq 1$,

$$P_{n+1} = \sum_{r=1}^{n} \binom{n}{r} P_r$$

where $\binom{n}{r}$ is the number of selections of r objects $(0 \leq r \leq n)$ from n distinct objects and this number is equal to

$$\frac{n!}{r!(n-r)!}$$

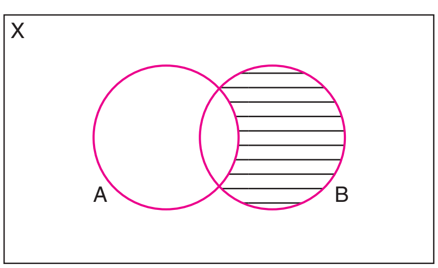
EXERCISES

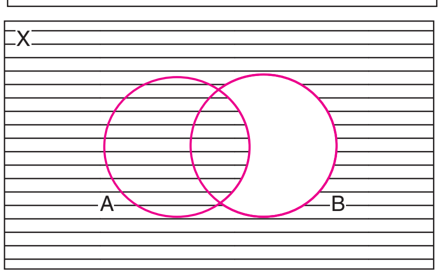
Single Correct Choice Type Questions

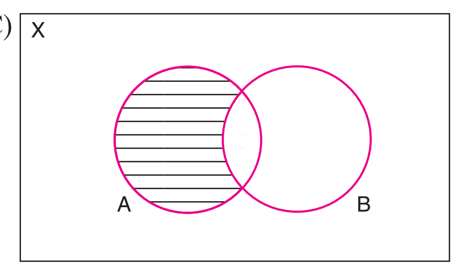
1. Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6, 8\}$ and $C = \{3, 4, 5, 6\}$. Then

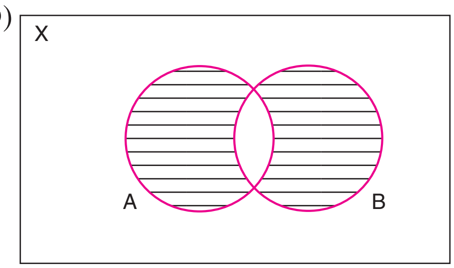
(A) $(B \cap C)^c = \{2, 4, 5, 6, 7\}$
(B) $(A \cap C)^c = \{1, 2, 3, 4, 5, 8, 9\}$
(C) $(B \cup C)^c = \{1, 7, 8, 9\}$
(D) $(A \cap B)^c = \{1, 3, 5, 6, 7, 8, 9\}$

2. If A and B are two non-empty subsets of a set X, then which one of the following shaded diagrams represent the complement of $B - A$ in X?

(A)


(B)


(C)


(D)


3. Let $A \,\Delta\, B$ denote the symmetric difference of A and B. Then, for any sets, A, B and C, which one of the following is not correct?

(A) $A \Delta B = C \Leftrightarrow A = B \Delta C$
(B) $A \Delta B = C \Delta B \Leftrightarrow A = C$
(C) $(A \Delta B) \Delta (B \Delta A) = \phi$
(D) $A \Delta B = \phi \Leftrightarrow A \subseteq B$

4. A, B, C are three finite sets such that $A \cap B \cap C$ has 10 elements. If the sets $A \Delta B$, $B \Delta C$ and $C \Delta A$ have 100, 150 and 200 elements, respectively, then the number of elements in $A \cup B \cup C$ is

(A) 325 (B) 352 (C) 235 (D) 253

5. In a class of 45 students, it is found that 20 students liked apples and 30 liked bananas. Then the least number of students who liked both apples and bananas is

(A) 5 (B) 10 (C) 15 (D) 8

6. In a class of 45 students, 25 play chess and 26 play cricket. If each student plays chess or cricket, then the number of students who play both is

(A) 5 (B) 6 (C) 7 (D) 4

7. The number of subsets of the empty set is

(A) 1 (B) 2 (C) 0 (D) 3

8. The number of non-empty subsets of the set {1, 2, 3, 4, 5} is

(A) 30 (B) 32 (C) 31 (D) 33

9. The number of subsets of a set A is of the form $10n + 4$, where n is a single-digit positive integer. Then n is equal to

(A) 8 (B) 4 (C) 5 (D) 6

10. If A and B are sets such that $n(A \cup B) = 40$, $n(A) = 25$ and $n(B) = 20$, then $n(A \cap B) =$

(A) 1 (B) 2 (C) 5 (D) 4

11. Let $\mathbb{N}$ be the set of all natural numbers and

$$\mathrm{R} = \{(a, b) \in \mathbb{N} \times \mathbb{N} \mid \text{g.c.d. of } \{a, b\} = 1\}$$

Then R is

(A) reflexive on $\mathbb{N}$
(B) symmetric
(C) transitive
(D) an equivalence relation

12. Let $\mathbb{Q}^*$ denote the set of non-zero rational numbers and

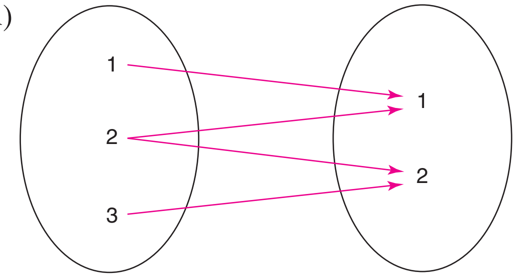
$$\mathrm{R} = \{(a, b) \in \mathbb{Q}^* \times \mathbb{Q}^* \mid ab = 1\}$$

Then R is

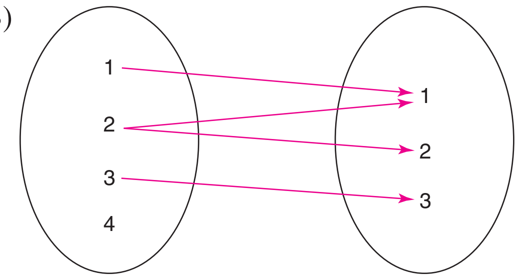
(A) symmetric
(B) reflexive on $\mathbb{Q}^*$
(C) an equivalence relation
(D) transitive

13. Which one of the following diagrams represents a function?

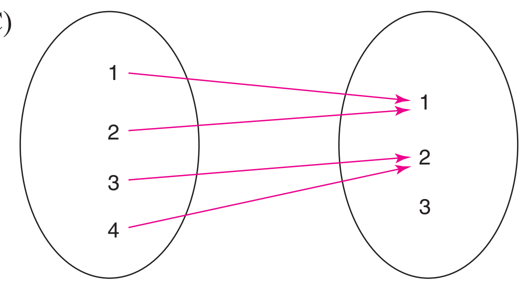
(A)



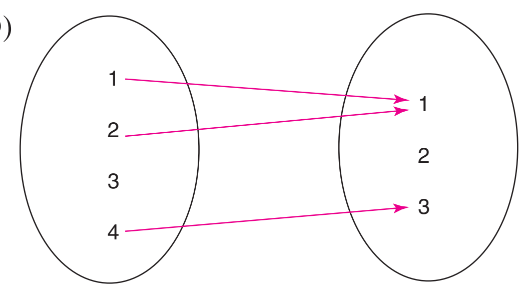
(B)



(C)



(D)

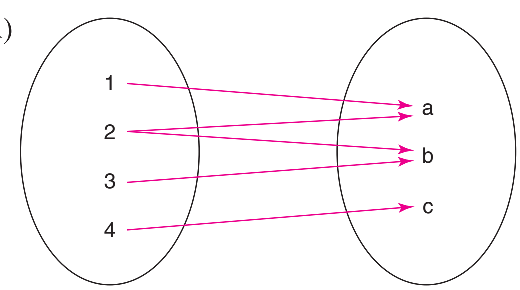


14. Let $A = \{1, 2, 3, 4\}$, $B = \{5, 6, 7\}$ and $c = \{a, b, c, d, e\}$. If $f = \{(1, 5), (2, 5), (3, 6), (4, 7)\}$ and $g = \{(5, a), (6, d), (7, c)\}$ are functions from A to B and from B to C, respectively, then

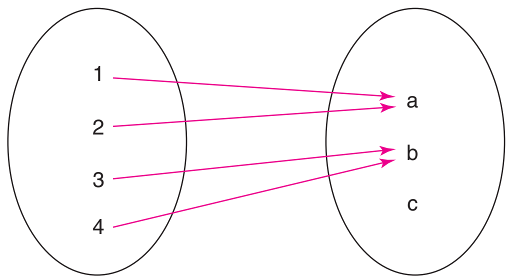
(A) $(g \circ f)(4) = d$ (B) $(g \circ f)(3) = a$
(C) $(g \circ f)(2) = c$ (D) $(g \circ f)(1) = a$

15. Which one of the following diagrams does not represent a function?

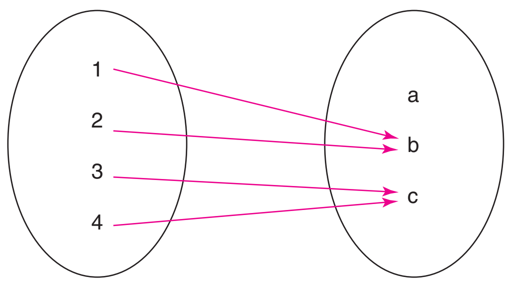
(A)



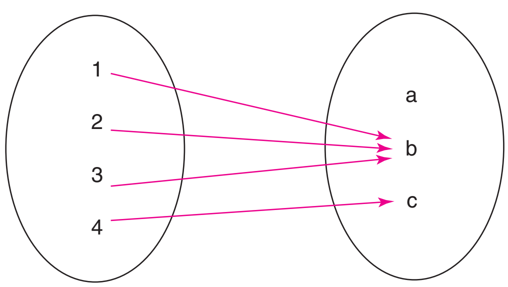
(B)



(C)

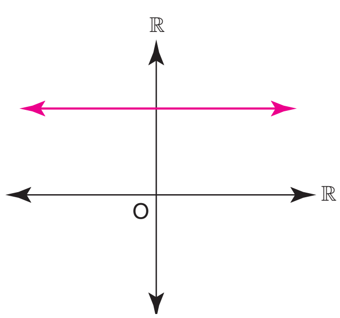


(D)

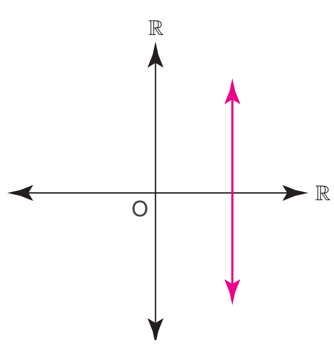


16. Which one of the following graphs does not represent a function from the real number set $\mathbb{R}$ into $\mathbb{R}$?

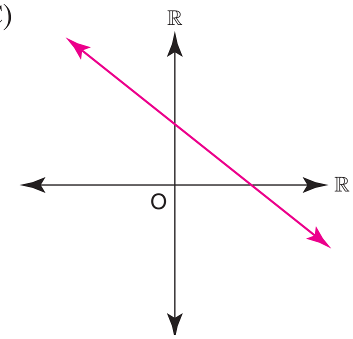
(A)



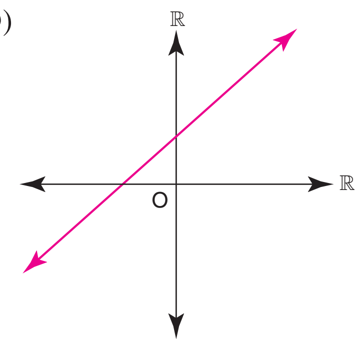
(B)



(C)



(D)



17. Let $f : [1, \infty) \to [2, \infty)$ be the function defined by

$$f(x) = x + \frac{1}{x}$$

If $g : [2, \infty) \to [1, \infty)$, is a function such that $(g \circ f)(x) = x$ for all $x \geq 1$, then $g(t) =$

(A) $t + \dfrac{1}{t}$ (B) $t - \dfrac{1}{t}$

(C) $\dfrac{t + \sqrt{t^2 - 4}}{2}$ (D) $\dfrac{t - \sqrt{t^2 - 4}}{2}$

18. Let f and g be the functions defined from $\mathbb{R}$ to $\mathbb{R}$ by

$$f(x) = \begin{cases} -2 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 2 & \text{if } x > 0 \end{cases} \quad \text{and} \quad g(x) = 1 + \{x\}$$

where $\{x\}$ is the fractional part of x. Then, for all $x \in \mathbb{R}$, $f(g(x))$ is equal to

(A) -2 (B) 0 (C) x (D) 2

19. The number of surjections of $\{1, 2, 3, 4\}$ onto $\{x, y\}$ is

(A) 16 (B) 8 (C) 14 (D) 6

20. If $f(x)$ is a polynomial function satisfying the relation

$$f(x) + f\left(\frac{1}{x}\right) = f(x)f\left(\frac{1}{x}\right)$$

for all $0 \neq x \in \mathbb{R}$ and if $f(2) = 9$, then $f(6)$ is

(A) 216 (B) 217 (C) 126 (D) 127

21. Let a be positive real number and n a positive integer. If $f(x) = (a - x^n)^{1/n}$, then $(f \circ f)(5)$ is

(A) 5 (B) 2 (C) 3 (D) 4

22. For any $0 \le x \le 1$, let $f(x) = \max\{x^2, (1-x)^2, 2x(1-x)\}$. Then which one of the following is correct?

(A) $f(x) = \begin{cases} 2x(1-x), & 0 \le x \le 1/3 \\ (1-x)^2, & 1/3 < x \le 2/3 \\ x^2, & 2/3 < x \le 1 \end{cases}$

(B) $f(x) = \begin{cases} (1-x)^2, & 0 \le x \le 1/3 \\ 2x(1-x), & 1/3 < x \le 2/3 \\ x^2, & 2/3 < x \le 1 \end{cases}$

(C) $f(x) = \begin{cases} x^2, & 0 \le x \le 1/3 \\ 2x(1-x), & 1/3 < x \le 2/3 \\ (1-x)^2, & 2/3 < x \le 1 \end{cases}$

(D) $f(x) = \begin{cases} (1-x)^2, & 0 \le x \le 1/3 \\ x^2, & 1/3 < x \le 2/3 \\ 2x(1-x), & 2/3 < x \le 1 \end{cases}$

23. Let $[x]$ denote the greatest integer $\le x$. Then the number of ordered pair (x, y), where x and y are positive integers less than 30 such that

$$\left[\frac{x}{2}\right] + \left[\frac{2x}{3}\right] + \left[\frac{y}{4}\right] + \left[\frac{4y}{5}\right] = \frac{7x}{6} + \frac{21y}{20}$$

is

(A) 1 (B) 2 (C) 3 (D) 4

24. Let $P : [0, \infty) \to \mathbb{N}$ be defined by

$$P(x) = \begin{cases} 13 & \text{if } 0 \le x < 1 \\ 13 + 15n & \text{if } n \le x < n+1, n \in \mathbb{N} \end{cases}$$

Then P is

(A) an injection
(B) a surjection
(C) a surjection but not an injection
(D) neither an injection nor a surjection

25. If $[x]$ and $\{x\}$ denote the integral part and the fractional part of a real number x, then the number of negative real numbers x for which $2[x] - \{x\} = x + \{x\}$ is

(A) 0 (B) 2 (C) 3 (D) infinite

26. The number of real numbers $x \ge 0$ which are solutions of $[x] + 3\{x\} = x + \{x\}$ is

(A) 1 (B) infinite (C) 0 (D) 2

27. The number of solutions of the equation $2x + \{x + 1\} = 4[x + 1] - 6$ is

(A) 1 (B) 2 (C) 3 (D)

28. Let $[x]$ denote the integral part of x. If a is a positive real number and $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x - [x - a]$, then a period of f is

(A) 1 (B) a (C) $2[a]$ (D) $2a$

29. If $f(x) = k$ (constant) for all $x \in \mathbb{R}$, then the least period of f is

(A) 1/3 (B) 1/2
(C) 2/3 (D) does not exist

30. Let $a > 0$ and $f : \mathbb{R} \to \mathbb{R}$ a function satisfying

$$f(x+a) = 1 + [2 - 3f(x) + 3f(x)^2 - f(x)^3]^{1/3}$$

for all $x \in \mathbb{R}$. Then a period of $f(x)$ is ka where k is a positive integer whose value is

(A) 1 (B) 2 (C) 3 (D) 4

31. Let $a < c < b$ such that $c - a = b - c$. If $f : \mathbb{R} \to \mathbb{R}$ is a function satisfying the relation

$$f(x+a) + f(x+b) = f(x+c) \quad \text{for all } x \in \mathbb{R}$$

then a period of f is

(A) $(b - a)$ (B) $2(b - a)$
(C) $3(b - a)$ (D) $4(b - a)$

32. If $f : \mathbb{R} - \{0, 1\} \to \mathbb{R}$ is a function such that

$$f(x) + f\left(\frac{1}{1-x}\right) = \frac{2(1-2x)}{x(1-x)} \quad \text{for all } x \ne 0, 1$$

then the value of $f(2)$ is

(A) 1 (B) 2 (C) 3 (D) 4

33. If $f : \mathbb{R} \to \mathbb{R}$ is a function satisfying the relations $f(2 + x) = f(2 - x)$ and $f(7 + x) = f(7 - x)$ for all $x \in \mathbb{R}$ then a period of f is

(A) 5 (B) 9 (C) 12 (D) 10

34. If $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = [x] + \left[x + \frac{1}{2}\right] + \left[x + \frac{2}{3}\right] - 3x + 5$$

where $[x]$ is the integral part of x, then a period of f is

(A) 1 (B) 2/3 (C) 1/2 (D) 1/3

35. If a function $f : \mathbb{R} \to \mathbb{R}$ satisfies the relation

$$f(x+1) + f(x-1) = \sqrt{3}\, f(x) \text{ for all } x \in \mathbb{R}$$

then a period of f is

(A) 10 (B) 12 (C) 6 (D) 4

36. The domain of $f(x) = 1/\sqrt{|x| - x}$ is

(A) $[0, 1)$ (B) $(0, \infty)$ (C) $(-\infty, 0)$ (D) $(1, \infty)$

37. The domain of the function defined by $f(x) = \min\{1 + x, \sqrt{1 - x}\}$ is

(A) $(1, \infty)$ (B) $(-\infty, \infty)$ (C) $[1, \infty)$ (D) $(-\infty, 1]$

38. The domain of definition of the function $f(x) = y$ given by the equation $2^x + 2^y = 2$ is

(A) $(-\infty, 1)$ (B) $(-\infty, 1)$ (C) $(-\infty, 0)$ (D) $(0, 1)$

39. The function $f : [1, \infty) \to [2, \infty)$ is defined by $f(x) = x + (1/x)$. Then $f^{-1}(x)$ is equal to

(A) $\dfrac{x}{1 + x^2}$ (B) $\dfrac{1}{2}(x + \sqrt{x^2 - 4})$

(C) $\dfrac{1}{2}(x - \sqrt{x^2 - 4})$ (D) $1 + \sqrt{x^2 - 4}$

40. Let $0 \neq a \in \mathbb{R}$ and $f(x) = ax/(x + 1)$ for all $-1 \neq x \in \mathbb{R}$. If $f(x) = f^{-1}(x)$ for all x, then the value of a is

(A) 1 (B) 2 (C) –1 (D) –2

41. If $f(x) = k$ (constant) for all real numbers x, then the least period of f is

(A) 1/6 (B) 1/4 (C) 1/3 (D) does not exist

42. Let $f(x) = (x + 1)^2$ for all $x \geq -1$. If $g(x)$ is the function whose graph is the reflection of the graph of $f(x)$ with respect to the line $y = x$, then $g(x)$ is equal to

(A) $\sqrt{x + 1}$ (B) $\sqrt{x} - 1$

(C) $\sqrt{x} + 1$ (D) $\dfrac{1}{(x + 1)^2}$

43. Let $f : \mathbb{R} \to A$ is defined by

$$f(x) = \frac{x - 1}{x^2 - 3x + 3}$$

If f is to be a surjection, then A should be

(A) $\left[0, \dfrac{1}{3}\right]$ (B) $\left[-\dfrac{1}{3}, 0\right]$

(C) $\left[-\dfrac{1}{3}, 1\right]$ (D) $\left[-\dfrac{1}{3}, 2\right]$

44. Let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = 1 + 2x$. If $g : \mathbb{R} \to \mathbb{R}$ is an even function such that $g(x) = f(x)$ for all $x \in [0, 1]$, then, for any $x \in \mathbb{R}$, $g(x)$ is equal to

(A) $1 - 2x$ (B) $2x - 1$

(C) $1 - 2|x|$ (D) $1 + 2|x|$

45. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{R}$ the set of real numbers. Let $f : \mathbb{N} \to \mathbb{R}$ be a function satisfying the following:

(i) $f(1) = 1$

(ii) $\sum_{r=1}^{n} r f(r) = n(n + 1) f(n)$ for all $n \geq 2$

Then the integral part of $f(2009)$ is

(A) 0 (B) 1 (C) 2 (D) 3

46. A school awarded 22 medals in cricket, 16 medals in football and 11 medals in kho-kho. If these medals went to a total of 40 students and only two students got medals in all the three games, then how many received medals in exactly two of the three games.

(A) 7 (B) 6 (C) 5 (D) 4

47. Let $P(x)$ be a polynomial of degree 98 such that $P(K) = 1/K$ for $K = 1, 2, 3, \ldots, 99$. Then $(50)P(100)$ equals

(A) 1 (B) 2 (C) 3 (D) 4

48. For any positive integer K, let $f_1(K)$ denote the square of the sum of the digits in K. For example $f_1(12) = (1 + 2)^2 = 9$. For $n \geq 2$, let $f_n(K) = f_1(f_{n-1}(K))$. Then $f_{2010}(11)$ is equal to

(A) 1005 (B) 256 (C) 169 (D) 201

Multiple Correct Choice Type Questions

1. Let $\wp(X)$ denote the power set of a set X. For any two sets A and B, if $\wp(A) = \wp(B)$, then

(A) $A \cup B = A \Delta B$ (B) $A = B$

(C) $A \cap B = \phi$ (D) $A \Delta B = \phi$

2. Let A and B be two sets such that the number of elements in $A \times B$ is 6. If three elements of $A \times B$ are $(x, a), (y, b)$ and (z, b) then

(A) $A = \{x, y, z\}$

(B) $B = \{a, b\}$

(C) $B = \{a, b, x, y\}$

(D) $A \times B = \{(x, a), (x, b), (y, a), (y, b), (z, a), (z, c)\}$

3. Let $A = \{1, 2, 3\}$, $B = \{3, 4\}$ and $C = \{1, 3, 5\}$. Then

(A) $n(A \times (B \cup C)) = 12$ (B) $n(A \times (B \cap C)) = 3$

(C) $n(A \times (B - C)) = 3$ (D) $n(B \times (A - C)) = 2$

4. For any three sets A, B and C,

(A) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(B) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(C) $A \times (B - C) = (A \times B) - (A \times C)$
(D) $A \times (B \Delta C) = (A \times B) \Delta (A \times C)$

5. Let A be the set of all non-degenerate triangles in the Euclidean plane and

$$\mathrm{R} = \{(x, y) \in A \times A \,|\, x \text{ is congruent to } y\}$$

Then R is

(A) reflexive on A
(B) transitive
(C) symmetric
(D) an equivalence relation on A

6. Let n be a positive integer and

$$\mathrm{R} = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \,|\, n \text{ divides } a - b\}$$

Then R is

(A) transitive
(B) reflexive on $\mathbb{Z}$
(C) symmetric
(D) an equivalence relation on $\mathbb{Z}$

7. Let A be the set of all human beings in a particular city at a given time and

$$\mathrm{R} = \{(x, y) \in A \times A \,|\, x \text{ and } y \text{ live in the same locality}\}$$

Then R is

(A) symmetric
(B) reflexive on A
(C) transitive
(D) not an equivalence relation

8. For any integer n, let I_n be the interval $(n, n+1)$. Define

$$\mathrm{R} = \{(x, y) \in \mathbb{R} \,|\, \text{both } x, y \in I_n \text{ for some } n \in \mathbb{Z}\}$$

Then R is

(A) reflexive on $\mathbb{R}$
(B) symmetric
(C) transitive
(D) an equivalence relation

9. Let $\mathbb{R}$ be the set of all real numbers and $S = \{(a, b) \in \mathbb{R} \times \mathbb{R} \,|\, a - b \le 0\}$. Then S is

(A) reflexive on $\mathbb{R}$
(B) transitive
(C) symmetric
(D) an equivalence relation on $\mathbb{R}$

10. For any ordered pairs (a, b) and (c, d) of real numbers, define a relation, denoted by R, as follows:

$$(a, b)\ \mathrm{R}\ (c, d) \text{ if } a < c \quad \text{or} \quad (a = c \text{ and } b \le d)$$

Then R is

(A) transitive
(B) an equivalence relation on $\mathbb{R} \times \mathbb{R}$
(C) symmetric
(D) reflexive on $\mathbb{R} \times \mathbb{R}$

11. Let M_2 be the set of square matrices of order 2 over the real number system and

$$\mathrm{R} = \{(A, B) \in M_2 \times M_2 \,|\, A = P^{\mathrm{T}} BP \text{ for some non-singular } P \in M\}$$

Then R is

(A) symmetric
(B) transitive
(C) reflexive on M_2
(D) not an equivalence relation on M_2

12. Let L be the set of all straight lines in the space and

$$\mathrm{R} = \{(l, m) \in L \times L \,|\, l \text{ and } m \text{ are coplanar}\}$$

Then R is

(A) reflexive on L
(B) not an equivalence relation on L
(C) symmetric
(D) transitive

13. Let $\mathbb{Q}^*$ be the set of all non-zero rational numbers and

$$\mathrm{R} = \{(a, b) \in \mathbb{Q}^* \times \mathbb{Q}^* \,|\, ab = 1\}$$

Then R is

(A) reflexive on $\mathbb{Q}^*$
(B) not reflexive on $\mathbb{Q}^*$
(C) symmetric
(D) not symmetric

14. Let $\mathbb{Q}$ be the set of all rational numbers, $\mathbb{Z}$ the set of all integers and

$$\mathrm{R} = \{(a, b) \in \mathbb{Q} \times \mathbb{Q} \,|\, a - b \in \mathbb{Z}\}$$

Then which of the following are true?

(A) $(x, 2x) \in \mathbb{R}$ for all $x \in \mathbb{Q}$
(B) $\mathbb{Z} \times \mathbb{Z} \subseteq \mathrm{R}$
(C) $(3{\cdot}5, 4{\cdot}5) \in \mathbb{R}$
(D) $(6{\cdot}3, 7{\cdot}2) \in \mathbb{R}$

15. Let $A = \{1, 2, 3, 4\}$, $B = \{5, 6, 7\}$ and $C = \{a, b, c, d, e\}$. Define mappings $f: A \to B$ and $g: B \to C$ by

$$f = \{(1, 5), (2, 6), (3, 5), (4, 7)\} \quad \text{and} \quad g = \{(5, b), (6, c), (7, a)\}$$

Then which of the following are true?

(A) $(g \circ f)(2) = c$
(B) $(g \circ f)(4) = b$
(C) $(g \circ f)(3) = b$
(D) $(g \circ f)(1) = a$

16. Let $A = \{1, 2, 3, 4\}$ and $f : A \to A$ and $g : A \to A$ be mappings defined by $f(1) = 2, f(2) = 3, f(3) = 4, f(4) = 1$; $g(1) = 1, g(2) = 3, g(3) = 4$ and $g(4) = 2$. Then which of the following are true?

(A) f is a bijection (B) g is an injection
(C) g is a surjection (D) f is an injection

17. Let $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ be mappings defined by $f(x) = x^2 + 3x + 2$ and $g(x) = 2x - 3$. Then which of the following are true?

(A) $(f\circ g)(1) = 0$ (B) $(g\circ f)(1) = 9$
(C) $(f\circ g)(3) = 20$ (D) $(g\circ f)(3) = 20$

18. Let $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ be mappings defined by $f(x) = x^2$ and $g(x) = 2x + 1$. If $(f\circ g)(x) = (g\circ f)(x)$, then x is equal to

(A) $-2+\dfrac{1}{\sqrt{2}}$ (B) -2
(C) $-2-\dfrac{1}{\sqrt{2}}$ (D) 0

19. Let $f:[-1,\infty)\to\mathbb{R}$ be defined by $f(x) = (x+1)^2 - 1$. If $(f\circ f)(x) = x$, then the value of x is

(A) 1 (B) 0 (C) -1 (D) -2

20. Let $f:\mathbb{R}\to\mathbb{R}$ be a function such that $f(x+y) = f(x) + f(y)$ for all $x, y\in\mathbb{R}$. Then which of the following hold?

(A) $f(0) = 0$
(B) f is an odd function
(C) $f(n) = nf(1)$ for $n\in\mathbb{Z}$
(D) f is an even function

21. If $f:\mathbb{R}\to\mathbb{R}$ is a function such that $f(0) = 1$ and $f(x + f(y)) = f(x) + y$ for all $x, y\in\mathbb{R}$, then

(A) 1 is a period of f
(B) $f(n) = 1$ for all integers n
(C) $f(n) = n$ for all integers n
(D) $f(-1) = 0$

22. Let $f, g : \mathbb{R}\to\mathbb{R}$ be functions defined by $f(x) = ax + b$ and $g(x) = cx + d$, where a, b, c, d are given real numbers and $c\neq 0$. If $(f\circ g)(x) = g(x)$, then

(A) $a = 1$ (B) $b = 0$
(C) $ab = 1$ (D) $f(4) = 4$

23. If $f:\mathbb{R}\to\mathbb{R}$ is defined by $f(x) = ax + b$, where a and b are given real numbers and $a\neq 0$, then

(A) f is an injection (B) f is a surjection
(C) f is not a bijection (D) f is a bijection

24. If $f : [0,\infty)\to[0,\infty)$ is the function defined by

$$f(x) = \frac{x}{x+1}$$

then

(A) f is an injection but not a surjection
(B) f is a bijection
(C) Each $0\le y<1$ has an inverse image under f
(D) f is a surjection

25. Let f be a real-valued function defined on the interval $[-1, 1]$. If the area of the equilateral triangle with $(0, 0)$ and $(x, f(x))$ as two vertices is $\sqrt{3}/4$, then $f(x)$ is equal to

(A) $\sqrt{1-x^2}$ (B) $\sqrt{1+x^2}$
(C) $-\sqrt{1-x^2}$ (D) $-\sqrt{1+x^2}$

26. Consider the equation $x^2 + y^2 = 1$. Then

(A) y in terms of x is a function with domain $[-1, 1]$
(B) $y = +\sqrt{1-x^2}$ is a function with domain $[-1, 1]$
(C) $y = +\sqrt{1-x^2}$ is an injection of $[0,1]$ into $[0, 1]$
(D) $y = +\sqrt{1-x^2}$ is a bijection of $[0,1]$ onto $[0, 1]$

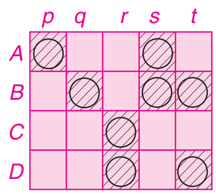
27. Let $f(x) = x^2$ for all $x\in[-2, 2]$. Then f is

(A) an even function
(B) not an even function
(C) a bijection
(D) not an injection

Matrix-Match Type Questions

In each of the following questions, statements are given in two columns, which have to be matched. The statements in Column I are labeled as **(A)**, **(B)**, **(C)** and **(D)**, while those in Column II are labeled as **(p)**, **(q)**, **(r)**, **(s)** and **(t)**. Any given statement in Column I can have correct matching with *one or more* statements in Column II. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following example.

Example: If the correct matches are (A) → (p), (s); (B) → (q), (s), (t); (C) → (r); (D) → (r), (t) that is if the matches are (A) → (p) and (s); (B) → (q), (s) and (t); (C) → (r); and (D) → (r), (t) then the correct darkening of bubbles will look as follows:



1. Let X be the universal set and A and B be subsets of X. Then match the items in Column I with Column II.

Column I	*Column II*
(A) $A - B = A \Leftrightarrow A \cap B =$	(p) ϕ
(B) $(A - B) \cap B =$	(q) $A = B$
(C) $(A - B) \cup (B - A) =$	(r) $A - B$
(D) $A \Delta B = \phi \Leftrightarrow$	(s) $B \subseteq A$
	(t) $(A \cup B) - (A \cap B)$

2. Let A, B and C be sets. Then match the items in Column I with those in Column II.

Column I	*Column II*
(A) $A \Delta B = C \Leftrightarrow$	(p) $A \cap B = \phi$
(B) $A - (B \cup C) =$	(q) $(A \cap B) - (A \cap C)$
(C) $A \cap (B - C) =$	(r) $B \Delta C = \phi$
(D) $A \Delta B = A \cup B \Leftrightarrow$	(s) $A = B \Delta C$
	(t) $(A - B) \cap (A - C)$

3. Let A, B, C and D be sets. Then match the items in Column I with those in Column II.

Column I	*Column II*
(A) $A \times (B \cup C)$	(p) $(A \times B) \cap (A \times C)$
(B) $(A \cup B) \times (C \cup D)$	(q) $(A \times C) \cap (B \times D)$
(C) $(A \cap B) \times (C \cap D)$	(r) $(A \times B) \cup (A \times C)$
(D) $A \times (B \cap C)$	(s) $(A \times C) \cup (B \times D)$
	(t) $(A \times C) \cup (A \times D) \cup (B \times C) \cup (B \times D)$

4. Let $f, g : \mathbb{R} \to \mathbb{R}$ be the functions defined by

$$f(x) = x^2 + 1 \quad \text{and} \quad g(x) = 2[x] - 1$$

where $[x]$ is the largest integer $\le x$. Then match the items given in Column I with those in Column II.

Column I	*Column II*
(A) $(g \circ f)\left(\frac{1}{2}\right)$	(p) 3
(B) $(f \circ g)\left(\frac{3}{2}\right)$	(q) 0
(C) $(f \circ g \circ f)\left(\frac{3}{4}\right)$	(r) –1
(D) $(g \circ f \circ g)\left(\frac{2}{3}\right)$	(s) 1
	(t) 2

5. Let S be the set of all square matrices of order 3 over the real number system. For $A \in S, |A|$ is the determinant value of A. Define $f : S \to \mathbb{R}$ by $f(A) = |A$. for all $A \in S$. Then match the items in Column I with those in Column II.

Column I	*Column II*
(A) If $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$ with $a + b + c = 0$, then $f(A) =$	(p) 1
(B) If $w \ne 1$ is a cube root of unity and $A = \begin{bmatrix} 1 & w & w^2 \\ w & w^2 & 1 \\ w^2 & 1 & w \end{bmatrix}$ then $f(A) =$	(q) –1
(C) If $A = \begin{bmatrix} 0 & 5 & -7 \\ -5 & 0 & 11 \\ 7 & -11 & 0 \end{bmatrix}$ then $f(A) =$	(r) $3abc - a^3 - b^3 - c^3$
(D) If $A \in S$ and $AA^T = I$ (the unit matrix) then $f(A) =$	(s) 2
	(t) 0

6. Match the items in Column I with those in Column II

Column I	*Column II*
(A) If f is a function such that $f(0)=2, f(1)=3$ and $f(x+2)=2f(x)-f(x+1)$, then $f(5)$ is equal to	(p) 4
(B) If $f(x)=\begin{cases} x^2, & \text{for } x\geq 0 \\ x, & \text{for } x<0 \end{cases}$ then $f(\sqrt{13})=$	(q) 3
(C) If $f(x)+2f(1-x)=x^2+2$ for all $x\in\mathbb{R}$, then $f(5)$ is	(r) 12
(D) If $f(x)=\dfrac{4^x}{4^x+2}$ for all $x\in\mathbb{R}$, then $\sum_{k=1}^{6} f\left(\dfrac{k}{7}\right)=$	(s) 11
	(t) 13

7. For any $0<a\in\mathbb{R}$, let

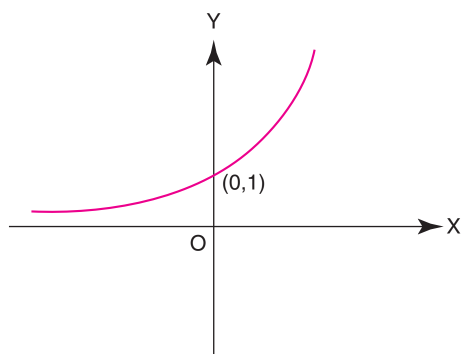
$$f_a(x)=\frac{a^x}{a^x+\sqrt{a}}$$

for all $x\in\mathbb{R}$. Then match the items in Column I with those in Column II.

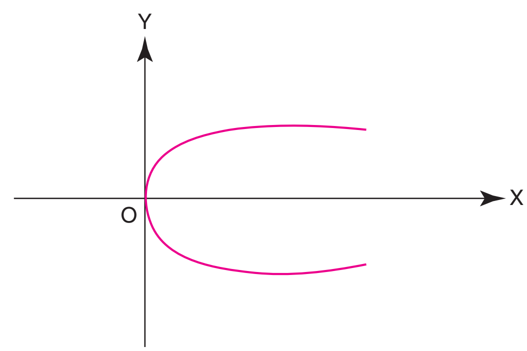
Column I	*Column II*
(A) $\sum_{k=1}^{1997} f_9\left(\dfrac{k}{1998}\right)=$	(p) 998.5
(B) $\sum_{k=1}^{1997} f_4\left(\dfrac{k}{1998}\right)=$	(q) 994
(C) $\sum_{k=1}^{2009} f_{16}\left(\dfrac{k}{2010}\right)=$	(r) 993
(D) $\sum_{k=1}^{2008} f_{25}\left(\dfrac{k}{2009}\right)=$	(s) 1004
	(t) 1004.5

8. Consider the following graphs G_1, G_2, G_3 and G_4 and match the items in Column I with those in Column II.

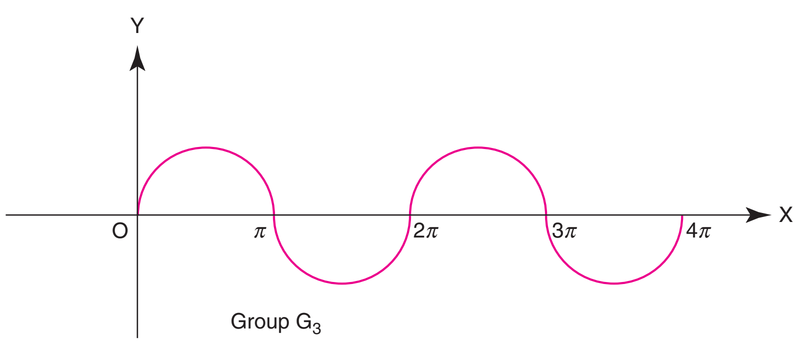
Column I	*Column II*
(A) G_1	(p) Does not represent a function
(B) G_2	(q) Represents an increasing function
(C) G_3	(r) Represents an increasing injection
	(s) Represents a periodic function
(D) G_4	(t) Represents a bijection



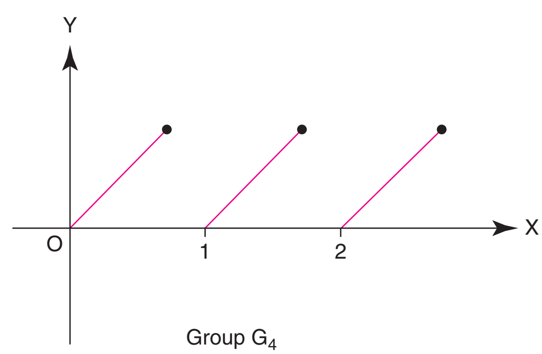
Group G_1



Group G_2



Group G_3



Group G_4

Comprehension-Type Questions

1. Passage: f is a real-valued function satisfying the functional relation:

$$2f(x)+3f\left(\frac{2x+29}{x-2}\right)=100x+80 \text{ for all } x\neq 2$$

Answer the following questions:

(i) $f(0)$ is equal to

(A) 754 (B) –754 (C) 854 (D) –854

(ii) $f\left(\frac{-29}{2}\right)$ is equal to

(A) 659 (B) –596 (C) 596 (D) –659

(iii) $f(-4)$ is equal to

(A) 34 (B) –34 (C) 43 (D) –43

2. Passage: Let $f:\mathbb{R}-\{0\}\to\mathbb{R}$ be a function satisfying

$$f(x)+2f\left(\frac{1}{x}\right)=3x$$

for all $0\neq x\in\mathbb{R}$. Answer the following questions.

(i) $xf(x)=$

(A) $2-x^2$ (B) x^2-2 (C) x^2-1 (D) $1-x^2$

(ii) The number of solution of the equation $f(x)=f(-x)$ is

(A) 1 (B) 2 (C) 3 (D) 0

(iii) The number of solutions of the equation $f(-x)=-f(x)$ is

(A) 1 (B) 2 (C) 0 (D) Infinite

3. Passage: It is given that $f(x)=2-|2x-5|$.

Answer the following questions.

(i) The range of the function f is

(A) $(-\infty,-1)$ (B) $(-\infty,2)$
(C) $(-\infty,2]$ (D) $(2,\infty)$

(ii) The sum of all positive possible values of x such that $f(x)=1$ is

(A) 4 (B) 6 (C) 8 (D) 5

(iii) The number of values of x such that $f(x)=3$ is

(A) 1 (B) 0 (C) 3 (D) 2

4. Passage: Let $f(x)=x+|x|$. Answer the following questions.

(i) The range of $f(x)$ is

(A) $[0,\infty)$ (B) $(-\infty,0]$ (C) $(0,\infty)$ (D) $\mathbb{R}$

(ii) The number of values of x such that $f(x)=x$ is

(A) 0 (B) 1 (C) 2 (D) infinite

(iii) The number of values of x such that $f(x)=0$ is

(A) 0 (B) 1 (C) 2 (D) infinite

5. Passage: Let $f:\mathbb{R}\to\mathbb{R}$ be a function satisfying the functional relation

$$(f(x))^y+(f(y))^x=2f(xy)$$

for all $x, y\in\mathbb{R}$ and it is given that $f(1)=1/2$. Answer the following questions.

(i) $f(x+y)=$

(A) $f(x)+f(y)$ (B) $f(x)f(y)$
(C) $f(x^y y^x)$ (D) $\dfrac{f(x)}{f(y)}$

(ii) $f(xy)=$

(A) $f(x)f(y)$ (B) $f(x)+f(y)$
(C) $(f(x))^y$ (D) $(f(xy))^{xy}$

(iii) $\sum_{k=0}^{\infty} f(k)=$

(A) 5/2 (B) 3/2 (C) 3 (D) 2

Assertion–Reasoning Type Questions

Statement I and statement II are given in each of the questions in this section. Your answers should be as per the following pattern:

(A) If both statements I and II are correct and II is a correct reason for I

(B) If both statements I and II are correct and II is not a correct reason for I

(C) If statement I is correct and statement II is false

(D) If statement I is false and statement II is correct

1. Statement I: In a survey of 1000 adults in a village, it is found that 400 drink coffee, 300 drink tea and 80 drink both coffee and tea. Then the number of adults who drink neither coffee nor tea is 380.

Statement II: If A and B are two finite sets, then

$$n(A\cup B)+n(A\cap B)=n(A)+n(B)$$

2. Statement I: In a class of 40 students, 22 drink Sprite, 10 drink Sprite but not Pepsi. Then the number of students who drink both Sprite and Pepsi is 15.

Statement II: For any two finite sets A and B,

$$n(A)=n(A-B)+n(A\cap B)$$

3. **Statement I:** In a class of 60, each student has to enroll for atleast one of History, Economics and Political Science. 20 students have enrolled for exactly two of these subjects and 8 enrolled for all the three. Then the number of students who have enrolled for exactly one subject is 32.

Statement II: For any three finite sets A, B and C.

$$\begin{aligned}&n(A \cup B \cup C)\\&\quad = n[A-(B \cup C)]\\&\qquad + n[B-(C \cup A)] + n[C-(A \cup B)]\\&\qquad + n[(A \cap B)-C] + n[(B \cap C)-A]\\&\qquad + n[(C \cap A)-B] + n(A \cap B \cap C)\end{aligned}$$

ANSWERS

Single Correct Choice Type Questions

1. (D)
2. (B)
3. (D)
4. (C)
5. (A)
6. (B)
7. (A)
8. (C)
9. (D)
10. (C)
11. (B)
12. (A)
13. (C)
14. (D)
15. (A)
16. (B)
17. (C)
18. (D)
19. (C)
20. (B)
21. (A)
22. (B)
23. (D)
24. (D)
25. (A)
26. (B)
27. (B)
28. (A)
29. (D)
30. (B)
31. (C)
32. (C)
33. (D)
34. (A)
35. (B)
36. (C)
37. (D)
38. (A)
39. (B)
40. (C)
41. (D)
42. (B)
43. (C)
44. (D)
45. (A)
46. (C)
47. (A)
48. (C)

Multiple Correct Choice Type Questions

1. (B), (D)
2. (A), (B), (D)
3. (A), (B), (C), (D)
4. (A), (B), (C), (D)
5. (A), (B), (C), (D)
6. (A), (B), (C), (D)
7. (A), (B), (C)
8. (B), (C)
9. (A), (B)
10. (A), (D)
11. (A), (B), (C)
12. (A), (B), (C)
13. (B), (C)
14. (B), (C)
15. (A), (C)
16. (A), (B), (C), (D)
17. (A), (B), (C)
18. (B), (D)
19. (B), (C)
20. (A), (B), (C)
21. (A), (B)
22. (A), (B), (D)
23. (A), (B), (D)
24. (A), (C)
25. (A), (C)
26. (B), (C), (D)
27. (A), (D)

Matrix-Match Type Questions

1. (A) → (p), (B) → (p), (C) → (t), (D) →(q)
2. (A) → (s), (B) → (t), (C) → (q), (D) → (p)
3. (A) → (r), (B) → (t), (C) → (q), (D) → (p)
4. (A) → (s), (B) → (t), (C) → (t), (D) → (p)
5. (A) → (r), (t), (B) → (t), (C) → (t), (D) → (p), (q)
6. (A) → (t), (B) → (t), (C) → (q), (D) → (q)
7. (A) → (p), (B) → (p), (C) → (t), (D) → (s)
8. (A) →(q), (r), (t), (B) → (p), (C) → (s), (D) → (s)

Comprehension-Type Questions

1. (i) (D); (ii) (C); (iii) (B)
2. (i) (A); (ii) (B); (iii) (D)
3. (i) (C); (ii) (D); (iii) (B)
4. (i) (A); (ii) (B); (iii) (D)
5. (i) (B); (ii) (C); (iii) (D)

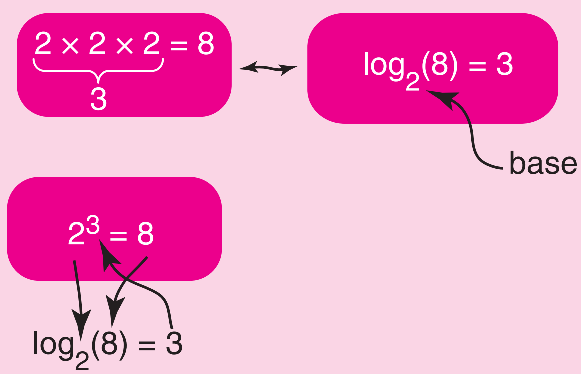
Assertion–Reasoning Type Questions

1. (A)
2. (D)
3. (A)

Exponentials and Logarithms

2

Exponentials and Logarithms



Contents

Exponential Function: For any positive real number a, the function $f(x) = a^x$ for $x \in \mathbb{R}$ is called exponential function with base a.

Logarithmic Function: Let $a > 0$ and $a \neq 1$. Consider the function $g : \mathbb{R}^+ \to \mathbb{R}$ defined by $g(y) = x \Leftrightarrow y = a^x$ for all $y \in \mathbb{R}^+$ and $x \in \mathbb{R}$. The function g is the logarithmic function denoted by $\log_a$.

In this chapter, we will discuss various properties of exponential and logarithmic functions which are often used in solving equations, systems of equations, and inequalities containing these functions.

2.1 | Exponential Function

For any positive real number a, we can define a^x for all real numbers x. This function is called *an exponential function*, whose domain is the set of all real numbers and codomain is also the set of real numbers.

DEFINITION 2.1 Let a be any positive real number. Then the function $f:\mathbb{R}\to\mathbb{R}$, defined by $f(x)=a^x$ for all real numbers x, is called the *exponential function with base a.*

As usual, we simply say that a^x is the exponential function with base a, with the idea that, as x varies over the set of real numbers, we get a function mapping x onto a^x. Note that a must be necessarily positive for a^x to be defined for all $x\in\mathbb{R}$. For example $(-1)^{1/2}$ is not defined in $\mathbb{R}$; for this reason, we take a to be positive.

Examples

(1) 2^x is the exponential function with base 2.
(2) $(0.02)^x$ is the exponential function with base 0.02.
(3) $(986)^x$ is the exponential function with base 986.
(4) The constant map which maps each x onto the real number 1 is also an exponential function with base 1, since $1^x=1$ for all $x\in\mathbb{R}$.

The following theorems are simple verifications and give certain important elementary properties of exponential function.

THEOREM 2.1 Let a be a positive real number. Then the following hold for all real numbers x and y:

1. $a^x a^y = a^{x+y}$
2. $a^x > 0$
3. $a^x / a^y = a^{x-y}$
4. $(a^x)^y = a^{xy}$
5. $a^{-x} = \dfrac{1}{a^x}$
6. $a^0 = 1$
7. $a^1 = a$
8. $1^x = 1$

THEOREM 2.2

1. If $a>1$, then a^x is an increasing function; that is, $x\le y \Rightarrow a^x \le a^y$.
2. If $0<a<1$, then a^x is a decreasing function; that is, $x \le y \Rightarrow a^x \ge a^y$.
3. If $a>0$ and $a\ne 1$, a^x is an injection; that is, $a^x \ne a^y$ for all $x\ne y$.
4. For any $a>0$ and $a\ne 1$, $a^x = 1$ if and only if $x=0$.

Examples

(1) The function $y = 2^x$ is increasing and its graph is given in Figure 2.1.

(2) The function $y = (1/2)^x$ is decreasing and its graph is given in Figure 2.2.

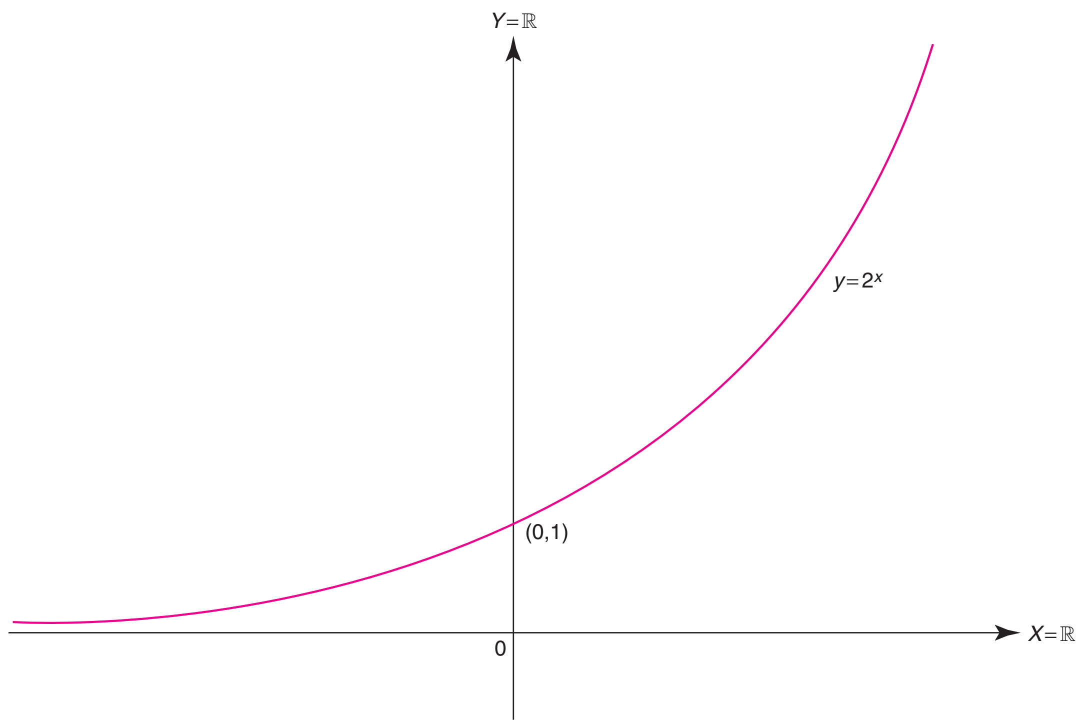


FIGURE 2.1 Graph of the function $y = 2^x$.

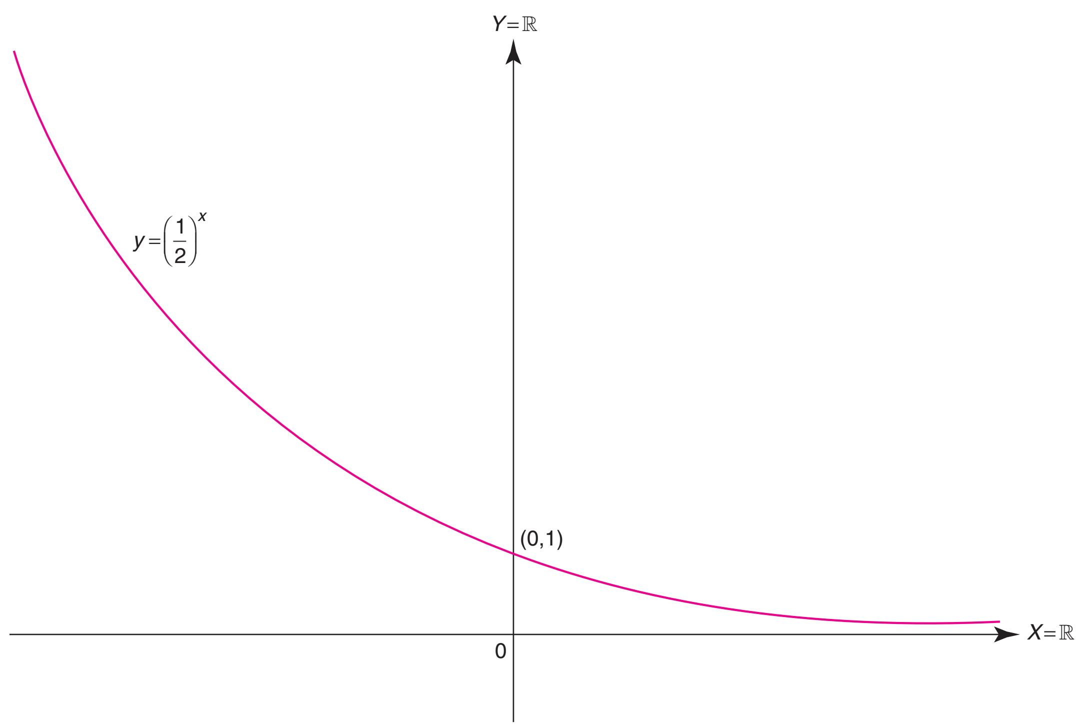


FIGURE 2.2 Graph of the function $y = (1/2)^x$.

2.2 | Logarithmic Function

We have observed in the previous section that, when $a > 0$ and $a \neq 1$, the exponential function with base a is an injection of $\mathbb{R}$ into $\mathbb{R}$ and its range is $\mathbb{R}^+ = (0, +\infty)$. Therefore the function $f : \mathbb{R} \to (0, +\infty)$, defined by $f(x) = a^x$, is a bijection and hence f has an inverse. This implies that there exists a function $g : (0, +\infty) \to \mathbb{R}$ such that

$$f(x) = y \Leftrightarrow x = g(y) \quad \text{or} \quad y = a^x \Leftrightarrow g(y) = x$$

for any $x \in \mathbb{R}$ and $0 < y \in \mathbb{R}$. This function g is called the logarithmic function with base a. Formally, we have the following definition.

DEFINITION 2.2 Let $0 < a \in \mathbb{R}$ and $a \neq 1$. Then the function $g : \mathbb{R}^+ \to \mathbb{R}$, defined such that

$$g(y) = x \Leftrightarrow y = a^x$$

for all $y \in \mathbb{R}^+$ and $x \in \mathbb{R}$, is called the *logarithmic function with base a* and is denoted by $\log_a$.

It is a convention to write $\log_a y$ instead of $\log_a(y)$. Note that $\log_a y$ is defined only when $a > 0$, $a \neq 1$ and $y > 0$ and that

$$\log_a y = x \Leftrightarrow y = a^x$$

for any $y \in \mathbb{R}^+$ and $x \in \mathbb{R}$.

The following are easy verifications and these are the working tools for solving exponential and logarithmic equations and inequalities.

THEOREM 2.3 Let $0 < a \in \mathbb{R}$, $a \neq 1$. Then the following hold for any $y, y_1, y_2 \in \mathbb{R}^+$ and $x, x_1, x_2 \in \mathbb{R}$:

1. $a^{\log_a y} = y$
2. $\log_a a^x = x$
3. $\log_a y = x \Leftrightarrow y = a^x$
4. $\log_a(y_1 y_2) = \log_a y_1 + \log_a y_2$
5. $\log_a(1/y) = -\log_a y$
6. $\log_a(y_1/y_2) = \log_a y_1 - \log_a y_2$
7. $\log_a(y^z) = z \log_a y$ for all $z \in \mathbb{R}$
8. $\log_a a = 1$ and $\log_a 1 = 0$

FORMULA FOR TRANSITION TO A NEW BASE

1. For any $a, b \in \mathbb{R}^+ - \{1\}$ and for any $y \in \mathbb{R}^+$,

$$\log_b y = \frac{\log_a y}{\log_a b} \quad \text{or} \quad \log_a y = \log_a b \log_b y$$

2. $\log_{a^\alpha}(y) = \dfrac{1}{\alpha} \log_a y$ for any $\alpha \neq 0$.

PROOF

1. Let $\log_a y = x$, $\log_b y = t$ and $\log_a b = z$. Then $a^x = y$, $b^t = y$ and $a^z = b$ and hence

$$a^{zt} = (a^z)^t = b^t = y$$

Therefore $\log_a y = zt = \log_a b \cdot \log_b y$.

2. For $\alpha \neq 0$, $(a^\alpha)^{(1/\alpha)\log_a y} = a^{\log_a y} = y$ and therefore

$$\log_{a^\alpha}(y) = \frac{1}{\alpha} \log_a y$$

■

Theorem 2.4
1. If $a > 1$, then $\log_a x$ is an increasing function.
2. If $0 < a < 1$, then $\log_a x$ is a decreasing function.

Proof This is a consequence of Theorem 2.2 and the fact that, where two functions f and g are inverses to each other and one function is increasing (decreasing), then so is the other. ∎

Theorem 2.5 For any $a > 0$ and $a \neq 1$, the function $\log_a x$ is a bijection from the set $\mathbb{R}^+$ onto $\mathbb{R}$.

Proof This follows from the fact that a^x and $\log_a x$ are functions which are inverses to each other. ∎

2.3 | Exponential Equations

It is known from the previous two sections that, for any $a > 0, a \neq 1$, the equation $a^x = b$ possesses a solution for any $b > 0$ and that the solution is unique. In general, the solution is written as $x = \log_a b$. If $a = 1$, then the equation $1^x = b$ has a solution for $b = 1$ only. Any real number x can serve as a solution for $1^x = 1$. Further, for any $a > 0, a \neq 1$, the equation $\log_a x = b$ has a solution for any $b \in \mathbb{R}$ and the solution is unique and is written as $x = a^b$. Since the exponential function a^x and the logarithmic function $\log_a x$ are inverses to each other, the exponential function is often called the *antilogarithmic function*.

We often make use of the two transformations, taking logarithms and taking antilogarithms for solving exponential and logarithmic equations. Taking logarithms to the base $a > 0, a \neq 1$ is a transition from the equality

$$x = y \tag{2.1}$$

to the equality

$$\log_a x = \log_a y \tag{2.2}$$

(x and y here can be numbers or the expressions containing the variables). If Eq. (2.1) is true and both sides are positive, then Eq. (2.2) is also true. Taking antilogarithms to the base $a > 0, a \neq 1$, is similar as transition from Eq. (2.2) to Eq. (2.1). If Eq. (2.2) is true, then Eq. (2.1) is true as well.

Example 2.1

Solve the equation

$$5^{x-1} + 5(0.2)^{x-2} = 26$$

Solution: First observe that

$$0.2 = \frac{2}{10} = 5^{-1}$$

and hence

$$(0.2)^{x-2} = 5^{-(x-2)} = 5^{2-x}$$

Therefore, the given equation reduces to

$$5^{x-1} + 5^{3-x} = 26$$

Put $5^{x-1} = t$. The equation reduces to

$$t + 25t^{-1} = 26$$
$$t^2 - 26t + 25 = 0$$
$$(t-1)(t-25) = 0$$
$$t = 1 \quad \text{or} \quad 25$$

Since $5^{x-1} = t$. We get $5^{x-1} = 1$ or 5^2.

Solving we get $x = 1$ or 3.

Example 2.2

Solve the equation

$$4 \times 9^{x-1} = 3\sqrt{2^{2x+1}}$$

Solution: First note that both sides of the given equation are positive. Taking logarithms with base 2, we get the equation

$$2 + (x-1)\log_2 9 = \log_2 3 + \frac{1}{2}(2x+1)$$

which has the same solutions as the original equation. Since $\log_2 3 = (1/2)\log_2 9$, we get that

$$x(\log_2 9 - 1) = \frac{3}{2}(\log_2 9 - 1)$$

Since $\log_2 9 \neq 1$, it follows that $x = 3/2$.

Example 2.3

Find the solution(s) of the equation

$$5^x \times 2^{(2x-1)/(x+1)} = 50$$

Solution: The given equation is equivalent to

$$5^{x-2} = 2^{[1-(2x-1)/(x+1)]}$$
$$5^{x-2} = 2^{(2-x)/(x+1)}$$

By transforming this into logarithmic equation (taking logarithms with base 5), we get

$$x-2 = \frac{-(x-2)}{x+1}\log_5 2$$

Then

$$x = 2$$

or

$$1 = \frac{-1}{x+1}\log_5 2$$
$$x+1 = \log_5 \frac{1}{2}$$
$$x = \log_5 \frac{1}{2} - \log_5 5 = \log_5 \frac{1}{10}$$

Therefore the given equation has two solutions, namely, 2 and $\log_5(1/10)$.

2.4 | Logarithmic Equations

Transforming a given logarithmic equation into an exponential equation, we can find solutions of the equations. For any $a > 0, a \neq 1,$ the logarithmic equation

$$\log_a x = \log_a y$$

is equivalent to $x = y$, where x and y are positive real numbers or expressions containing the variable. We simply write $\log x$ for $\log_{10} x$ or $\log_e x$. One has to take it depending on the context. Since

$$\log_{10} x = \frac{1}{\log_e 10}(\log_e x)$$

it is easy to pass from logarithms with base 10 to those with base e.

Example 2.4

Find the solution(s) of the equation

$$2\log(2x) = \log(x^2+75) \qquad (2.3)$$

Solution: The equation is meaningful only when $x > 0$. The given equation can be transformed into

$$\log(4x^2) = \log(x^2+75) \qquad (2.4)$$

Note that this is meaningful for all $x \neq 0$, whereas the given equation is valid only when $x > 0$. It follows that

$$4x^2 = x^2 + 75$$
$$x^2 = 25$$

Therefore $x = 5$ or -5. Equation (2.3) has only 5 as a solution, whereas Eq. (2.4) has two solutions, namely 5 and -5.

Example 2.5

Find the solution(s) of the equation $\log_2 x + \log_2(x-1) = 1$.

Solution: The equation is meaningful only when $x > 1$. Transforming the sum of logarithms to the logarithm of a product, we have

$$\log_2[x(x-1)] = 1 = \log_2 2$$

Therefore

$$x(x-1) = 2$$

or

$$x^2 - x - 2 = 0$$

or

$$(x-2)(x+1) = 0$$

Now we have

$$x - 2 = 0 \Rightarrow x = 2$$
$$x + 1 = 0 \Rightarrow x = -1$$

Therefore this has two solutions, namely, 2 and -1. However, for the given equation to be meaningful, we should have $x > 1$. Therefore, 2 is the only solution of the given equation.

Example 2.6

Find the solution(s) of the equation

$$\log_3(3^x - 8) = 2 - x$$

Solution: Taking antilogarithms with the base 3 of the given equation, we get

$$3^x - 8 = 3^{2-x}$$

$$3^{2x} - 8 \times 3^x - 9 = 0$$

$$(3^x - 9)(3^x + 1) = 0$$

This gives

$$3^x = 9 \quad \text{or} \quad 3^x = -1$$

The equation $3^x = -1$ has no solution and the equation $3^x = 9$ has unique solution, namely 2. Thus, 2 is the only solution of the given equation.

Example 2.7

Find the solution(s) of the equation

$$x^{\log 2x} = 5$$

Solution: By taking logarithms with base 10, we get an equation

$$\log 2x \times \log x = \log 5$$
$$\log x(\log 2 + \log x) = \log 5$$

This gives

$$\log^2 x + \log 2 \times \log x - \log 5 = 0$$

This is equivalent to the original equation and is meaningful only when $x > 0$. Also, the above equation is a quadratic equation with respect to $\log x$. Therefore

$$\log x = \frac{1}{2}(-\log 2 \pm \sqrt{\log^2 2 + 4\log 5})$$

Since $\log 5 = 1 - \log 2$, we find that

$$\log^2 2 + 4\log 5 = (\log 2 - 2)^2$$

and therefore,

$$\log x = \frac{1}{2}[-\log 2 \pm (\log 2 - 2)]$$

Therefore, $\log x = -1$ or $1 - \log 2 (= \log 5)$. Thus 1/10 and 5 are solutions of the given equation.

2.5 | Systems of Exponential and Logarithmic Equations

In this section we consider finding solutions simultaneously satisfying a given system of exponential and logarithmic equations.

Example 2.8

Solve the simultaneous equations

$$\log_x y + \log_y x = 2.5$$

$$xy = 27$$

Solution: We have to find a common solution to both the above equations. Note that $0 < x \neq 1$ and $0 < y \neq 1$. By taking $\log_x y = t$ in the first equation we get that

$$t + \frac{1}{t} = \frac{5}{2}$$

$$2t^2 + 2 = 5t$$

$$(t-2)(2t-1) = 0$$

Therefore $t = 2$ or 1/2. Now

$$t = 2 \Rightarrow \log_x y = 2 \Rightarrow y = x^2$$

$$t = 1/2 \Rightarrow \log_y x = 2 \Rightarrow x = y^2$$

From the equation $xy = 27$, it follows that when $y = x^2$ we get

$$x^3 = 27 \Rightarrow x = 3$$

Substituting this value of x we get $y = (3)^2 = 9$. Therefore (3, 9) is one solution. Similarly (9, 3) is another solution. Therefore, (3,9) and (9,3) are common solutions for the given two equations.

Example 2.9

Solve the simultaneous equations

$$x^{\log_3 y} = 27y$$

$$y^{\log_3 x} = 81x$$

Solution: Taking logarithms with base 3, these equations can be transformed into

$$\log_3 y \log_3 x = 3 + \log_3 y \tag{2.5}$$

$$\log_3 x \log_3 y = 4 + \log_3 x \tag{2.6}$$

Comparing Eqs. (2.5) and (2.6) we get

$$3 + \log_3 y = 4 + \log_3 x$$

$$\log_3 y = 1 + \log_3 x$$

From Eq. (2.6), we get

$$(1 + \log_3 x)\log_3 x = 4 + \log_3 x$$

$$(\log_3 x)^2 = 4$$

$$\log_3 x = \pm 2$$

Now two situations occur:

(1) $\log_3 x = 2 \Rightarrow x = 9,\ \log_3 y = 3,\ y = 27$

(2) $\log_3 x = -2 \Rightarrow x = 3^{-2} = \dfrac{1}{9},\ \log_3 y = -1,\ y = \dfrac{1}{3}$

Thus, (9, 27) and (1/9, 1/3) are the solutions of the given system of equations.

Example 2.10

Solve the system of equations

$$\log_8(xy) = 3(\log_8 x \cdot \log_8 y)$$

$$4\log_8\left(\frac{x}{y}\right) = \frac{\log_8 x}{\log_8 y}$$

Solution: This system of equations can be transformed to

$$\log_8 x + \log_8 y = 3\log_8 x \times \log_8 y$$

$$4(\log_8 x - \log_8 y) = \frac{\log_8 x}{\log_8 y}$$

By putting $s = \log_8 x$ and $t = \log_8 y$, we get

$$s + t = 3st$$

$$4(s - t) = s/t$$

By solving these two equations, we get that $t = 1/2$ or $1/6$. Therefore, we have

$$t = 1/2 \Rightarrow \log_8 y = 1/2 \Rightarrow y = 2\sqrt{2}$$

$$s = 1 \Rightarrow \log_8 x = 1 \Rightarrow x = 8$$

$$t = 1/6 \Rightarrow \log_8 y = 1/6 \Rightarrow y = 8^{1/6} = \sqrt{2}$$

$$s = -1/3 \Rightarrow \log_8 x = -1/3 \Rightarrow x = 8^{-1/3} = 2^{-1} = 1/2$$

Therefore, $(8, 2\sqrt{2})$ and $(1/2, \sqrt{2})$ are the solutions of the given system of equations.

2.6 | Exponential and Logarithmic Inequalities

Let us recall that, if $a > 1$, the function a^x increases and that, $0 < a < 1$, the function a^x decreases. Also, the function $\log_a x$ increases if $a > 1$, and decreases if $0 < a < 1$. These properties can be used to solve some exponential and logarithmic inequalities.

Example 2.11

Solve the inequality

$$\frac{1}{2} < \log_9 \frac{2x}{x+1} \tag{2.7}$$

Solution: This can be written as

$$\log_9 3 < \log_9 \frac{2x}{x+1} \tag{2.8}$$

These expressions are meaningful only when $2x/(x+1) > 0$. Also, the function $\log_9 x$ is increasing and hence the inequality (2.7) is equivalent to the inequality

$$3 < \frac{2x}{x+1} \tag{2.9}$$

Now x cannot be positive [for, if $x > 0$, then $x + 1 > 0$ and hence, by Eq. (2.9), $3(x+1) < 2x$ and hence $x + 3 < 0$, a contradiction to the fact that $2x/(x+1)>0$]. Therefore $x<0$. Then $x+1<0$ and hence $x<-1$. Again by Eq. (2.9)

$$3(x+1)>2x$$

and therefore, $x>-3$. Thus, the interval $(-3,-1)$ is the set of solutions of the given inequality.

Example 2.12

Solve the inequality

$$(x^2-2.5x+1)^{x+1}\le 1 \tag{2.10}$$

Solution: This is equivalent to the collection of two systems of inequalities

$$\left.\begin{aligned} 0<x^2-2.5x+1\le 1\\ x+1\ge 0\end{aligned}\right\} \tag{2.11}$$

$$\left.\begin{aligned} x^2-2.5x+1\ge 1\\ x+1\le 0\end{aligned}\right\} \tag{2.12}$$

The system Eq. (2.11) of inequalities has solutions $0\le x<0.5$ and $2<x\le 2.5$. The system Eq. (2.12) has solutions $x\le -1$. Therefore, the set of solutions of the inequality Eq. (2.10) is

$$[-\infty,-1]\cup\left[\left(0,\frac{1}{2}\right)\cup\left(2,\frac{5}{2}\right)\right]$$

Example 2.13

Solve the inequality

$$2^x<3^{1/x} \tag{2.13}$$

Solution: First note that both sides of this inequality are positive for all $x\ne 0$ and therefore, their logarithms are defined with respect to any base. In particular, since the function $\log_2 x$ is increasing, the inequality (2.13) is equivalent to the inequality

$$\log_2(2^x)<\log_2 3^{1/x} \tag{2.14}$$

This implies

$$x<\frac{1}{x}\log_2 3$$

$$\frac{x^2-\log_2 3}{x}<0 \tag{2.15}$$

If x is a solution of Eq. (2.15) and $x>0$, then $x^2-\log_2 3<0$ and hence $0<x<\sqrt{\log_2 3}$. If $x<0$ and is a solution of Eq. (2.15), then $x^2-\log_2 3>0$ and hence $x<-\sqrt{\log_2 3}$. Therefore, the set of solutions of the inequality (2.13) is

$$(-\infty,-\sqrt{\log_2 3})\cup(0,\sqrt{\log_2 3})$$

WORKED-OUT PROBLEMS

Single Correct Choice Type Questions

1. $(0.16)^{\log_{2.5}[(1/3)+(1/3^2)+\cdots+\infty]}=$

(A) $2\sqrt{2}$ (B) 2 (C) $4\sqrt{2}$ (D) 4

Solution: We know that, for any $-1<r<1$,

$$a+ar+ar^2+\cdots+\infty=\frac{a}{1-r}$$

Therefore

$$\frac{1}{3}+\frac{1}{3^2}+\cdots+\infty=\frac{1/3}{1-1/3}=\frac{1}{2}$$

Finally we have

$$(0.16)^{\log_{2.5}[(1/3)+(1/3^2)+\cdots+\infty]}=\left(\frac{2}{5}\right)^{2\log_{2.5}(1/2)}$$

$$=\left(\frac{2}{5}\right)^{-2\log_{2.5}(2)}=\left(\frac{5}{2}\right)^{\log_{5/2}(4)}=4$$

Answer: (D)

2. If $\log_{12}27=a$, then $\log_6 16=$

(A) $4\left(\dfrac{3+a}{3-a}\right)$ (B) $4\left(\dfrac{3-a}{3+a}\right)$

(C) $2\left(\dfrac{3-a}{3+a}\right)$ (D) $2\left(\dfrac{3+a}{3-a}\right)$

Solution:

$$\log_6 16=4\log_6 2=\frac{4}{\log_2 6}=\frac{4}{1+\log_2 3} \tag{2.16}$$

Now,

$$a = \log_{12} 27 = 3\log_{12} 3 = \frac{3}{\log_3 12} = \frac{3}{1+2\log_3 2}$$

Therefore

$$a(1+2\log_3 2) = 3$$
$$2\log_3 2 = \frac{3}{a} - 1 = \frac{3-a}{a}$$
$$\frac{2}{\log_2 3} = \frac{3-a}{a}$$
$$\log_2 3 = \frac{2a}{3-a}$$

Substituting in Eq. (2.16), we get that

$$\log_6 16 = \frac{4}{1+[2a/(3-a)]} = 4\left(\frac{3-a}{3+a}\right)$$

Answer: (B)

3. If $\log(a+c)+\log(a-2b+c) = 2\log(a-c)$, then

(A) $2b = a+c$ (B) $a^2+c^2 = 2b^2$

(C) $b^2 = ac$ (D) $\frac{2ac}{a+c} = b$

Solution:

$$\log[(a+c)(a-2b+c)] = \log(a-c)^2$$
$$(a+c)(a+c-2b) = (a-c)^2$$
$$(a+c)^2 - 2b(a+c) = (a-c)^2$$
$$b = \frac{2ac}{a+c}$$

Answer: (D)

4. The solution of the equation $\log_7 \log_5(\sqrt{x+5}+\sqrt{x}) = 0$ is

(A) 2 (B) 3 (C) 4 (D) 1

Solution:

$$\log_7 \log_5(\sqrt{x+5}+\sqrt{x}) = 0$$
$$\log_5(\sqrt{x+5}+\sqrt{x}) = 7^0 = 1$$
$$\sqrt{x+5}+\sqrt{x} = 5^1 = 5$$
$$x+5 = 25 - 10\sqrt{x} + x$$
$$10\sqrt{x} = 20$$
$$\sqrt{x} = 2$$
$$x = 4$$

Therefore, $x = 4$ satisfies the given equation.

Answer: (C)

5. If $\log_3 2 + \log_3(2^x - 7/2) = 2\log_3(2^x - 5)$, then the value of x is

(A) 3 (B) 2 (C) 1 (D) 4

Solution: First note that $2^x > 7/2$ and $2^x > 5$. Therefore $x > 2$. From the hypothesis, we have

$$2(2^x - 7/2) = (2^x - 5)^2$$

Therefore

$$2\times 2^x - 7 = 2^{2x} - 10\times 2^x + 25$$

Put $a = 2^x$. Then $2a - 7 = a^2 - 10a + 25$. Therefore

$$a^2 - 12a + 32 = 0$$
$$(a-8)(a-4) = 0$$

Now $a = 4$ or 8. That is

$$2^x = 4 \quad \text{or} \quad 8$$
$$x = 2 \quad \text{or} \quad 3$$

But $x > 2$. Therefore $x = 3$.

Answer: (A)

6. If $\log_{(2x+3)}(6x^2+23x+21) = 4 - \log_{(3x+7)}(4x^2+12x+9)$, then the value of $-4x$ is

(A) 0 (B) 1 (C) 2 (D) $-1/4$

Solution: First note that $2x+3 > 0$ and $2x+3 \neq 1$, that is, $x > -3/2$ and $x \neq -1$. Also, $3x+7 > 0$ and $3x+7 \neq 1$, that is, $x > -7/3$ and $x \neq -2$. Suppose $x > -3/2, x \neq -1$. Then the given equation can be written as

$$\frac{\log[(2x+3)(3x+7)]}{\log(2x+3)} = 4 - \frac{2\log(2x+3)}{\log(3x+7)}$$
$$1 + \frac{\log(3x+7)}{\log(2x+3)} = 4 - \frac{2\log(2x+3)}{\log(3x+7)}$$

Put

$$\frac{\log(3x+7)}{\log(2x+3)} = y$$

Then

$$1 + y = 4 - \frac{2}{y}$$

Therefore

$$y = 3 - \frac{2}{y}$$
$$y^2 - 3y + 2 = 0$$
$$(y-1)(y-2) = 0$$

This gives $y = 1$ or 2.

Case 1: Suppose that $y = 1$. Then

$$\log(3x+7) = \log(2x+3)$$
$$3x+7 = 2x+3$$
$$x = -4$$

This is rejected because $x > -3/2$.

Case 2: Suppose that $y = 2$. Then

$$\log(3x+7) = 2\log(2x+3) = \log(2x+3)^2$$

Therefore

$$3x+7 = 4x^2+12x+9$$
$$4x^2+9x+2 = 0$$
$$(4x+1)(x+2) = 0$$
$$x = -1/4 \quad \text{or} \quad -2$$

Here $x = -1/4$ (since $x > -3/2$). So

$$-4x = 1$$

Answer: (B)

7. The number of the solutions of the equation $\log(x^2 - 6x + 7) = \log(x - 3)$ is

(A) 6 (B) 5 (C) 7 (D) 4

Solution: We have, for the term in parentheses on the RHS of the given equation,

$$x^2 - 6x + 7 = (x-3)^2 - 2 > 0 \Leftrightarrow |x-3| > \sqrt{2}$$

Also, $\log(x - 3)$ is defined for all $x > 3$. From the given equation, $x^2 - 6x + 7 = x - 3$, $x > 3$. Therefore

$$x^2 - 7x + 10 = 0, \ x > 3$$
$$(x-2)(x-5) = 0, \ x > 3$$
$$x = 5$$

Answer: (B)

8. The number of solutions of the equation

$$|x-3|^{(x^2-8x+15)/(x-2)} = 1$$

is

(A) 1 (B) 2 (C) 0 (D) 4

Solution:

$$|x-3|^{(x^2-8x+15)/(x-2)} = 1$$

$$\Rightarrow x \neq 3, x \neq 2 \quad \text{and} \quad \frac{x^2-8x+15}{x-2}\log|x-3| = 0$$

$$\Rightarrow x \neq 2, x \neq 3 \quad \text{and} \quad |x-3| = 1 \quad \text{or} \quad x^2-8x+15 = 2$$

$$\Rightarrow x \neq 2, x \neq 3 \quad \text{and} \quad [x = 2 \text{ or } 4 \quad \text{or} \quad (x-3)(x-5) = 0]$$

$$\Rightarrow x = 4 \quad \text{or} \quad x = 5$$

Therefore, the number of the solutions of the given equation is 2.

Answer: (B)

Alternative Method

$$|x-3|^{(x^2-8x+15)/(x-2)} = 1$$

$$\Rightarrow x \neq 2, x \neq 3 \quad \text{and} \quad |x-3| = 1 \quad \text{or} \quad x^2-8x+15 = 0$$

$$\Rightarrow x \neq 2, x \neq 3 \quad \text{and} \quad (x = 4 \text{ or } 2 \quad \text{or} \quad x = 3 \text{ or } 5)$$

$$\Rightarrow x = 4 \quad \text{or} \quad x = 5$$

9. If (x_1, y_1) and (x_2, y_2) are solutions of the system of simultaneous equations

$$\log_8(xy) = 3\log_8 x \cdot \log_8 y$$
$$4\log_8\left(\frac{x}{y}\right) = \frac{\log_8 x}{\log_8 y}$$

then $x_1x_2 + y_1y_2$ equals to

(A) 4 (B) 6 (C) 2 (D) 8

Solution: Clearly $x > 0$, $y > 0$ and $y \neq 1$, so as to make the equations meaningful. The given equations are equivalent to

$$\log_8 x + \log_8 y = 3\log_8 x \log_8 y$$
$$4(\log_8 x - \log_8 y) = \log_8 x / \log_8 y$$

Put $\log_8 x = m$ and $\log_8 y = n \neq 0$. Then the equivalent system is

$$\left.\begin{aligned} m+n &= 3mn \\ 4(m-n) &= m/n \end{aligned}\right\} \tag{2.17}$$

Multiplying both the equations of the equivalent system we get

$$4(m^2 - n^2) = 3m^2$$

Therefore

$$m^2 = 4n^2 \quad \text{or} \quad m = \pm 2n$$

Putting $m = 2n$ in Eq. (2.17), we get that

$$3n = 6n^2 \quad \text{or} \quad n = \frac{1}{2} \quad (\text{since } n \neq 0) \quad \text{and} \quad m = 1$$

Now

$$m = 1 \Rightarrow \log_8 x = 1 \Rightarrow x = 8$$
$$n = \frac{1}{2} \Rightarrow \log_8 y = \frac{1}{2} \Rightarrow y = 2\sqrt{2}$$

Therefore

$$x_1 = 8,\ y_1 = 2\sqrt{2}$$

Again by taking $m = -2n$, we get that

$$n = 6n^2 \quad \text{or} \quad n = 1/6 \quad \text{and} \quad m = -1/3$$

$$-1/3 = m = \log_8 x \Rightarrow x = 8^{-1/3} = (2^3)^{-1/3} = \frac{1}{2}$$

$$1/6 = n = \log_8 y \Rightarrow y = 8^{1/6} = (2^3)^{1/6} = \sqrt{2}$$

For $x_2 = 1/2$ and $x_2 = \sqrt{2}$. Therefore

$$x_1x_2 + y_1y_2 = 8 \times \frac{1}{2} + 2\sqrt{2} \times \sqrt{2} = 4 + 4 = 8$$

Answer: (D)

10. If

$$\log_{10}\left(\frac{1}{2^x + x - 1}\right) = x(\log_{10} 5 - 1)$$

then x is equal to

(A) 1 (B) 2 (C) 3 (D) 0

Solution: Given equation is equivalent to

$$\log_{10}\left(\frac{1}{2^x + x - 1}\right) = x(\log_{10} 5 - \log_{10} 10)$$

$$= x\log_{10}\left(\frac{5}{10}\right)$$

$$= \log_{10}\frac{1}{2^x}$$

Therefore

$$\frac{1}{2^x + x - 1} = \frac{1}{2^x}$$

This gives $x - 1 = 0$ or $x = 1$ which satisfies the equation.

Answer: (A)

11. The set of all values of x satisfying the inequality $\log_2 \sqrt{x} - 2(\log_{1/4} x)^2 + 1 > 0$ is the interval

(A) $(0, 1)$ (B) $(4, \infty)$

(C) $\left(\frac{1}{2}, 4\right)$ (D) $\left(\frac{1}{4}, \frac{1}{2}\right)$

Solution: The given inequality is meaningful for $x > 0$ and is equivalent to

$$\frac{1}{2}\log_2 x - 2\left[-\frac{1}{2}\log_2 x\right]^2 + 1 > 0$$

$$\frac{1}{2}\log_2 x - \frac{1}{2}(\log_2 x)^2 + 1 > 0$$

$$(\log_2 x)^2 - \log_2 x - 2 < 0$$

$$(\log_2 x - 2)(\log_2 x + 1) < 0$$

$$-1 < \log_2 x < 2$$

$$\frac{1}{2} < x < 2^2$$

Answer: (C)

12. If $\log_3 x(x+2) = 1$, then x is equal to

(A) 3 or –1 (B) 1 or –4

(C) –3 or –1 (D) 1 or –3

Solution: $\log_3 x(x+2) = 1$ is meaningful if $x(x+2) \neq 0$ and $x(x+2) > 0$. Also, this equation implies

$$x(x+2) = 3$$

$$x^2 + 2x - 3 = 0$$

$$(x+3)(x-1) = 0$$

$$x = -3 \quad \text{or} \quad 1$$

Answer: (D)

13. A solution of the equation

$$\log(2x) = \frac{1}{4}\log(x-15)^4$$

is

(A) 4 (B) 5 (C) 2 (D) –15

Solution: The given equation is meaningful if $x > 0$ and $x \neq 15$. If $x > 15$, then the given equation is equivalent to

$$\log 2x = \log(x - 15)$$

and hence $2x = x - 15$ and therefore $x = -15$, which is false (since $x > 0$). Therefore $0 < x < 15$. Then, from the given equation

$$\log(2x) = \frac{1}{4}\log(15 - x)^4 = \log(15 - x)$$

and hence $2x = 15 - x$, so that $x = 5$.

Answer: (B)

Multiple Correct Choice Type Questions

1. Which of the following are true?

(A) $\dfrac{1}{\log_2 n}+\dfrac{1}{\log_3 n}+\cdots+\dfrac{1}{\log_{43} n}=\log_n(43)!$

(B) $\dfrac{1}{\log_{xy}(xyz)}+\dfrac{1}{\log_{yz}(xyz)}+\dfrac{1}{\log_{zx}(xyz)}=2$

(C) If $n=(2009)!$, then

$$\frac{1}{\log_2 n}+\frac{1}{\log_3 n}+\cdots+\frac{1}{\log_{2009} n}=1$$

(D) $\dfrac{\log_a n}{\log_{ab} n}=\log_a b-1$

Solution:

(A) $\displaystyle\sum_{k=2}^{43}\frac{1}{\log_k n}=\sum_{k=2}^{43}\log_n k=\sum_{k=2}^{43}\log_n k=\log_n(43)!$

(B) $\dfrac{1}{\log_{xy}(xyz)}+\dfrac{1}{\log_{yz}(xyz)}+\dfrac{1}{\log_{zx}(xyz)}$

$$=\log_{xyz}(xy)+\log_{xyz}(yz)+\log_{xyz}(zx)$$

$$=\log_{xyz}(xy\cdot yz\cdot zx)=2$$

(C) By (A), the given sum is $\log_n(2009)!=\log_n n=1$.

(D) $\dfrac{\log_a n}{\log_{ab} n}=\dfrac{\log_n ab}{\log_n a}=\dfrac{\log_n a+\log_n b}{\log_n a}$

$$=1+\frac{\log_n b}{\log_n a}=1+\log_a b$$

Answers: (A), (B), (C)

2. Which of the following are correct?

(A) $\log_b a\cdot\log_c b\cdot\log_d c\cdot\log_a d=1$

(B) $2^2\cdot 2^{-\log_2 5}=\dfrac{4}{5}$

(C) $3^{4\log_3 5}+(27)^{\log_9 36}=741$

(D) $8^{\log_2\sqrt[3]{121}+(1/3)}=242$

Solution:

(A) $\log_b a\cdot\log_c b\cdot\log_d c\cdot\log_a d=\log_c a\cdot\log_d c\cdot\log_a d$

$$=\log_d a\cdot\log_a d=1$$

(B) $2^2\cdot 2^{-\log_2 5}=4\cdot 2^{\log_2(5^{-1})}=\dfrac{4}{5}$

(C) $3^{4\log_3 5}+(27)^{\log_9 36}=5^4+(3^3)^{(1/2)\log_3(36)}$

$$=625+(36)^{3/2}=625+216=841$$

(D) $8^{\log_2\sqrt[3]{121}+(1/3)}=2^{3[\log_2(121)^{1/3}+1/3]}$

$$=2^{\log_2 121+1}=121\times 2=242$$

Answers: (A), (B), (D)

3. If x, y, z simultaneously satisfy the equations

$$\log_2 x+\log_4 y+\log_4 z=2$$

$$\log_3 y+\log_9 z+\log_9 x=2$$

$$\log_4 z+\log_{16} x+\log_{16} y=2$$

then which of the following is (are) true?

(A) $xy=9/4$ (B) $yz=36$

(C) $zx=64/9$ (D) $x+y+z=xyz$

Solution: First observe that

$$\log_2 x=\log_4(x^2)$$

$$\log_3 y=\log_9(y^2)$$

$$\log_4 z=\log_{16}(z^2)$$

From $\log_2 x+\log_4 y+\log_4 z=2$, we get that

$$\log_4 x^2yz=2$$

and hence

$$x^2yz=4^2=16 \tag{2.18}$$

Similarly,

$$y^2zx=9^2=81 \tag{2.19}$$

$$z^2xy=16^2=256 \tag{2.20}$$

From Eqs. (2.18)–(2.20), we get that $x^4y^4z^4=16\times 81\times 256$. Therefore

$$xyz=2\times 3\times 4=24$$

Since $x^2yz=16$ and $xyz=24$, we get that

$$x=\frac{16}{24}=\frac{2}{3}$$

Similarly, $y=27/8$ and $z=32/3$. Therefore, $xy=9/4$, $yz=36$ and $zx=64/9$.

Answers: (A), (B), (C)

Matrix-Match Type Questions

1. Match the items in Column I with those in Column II.

Column I	Column II
(A) The number of real solutions of the equation $\log_4(x-1) = \log_2(x-3)$ is	(p) 3
(B) The number of solutions of the equation $x^{[3/4(\log_2 x)^2+\log_2 x-5/4]} = x\sqrt{2}$ is	(q) 0
	(r) 2
(C) The smallest positive integer x such that $\log_{0.3}(x-1) < \log_{0.09}(x-1)$ is	(s) 4
(D) The minimum value of $\log_a x + \log_x a$, where $1 < a < x$ is	(t) 1

Solution:

(A)

$$\log_4(x-1) = \log_2(x-3)$$

$$\Rightarrow \frac{1}{2}\log_2(x-1) = \log_2(x-3)$$

$$\Rightarrow x-1 = (x-3)^2$$

$$\Rightarrow x^2 - 7x + 10 = 0$$

$$\Rightarrow x = 2 \quad \text{or} \quad 5$$

But the given equation is defined for $x > 3$. Therefore $x = 5$.

Answer: (A) → (t)

(B)

$$x^{[3/4(\log_2 x)^2+\log_2 x-5/4]} = \sqrt{2}$$

Taking logarithms on both sides to the base 2,

$$\left[\frac{3}{4}(\log_2 x)^2 + \log_2 x - \frac{5}{4}\right]\log_2 x = \frac{1}{2}$$

Put $\log_2 x = t$. Then

$$\left[\frac{3}{4}t^2 + t - \frac{5}{4}\right]t = \frac{1}{2}$$

Therefore

$$3t^3 + 4t^2 - 5t - 2 = 0$$

Clearly, $t = 1$ is root of this equation. Now,

$$(t-1)(3t^2 + 7t + 2) = 0$$

$$t = 1, -2, -1/3$$

Therefore

$$x = 2, 2^{-2}, 2^{-1/3}$$

Answer: (B) → (p)

(C)

$$\log_{0.3}(x-1) < \log_{0.09}(x-1) = \log_{(0.3)^2}(x-1) = \frac{1}{2}\log_{0.3}(x-1)$$

Therefore

$$\frac{2\log_{10}(x-1)}{\log_{10}(0.3)} < \frac{\log_{10}(x-1)}{\log_{10}(0.3)}$$

$$2\log_{10}(x-1) > \log_{10}(x-1)$$

$$\log_{10}(x-1) > 0$$

$$x - 1 > 1 \quad \text{or} \quad x > 2$$

Therefore, the smallest integer x satisfying the given equation is 3.

Answer: (C) → (p)

(D)

$$1 < a \le x \Rightarrow \log_a x > 0, \log_x a > 0$$

Therefore

$$\log_a x + \log_x a \ge 2(\log_a x \cdot \log_x a)^{1/2} = 2$$

and equality occurs if and only if $x = a$. Therefore minimum value is 2.

Answer: (D) → (r)

2. Match the items in Column I with those in Column II.

Column I	Column II
(A) $\log_2(\log_3 81) =$	(p) 0
(B) $3^{4\log_9 7} = 7^k$, then $k =$	(q) 1
	(r) 3
(C) $2^{\log_3 5} - 5^{\log_3 2} =$	(s) 2
(D) $\log_3[\log_2(512)] =$	(t) 4

Solution:

(A) $\log_2(\log_3 81) = \log_2(\log_3 3^4) = \log_2 4 = 2$

Answer: (A) → (s)

(B)

$$3^{4\log_9 7} = 7^k$$

$$\Rightarrow 3^{4\times(1/2)\log_3 7} = 7^k$$

$$\Rightarrow (3^{\log_3 7})^2 = 7^k$$
$$\Rightarrow 7^2 = 7^k$$
$$\Rightarrow k = 2$$

Answer: (B) → (s)

(C)

$$2^{\log_3 5} - 5^{\log_3 2} = 2^{\log_2 5 \cdot \log_3 2} - 5^{\log_3 2}$$
$$= (2^{\log_2 5})^{\log_3 2} - 5^{\log_3 2}$$
$$= 5^{\log_3 2} - 5^{\log_3 2} = 0$$

Answer: (C) → (p)

(D)

$$\log_3[\log_2(512)] = \log_3(\log_2 2^9)$$
$$= \log_3 9 = 2$$

Answer: (D) → (s)

Comprehension-Type Questions

1. Passage: It is given that

$$\log_a(bc) = \log_a b + \log_a c, a \neq 1, a > 0, b > 0, c > 0$$

$$\log_a\left(\frac{b}{c}\right) = \log_a b - \log_a c, a \neq 1, a > 0, b > 0, c > 0$$

$$\log_{a^m} b^n = \frac{n}{m}\log_a b, a \neq 1, a > 0, b > 0, m \neq 0$$

$$\log_a b = \log_c b/\log_c a, a \neq 1, c \neq 1, a > 0, b > 0, c > 0$$

$$\log_a b = \frac{1}{\log_b a}, a \neq 1, b \neq 1, a > 0, b > 0$$

Answer the following questions:

(i) If $a > 0, b > 0$ and $a^2 + b^2 = 7ab$, then

(A) $2\log\left(\frac{a+b}{3}\right) = \log(ab)$

(B) $\log\left(\frac{a+b}{3}\right) = \log(ab)$

(C) $\log\left(\frac{a+b}{3}\right) = \log\left(\frac{a}{b}\right)^2$

(D) $\log\left(\frac{|a-b|}{3}\right) = \log a + \log b$

(ii) $\dfrac{\log_3 135}{\log_{15} 3} - \dfrac{\log_3 5}{\log_{405} 3}$ is equal to

(A) 4 (B) 5 (C) 3 (D) 0

(iii) $\log_p\left(\log_p \sqrt[p]{\sqrt[p]{\sqrt[p]{\cdots\cdots\sqrt[p]{p}}}}\right)$ (n radicals) =

(A) np (B) $-n$ (C) $-np$ (D) n

Solution:

(i)

$$a^2 + b^2 = 7ab$$
$$(a+b)^2 = 9ab$$
$$2\log(a+b) = 2\log 3 + \log(ab)$$
$$2\log\left(\frac{a+b}{3}\right) = \log(ab)$$

Answer: (A)

(ii) The given number can be written as

$$\log_3(135)\log_3(15) - \log_3 5 \cdot \log_3 405$$
$$= (\log_3 5 + 3)(1 + \log_3 5) - (\log_3 5)(\log_3 5 + 4) = 3$$

Answer: (C)

(iii) $\log_p\left(\log_p \underbrace{\sqrt[p]{\sqrt[p]{\sqrt[p]{\cdots\cdots\sqrt[p]{p}}}}}_{n}\right) = \log_p \log_p (p^{1/p^n})$

$$= \log_p\left(\frac{1}{p^n}\right) = -n$$

Answer: (B)

Assertion–Reasoning Type Questions

1. Statement I: If a, b, c are the sides of a right-angled triangle with c as the hypotenuse and both $c + b$ and $c - b$ are not equal to unity, then

$$\log_{c+b} a + \log_{c-b} a = 2\log_{c+b} a \times \log_{c-b} a$$

Statement II: $a^2 = c^2 - b^2$

(A) Both Statements I and II are correct and Statement II is a correct explanation of Statement I.

(B) Both Statements I and II are correct and Statement II is not a correct explanation of Statement I.

(C) Statement I is true, but Statement II is false.

(D) Statement I is false, but Statement II is correct.

Solution: In a right-angled triangle, it is known that the square of the hypotenuse is equal to the sum of the squares of the other two sides. Therefore Statement II is correct. Also,

$$\log_{c+b} a + \log_{c-b} a = \frac{1}{\log_a(c+b)} + \frac{1}{\log_a(c-b)}$$

$$= \frac{\log_a(c+b) + \log_a(c-b)}{\log_a(c+b)\log_a(c-b)}$$

$$= \frac{\log_a(c^2-b^2)}{\log_a(c+b)\log_a(c-b)}$$

$$= \frac{\log_a a^2}{\log_a(c+b)\log_a(c-b)}$$

$$= 2\log_{c+b} a \times \log_{c-b} a$$

Answer: (A)

SUMMARY

2.1 ***Exponential function:*** For any positive real number a, the function $f(x) = a^x$ for $x \in \mathbb{R}$ is called exponential function with base a.

2.2 ***Properties of a^x:***

(1) $a^x \cdot a^y = a^{x+y}$

(2) $a^x > 0$

(3) $\dfrac{a^x}{a^y} = a^{x-y}$

(4) $(a^x)^y = a^{xy}$

(5) $a^{-x} = 1/a^x$

(6) $a^0 = 1$

(7) $a^1 = a$

(8) $1^x = 1$

(9) For $a > 1$, if $x \le y$, then $a^x \le a^y$ (i.e., a^x is an increasing function).

(10) If $0 < a < 1$, then $x \le y \Rightarrow a^x \ge a^y$ (i.e., a^x is a decreasing function).

(11) If $a > 0$, then a^x is an infection.

(12) For $a > 0$ and $a \ne 1$, then $a^x = 1 \Leftrightarrow x = 0$.

2.3 ***Logarithmic function:*** Let $a > 0$ and $a \ne 1$. Consider the function $g: \mathbb{R}^+ \to \mathbb{R}$ defined by $g(y) = x \Leftrightarrow y = a^x$ for all $y \in \mathbb{R}^+$ and $x \in \mathbb{R}$. This function g is denoted by $\log_a$ meaning that $\log_a y = x \Leftrightarrow y = a^x$. Note that $\log_a y$ is defined only $0 < a \ne 1$ and $y > 0$.

2.4 ***Properties of logarithmic function:***

(1) $a^{\log_a y} = y$

(2) $\log_a(a^x) = x$

(3) $\log_a y = x \Leftrightarrow y = a^x$

(4) $\log_a(y_1 y_2) = \log_a y_1 + \log_a y_2$

(5) $\log_a(1/y) = -\log_a y$

(6) $\log_a(y_1/y_2) = \log_a y_1 - \log_a y_2$

(7) $\log_a(y^z) = z\log_a y$ for all $z \in \mathbb{R}$

(8) $\log_a a = 1$ and $\log_a 1 = 0$

2.5 ***Some more important formulae:***

(1) **Change of base:** If a, b are both positive and different from 1, and y is positive, then

$$\log_a y = \log_b y \times \log_a b$$

(2) $\log_b a \times \log_a b = 1$ or $\log_b a = \dfrac{1}{\log_a b}$

(3) $\log_x y = \log y / \log x$ where both numerator and denominator have common base.

(4) $\log_{a^n}(y) = \dfrac{1}{n}\log_a y$

(5) If $0 < a < 1$, then $\log_a x$ is a decreasing function.

(6) If $a > 1$, then $\log_a x$ is an increasing function.

EXERCISES

Single Correct Choice Type Questions

1. If $a > 0$, $b > 0$ and $a^2 + 4b^2 = 12ab$, then $\log(a + 2b) - 2\log 2$ is equal to

(A) $\log a + \log b$

(B) $2(\log a + \log b)$

(C) $3(\log a + \log b)$

(D) $\dfrac{1}{2}(\log a + \log b)$

2. If $1 < a \le b$, then

$$2[\sqrt{\log_a \sqrt[4]{ab} + \log_b \sqrt[4]{ab}} - \sqrt{\log_a \sqrt[4]{b/a} + \log_b \sqrt[4]{a/b}}]\sqrt{\log_a b} =$$

(A) 1 (B) 2 (C) 3 (D) 4

3. $\log_3 2 \cdot \log_4 3 \cdot \log_5 4 \cdot \log_6 5 \cdot \log_7 6 \cdot \log_8 7 =$

(A) $\frac{1}{2}$ (B) 3 (C) $\frac{1}{3}$ (D) 2

4. $\log_2(2x^2) + (\log_2 x) \cdot x^{\log_x(\log_2 x+1)} + \frac{1}{2}(\log_4 x^4)^2 + 2^{-3\log_{1/2}\log_2 x} =$

(A) $(1+\log_2 x)^3$ (B) $1+\log_2 x$
(C) $(1+\log_2 x)^2$ (D) $(1+\log_2 x)^4$

5. The number of pairs (x, y) satisfying the equations $\log_y x + \log_x y = 2$ and $x^2 = 20 + y$ is

(A) Infinite (B) 2 (C) 0 (D) 1

6. The set of solutions of the inequality $\log_x(2x - 3/4) > 2$ is

(A) $\left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$ (B) $\left(\frac{3}{8}, \frac{1}{2}\right) \cup \left(1, \frac{3}{2}\right)$
(C) $\left(0, \frac{3}{8}\right) \cup \left(1, \frac{3}{2}\right)$ (D) $(0, 1) \cup \left(1, \frac{3}{2}\right)$

7. The set of solutions of the inequality $2\log_2(x-1) > \log_2(5-x) + 1$ is

(A) $(1, 5)$ (B) $(5, \infty)$
(C) $(3, 5)$ (D) $(-\infty, -3)$

8. If $\log_a 2 = m$ and $\log_a 5 = n$, where $0 < a \ne 1$, then $\log_a 500 =$

(A) $2m + 3n$ (B) $3m + 2n$
(C) $3m + 3n$ (D) $2m + 2n$

9. The domain of the function $f(x) = [1/\log_{10}(1-x)] + \sqrt{x+2}$ is

(A) $(-\infty, -3)$ (B) $(2, \infty)$
(C) $(-2, -1)$ (D) $(-2, 0) \cup (0, 1)$

10. If $|\log_2(x^2/2)| \le 1$, then x lies in

(A) $(0, 1)$ (B) $[-2, -1] \cup [1, 2]$
(C) $(3, \infty)$ (D) $(-\infty, -2)$

11. The domain of the function

$$f(x) = \frac{\log_2(x+3)}{x^2+3x+2}$$

is

(A) $\mathbb{R} - \{-1, -2\}$ (B) $(-2, \infty)$
(C) $\mathbb{R} - \{-1, -2, -3\}$ (D) $(-3, \infty) - \{-1, -2\}$

12. Let $f: [1, \infty) \to [1, \infty)$ be defined by $f(x) = 2^{x(x-1)}$. Then $f^{-1}(x)$ is equal to

(A) $2^{-x(x-1)}$ (B) $\frac{1}{2}(1+\sqrt{1+4\log_2 x})$
(C) $\frac{1}{2}(1-\sqrt{1+4\log_2 x})$ (D) $f^{-1}(x)$ does not exist

13. Let $f(x) = x^2 + x + \log(1+|x|)$ for $0 \le x \le 1$. If $F(x)$ is defined on $[-1, 1]$ such that $F(x)$ is odd and $F(x) = f(x)$ for $0 \le x \le 1$, then

(A) $F(x) = \begin{cases} f(x) & \text{for} \quad 0 \le x \le 1 \\ -x^2 + x - \log(1+|x|) & \text{for} \quad -1 \le x \le 0 \end{cases}$

(B) $F(x) = x^2 + x - \log(1+|x|)$ for $-1 \le x \le 0$
(C) $F(x) = -f(x)$ for $-1 \le x \le 0$
(D) $F(x) = -x^2 + x + \log(1+|x|)$ for $-1 \le x \le 0$

14. Let W be the set of whole numbers and $f: W \to W$ be defined by

$$f(x) = \begin{cases} \left(x - 10\left[\dfrac{x}{10}\right]\right)10^{[\log_{10} x]} + f\left(\left[\dfrac{x}{10}\right]\right) & \text{if} \quad x > 0 \\ 0 \quad \text{if} \quad x = 0 \end{cases}$$

where $[y]$ denotes the largest integer $\le y$. Then $f(7752) =$

(A) 7527 (B) 5727 (C) 7257 (D) 2577

Multiple Correct Choice Type Questions

1. If $\log_x(6x-1) > \log_x(2x)$, then x belongs to

(A) $\left(\frac{1}{6}, \frac{1}{4}\right)$ (B) $\left(\frac{1}{6}, +\infty\right)$
(C) $(1, +\infty)$ (D) $\left(\frac{1}{8}, +\infty\right)$

2. If $\dfrac{x(y+z-x)}{\log x} = \dfrac{y(z+x-y)}{\log y} = \dfrac{z(x+y-z)}{\log z}$ then

(A) $x^y y^x = y^z z^y$ (B) $y^z z^y = x^z z^x$
(C) $x^z z^y = y^z z^x$ (D) $x^y y^x = z^x x^z$

3. A solution of the equation $x^{\log 2x} = 5$ is

(A) 0.2 (B) 0.1 (C) 5 (D) 4

4. A solution of the system of equations

$$x^{x-y} = y^{x+y} \quad \text{and} \quad \sqrt{x}\cdot y = 1$$

is

(A) $(1, 1)$ (B) $(1, \sqrt[3]{3})$

(C) $(1/\sqrt[3]{9}, 1)$ (D) $(1/\sqrt[3]{9}, \sqrt[3]{3})$

5. A solution of the inequality $\log_{0.2}(x^2 - 4) \geq -1$ satisfies

(A) $1 < |x| < 2$ (B) $2 < |x| \leq 3$

(C) $3 < |x| \leq 4$ (D) $1 < |x| \leq 3$

6. If $f(x) = \log_{10}(3x^2 - 4x + 5)$, then

(A) Domain of f is $\mathbb{R}$

(B) Range of f is $[\log_{10}(11/3), +\infty)$

(C) f is defined in $(0, +\infty)$

(D) Range of f is $(-\infty, \log_{10}(11/3)]$

7. If $e^x + e^{g(x)} = e$, then

(A) Domain of g is $(-\infty, 1)$

(B) Range of g is $(-\infty, 1)$

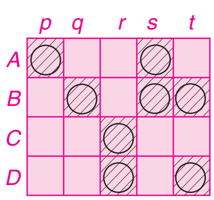
(C) Domain of g is $(-\infty, 0]$

(D) Range of g is $(-\infty, 1]$

Matrix-Match Type Questions

In each of the following questions, statements are given in two columns, which have to be matched. The statements in Column I are labeled as **(A)**, **(B)**, **(C)** and **(D)**, while those in Column II are labeled as **(p)**, **(q)**, **(r)**, **(s)** and **(t)**. Any given statement in Column I can have correct matching with *one or more* statements in Column II. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following example.

Example: If the correct matches are (A) → (p), (s); (B) → (q), (s), (t); (C) → (r), (D) → (r), (t); that is if the matches are (A) → (p) and (s); (B) → (q), (s) and (t); (C) → (r); and (D) → (r), (t); then the correct darkening of bubbles will look as follows:



1. Match the items in Column I with those in Column II.

Column I	*Column II*
(A) The number of solutions of the equation $2 - x + 3\log_5 2 = \log_5(3^x - 5^{2-x})$ is	(p) 3
(B) The number of values of x satisfying the equation $(\log_2 x)^2 - 5(\log_2 x) + 6 = 0$ is	(q) 1
	(r) 4
(C) The number of roots of the equation $\log_{10}\sqrt{x-1} + \frac{1}{2}\log_{10}(2x+15) = 1$ is	(s) 0
(D) The number of solutions of the equation $\log_7(x+2) = 6 - x$ is	(t) 2

2. Match the items in Column I with those in Column II.

Column I	*Column II*
(A) The number of solutions of the equation $\log_{10}(3x^2 + 12x + 19) - \log_{10}(3x + 4) = 1$ is	(p) 0
	(q) 3
(B) $\log_{\sqrt{5}}(4^x - 6) - \log_{\sqrt{5}}(2^x - 2) = 2$ is satisfied by x whose number is	(r) 2
(C) The number of solutions of the equation $\log_3(3^x - 8) = 2 - x$ is	(s) 4
(D) The number of values of x that satisfy the equation $2\log_3(x-2) + \log_3(x-4)^2 = 0$ is	(t) 1

3. Match the items in Column I with those in Column II.

Column I	*Column II*
(A) $f(x) = \dfrac{\sqrt{\log_{0.3}\lvert x-2\rvert}}{\lvert x\rvert}$ is defined for x belonging to	(p) $[1, 2)$
(B) Domain of the function	(q) $(-2, 1)$
$f(x) = \log[1 - \log_{10}(x^2 - 5x + 16)]$ is	(r) $(2, 3)$
(C) $f(x) = (\sqrt{\log_{0.5}(x^2 - 7x + 13)})^{-1}$ is defined for x belonging to	(s) $(3, 4)$
(D) Domain of the function $f(x) = \log\left(\dfrac{\sqrt{4-x^2}}{1-x}\right)$ is	(t) $(2, 3]$

Assertion–Reasoning Type Questions

Statement I and Statement II are given in each of the questions in this section. Your answers should be as per the following pattern:

(A) If both Statements I and II are correct and II is a correct reason for I

(B) If both Statements I and II are correct and II is not a correct reason for I

(C) If Statement I is correct and Statement II is false

(D) If Statement I is false and Statement II is correct.

1. Statement I: If $a = x^2$, $b = y^2$ and $c = z^2$, where x, y, z are non-unit positive reals, then $8(\log_a x^3)(\log_b y^3)(\log_c z^3) = 27$.

Statement II: $\log_b a \cdot \log_a b = 1$

2. Statement I: If $x^{\log_x (1-x)^2} = 9$, then $x = 3$.

Statement II: $a^{\log_a x} = x$ where $0 < a \neq 1$ and $x > 0$

3. Statement I: The equation $\log_{(2+x)^{-1}}(5+x^2) = \log_{3+x^2}(15+\sqrt{x})$ has no solution.

Statement II: $\log_{b^m} a = \dfrac{1}{m}\log_b a$

4. Statement I: The equation $9^{\log_3(\log_2 x)} = \log_2 x - (\log_2 x)^2 + 1$ has only one solution.

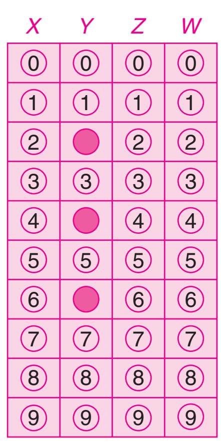
Statement II: $a^{\log_a x} = x$ and $\log_a x^n = n\log_a x$, where $x > 0$.

5. Statement I: If n is a natural number greater than 1 such that $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where $p_1, p_2, \ldots, p_k$ are distinct primes and $\alpha_1, \alpha_2, \ldots, \alpha_k$ are positive integers, then $\log n \geq k \log 2$.

Statement II: $\log_a x > \log_a y$ when $x > y$ and $a > 1$.

Integer Answer Type Questions

The answer to each of the questions in this section is a non-negative integer. The appropriate bubbles below the respective question numbers have to be darkened. For example, as shown in the figure, if the correct answer to the question number Y is 246, then the bubbles under Y labeled as 2, 4, 6 are to be darkened.



1. $5^{\log_{1/5}(1/2)} + \log_{\sqrt{2}}\left(\dfrac{4}{\sqrt{3}+\sqrt{7}}\right) + \log_{1/2}\left(\dfrac{1}{10+2\sqrt{21}}\right) =$ ______.

2. $\dfrac{(81)^{1/\log_5 9} + 3^{3/\log_{\sqrt{6}} 3}}{409}\left[(\sqrt{7})^{2/\log_{25} 7} - (125)^{\log_{25} 6}\right] =$ ______.

3. The value of x satisfying the equation $6^{2x+4} = (3^{3x})(2^{x+8})$ is ______.

4. The number of solutions of the equation $|x-2|^{10x^2-1} = |x-2|^{3x}$ is ______.

5. The number of ordered pairs (x, y) satisfying the two equations $8(\sqrt{2})^{x-y} = (0.5)^{y-3}$ and $\log_3(x-2y) + \log_3(3x+2y) = 3$ is ______.

6. If (x_1, y_1) and (x_2, y_2) are the solutions of the simultaneous equations $x + y = 12$ and $2(2\log_{y^2} x - \log_{1/x} y) = 5$, then $x_1 x_2 - y_1 y_2$ is equal to ______.

7. The number of solutions of the system of equations $y = 1 + \log_4 x$, $x^y = 4^6$ is ______.

8. The number of integers satisfying the inequality $3^{(5/2)\log_3(12-3x)} - 3^{\log_2 x} > 83$ is ______.

9. The number of integer values of x satisfying the inequality $2x + 1 < 2\log_2(x+3)$ is ______.

ANSWERS

Single Correct Choice Type Questions

1. (D)
2. (B)
3. (C)
4. (A)
5. (D)
6. (B)
7. (C)
8. (A)
9. (D)
10. (B)
11. (D)
12. (B)
13. (A)
14. (D)

Multiple Correct Choice Type Questions

1. (A), (C)
2. (A), (B), (D)
3. (B), (C)
4. (A), (D)
5. (B), (D)
6. (A), (B), (C)
7. (A), (B)

Matrix-Match Type Questions

1. (A) $\to$ (q), (B) $\to$ (t), (C) $\to$ (q), (D) $\to$ (q)
2. (A) $\to$ (r), (B) $\to$ (t), (C) $\to$ (t), (D) $\to$ (t)
3. (A) $\to$ (p), (r), (t); (B) $\to$ (r), (C) $\to$ (s), (D) $\to$ (q)

Assertion–Reasoning Type Questions

1. (A)
2. (D)
3. (A)
4. (A)
5. (A)

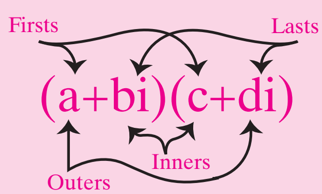
Integer Answer Type Questions

1. 6
2. 1
3. 4
4. 2
5. 1
6. 0
7. 2
8. 2
9. 4

Complex Numbers

3

A: What do you mean?
B: Well, what if we make up a number, say ‘i’, so that
$i \times i = -1$
A: Can we do that?
B: Why not!
A: But there is no such number that has that size.
B: I know, but the idea can exist in our imagination! I think we should call it an *imaginary* number.



Complex Numbers

Contents

Any ordered pair (a, b) where a and b are real numbers is called a **complex number.** The set of all complex numbers is denoted by $\mathbb{C}$ which is $\mathbb{R} \times \mathbb{R}$.

It is well known that there is no real number a for which $a^2 = -1$. In other words, the equation $x^2 + 1 = 0$ has no root in the real number system $\mathbb{R}$. Likewise, the equation $x^2 + x + 1 = 0$ has no root in $\mathbb{R}$. For this reason, the real number system $\mathbb{R}$ is enlarged to a system $\mathbb{C}$ in such a way that every polynomial equation, with coefficients in $\mathbb{C}$, has a root in $\mathbb{C}$. The members of $\mathbb{C}$ are called complex numbers. Infact, the system $\mathbb{C}$ of complex numbers is the smallest expansion of the real number system $\mathbb{R}$ satisfying the above property. In this chapter we will discuss the construction and several properties of the system of the complex numbers.

3.1 | Ordered Pairs of Real Numbers

A complex number can be defined as an ordered pair of real numbers. Let $\mathbb{R}$ denote the set of real numbers and

$$\mathbb{C} = \mathbb{R} \times \mathbb{R}$$

That is, $\mathbb{C}$ is the set of all ordered pairs (a, b) such that a and b are real numbers. We will introduce all the arithmetical concepts of addition, subtraction, multiplication, and division among members of $\mathbb{C}$. The members of $\mathbb{C}$ are called *complex numbers*. First let us recall that two ordered pairs (a, b) and (c, d) are said to be equal if $a = c$ and $b = d$.

Mathematical Operations on Complex Numbers

DEFINITION 3.1 For any complex numbers (a, b) and (c, d), let us define

$$(a, b) + (c, d) = (a + c, b + d)$$

$$(a, b) - (c, d) = (a - c, b - d)$$

$(a, b) + (c, d)$ is called the ***sum*** of (a, b) and (c, d) and the process of taking sum is called the ***addition***. Similarly $(a, b) - (c, d)$ is called the ***difference*** of (c, d) with (a, b) and the process of taking difference is called the ***subtraction***.

Try it out Verify the following properties:

1. $((a, b) + (c, d)) + (s, t) = (a, b) + ((c, d) + (s, t))$
2. $(a, b) + (c, d) = (c, d) + (a, b)$
3. $(a, b) + (0, 0) = (a, b)$
4. $(a, b) + (-a, -b) = (0, 0)$
5. $(a, b) + (c, d) = (s, t) \Leftrightarrow (a, b) = (s, t) - (c, d)$
   $\Leftrightarrow (c, d) = (s, t) - (a, b)$

DEFINITION 3.2 For any complex numbers (a, b) and (c, d), let us define

$$(a, b)\cdot(c, d) = (ac - bd, ad + bc)$$

This is called the ***product*** of (a, b) and (c, d) and the process of taking products is called ***multiplication***.

Try it out Verify the following properties for any complex numbers (a, b), (c, d) and (s, t).

1. $[(a, b)\cdot(c, d)]\cdot(s, t) = (a, b)\cdot[(c, d)\cdot(s, t)]$
2. $(a, b)\cdot(c, d) = (c, d)\cdot(a, b)$
3. $(a, b)\cdot[(c, d) + (s, t)] = (a, b)\cdot(c, d) + (a, b)\cdot(s, t)$
4. $(a, b)\cdot(1, 0) = (a, b)$
5. $(a, 0)\cdot(c, d) = (ac, ad)$
6. $(a, 0)\cdot(c, 0) = (ac, 0)$
7. $(a, 0) + (c, 0) = (a + c, 0)$

Properties 6 and 7 in "Try it out" suggest that, when we identify any real number a with the complex number $(a, 0)$, then the usual arithmetics of real numbers are carried over to the complex numbers of the form $(a, 0)$. Further one can easily observe that the mapping $a \mapsto (a, 0)$ is an injection of $\mathbb{R}$ into $\mathbb{C}$. Therefore, we can identify $\mathbb{R}$ with the subset $\mathbb{R} \times \{0\}$ of $\mathbb{C}$. This also suggests that any real number a can be considered as a complex number $(a, 0)$. Thus $\mathbb{C}$ is an enlargement of $\mathbb{R}$ without disturbing the arithmetics in $\mathbb{R}$.

Examples

Let $z_1 = (2, 3)$ and $z_2 = \left(\frac{1}{2}, \frac{2}{3}\right)$, then

(1) $z_1 + z_2 = (2, 3) + \left(\frac{1}{2}, \frac{2}{3}\right) = \left(2 + \frac{1}{2}, 3 + \frac{2}{3}\right) = \left(\frac{5}{2}, \frac{11}{3}\right)$

(2) $z_1 - z_2 = (2, 3) - \left(\frac{1}{2}, \frac{2}{3}\right) = \left(2 - \frac{1}{2}, 3 - \frac{2}{3}\right) = \left(\frac{3}{2}, \frac{7}{3}\right)$

(3)
$$\begin{aligned} z_1 \cdot z_2 &= (2, 3) \cdot \left(\frac{1}{2}, \frac{2}{3}\right) \\ &= \left(2 \times \frac{1}{2} - 3 \times \frac{2}{3}, 2 \times \frac{2}{3} + 3 \times \frac{1}{2}\right) \\ &= \left(1 - 2, \frac{4}{3} + \frac{3}{2}\right) \\ &= \left(-1, \frac{17}{6}\right) \end{aligned}$$

Zero and Unity in Complex Numbers

DEFINITION 3.3 The complex numbers (0, 0) and (1, 0) are called the zero and unity, respectively, and are simply denoted by 0 and 1. Note that these are the real numbers 0 and 1 also, since, for any real number a, we identify a with the complex number $(a, 0)$.

THEOREM 3.1 For any non-zero complex number z, there exists a unique complex number s such that $z \cdot s = 1\ [= (1, 0)]$.

PROOF Let $z = (a, b)$ be a non-zero complex number; that is, $z \neq (0, 0)$ and hence either $a \neq 0$ or $b \neq 0$ so that $a^2 + b^2$ is a positive real number. Put

$$s = \left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2}\right)$$

Then

$$\begin{aligned} z \cdot s &= (a, b) \cdot \left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2}\right) \\ &= \left(\frac{a^2}{a^2 + b^2} - \frac{b(-b)}{a^2 + b^2}, \frac{a(-b)}{a^2 + b^2} + \frac{ba}{a^2 + b^2}\right) \\ &= (1, 0) = 1 \end{aligned}$$

If (c, d) is any complex number such that

$$(a, b) \cdot (c, d) = (1, 0)$$

then $ac - bd = 1$ and $ad + bc = 0$. From these we can derive that

$$c = \frac{a}{a^2 + b^2} \quad \text{and} \quad d = \frac{-b}{a^2 + b^2}$$

Thus, s is the unique complex number such that $z \cdot s = 1$. ∎

Multiplicative Inverse

DEFINITION 3.4 The unique complex number s such that $z \cdot s = 1$ is called the ***multiplicative inverse*** of z and is denoted by $1/z$ or z^{-1}. Also, $z_1 \cdot (1/z_2)$ will be simply expressed as z_1/z_2.

COROLLARY 3.1 For any complex numbers z_1 and z_2,

$$z_1 \cdot z_2 = 0 \Leftrightarrow z_1 = 0 \quad \text{or} \quad z_2 = 0$$

Examples

(1) If $z = (2, 3)$, then

$$\frac{1}{z} = \left(\frac{2}{2^2+3^2}, \frac{-3}{2^2+3^2}\right) = \left(\frac{2}{13}, \frac{-3}{13}\right)$$

(2) If $z = (4, 0)$, then

$$\frac{1}{z} = \left(\frac{4}{4^2+0^2}, \frac{-0}{4^2+0^2}\right) = \left(\frac{1}{4}, 0\right)$$

Infact, if $z = (a, 0)$, then

$$\frac{1}{z} = \left(\frac{1}{a}, 0\right)$$

(3) If $z = (0, 1)$, then

$$\frac{1}{z} = (0, -1)$$

Infact, if $z = (0, b)$, then

$$\frac{1}{z} = \left(0, \frac{-1}{b}\right)$$

(4) $(0, 1) \cdot (0, 1) = (-1, 0)$

3.2 | Algebraic Form $a + ib$

Even though there is no real number a such that $a^2 = -1$, there is a complex number z such that $z^2 (= z \cdot z) = -1$; for consider the complex number $(0, 1)$. We have

$$(0, 1) \cdot (0, 1) = (-1, 0) = -1$$

Also,

$$(0, -1) \cdot (0, -1) = (-1, 0) = -1$$

Infact, $(0, 1)$ and $(0, -1)$ are the only complex numbers satisfying the equation $z^2 = -1$. For if $z = (a, b)$ and $z^2 = -1$, then

$$(-1, 0) = -1 = (a, b) \cdot (a, b) = (a^2 - b^2, 2ab)$$

and hence $a^2 - b^2 = -1$ and $2ab = 0$. Since $b \neq 0$ (for, if $b = 0$, then a is a real number such that $a^2 = -1$), it follows that $a = 0$ and $b = \pm 1$ and hence $z = (0, 1)$ or $(0, -1)$.

Note: We will denote the complex number $(0, 1)$ by the symbol i (indicating that it is an imaginary number). By the above discussion, we have $i^2 = -1 = (-i)^2$. Recall that we are identifying a real number a with the complex number $(a, 0)$. With this notation, we have the following theorem.

THEOREM 3.2 Any complex number z can be uniquely expressed as

$$z = a + ib$$

where a and b are real numbers and $i = (0, 1)$. This expression is called the *algebraic form* of z.

PROOF Let z be a complex number. Then $z = (a, b)$ where a and b are real numbers. Now consider

$$z = (a, b) = (a, 0) + (0, 1)(b, 0) = a + ib$$

Clearly a and b are unique real numbers such that $z = a + ib$. ■

Note: We can perform the algebraic operations addition and multiplication with much ease when we consider the complex numbers in the form $a + ib$. We can sum or multiply as in the real number system by substituting -1 for i^2.

DEFINITION 3.5 Let z be a complex number and $z = a + ib$, where a and b are real numbers. Then a is called the ***real part* of z** and is denoted by **Re(z)**. Also, b is called the ***imaginary part* of z** and is denoted by **Im(z)**.

By the uniqueness of the real and imaginary parts of a complex number, it follows that, for any complex numbers z_1 and z_2,

$$z_1 = z_2 \Leftrightarrow \mathrm{Re}(z_1) = \mathrm{Re}(z_2) \quad \text{and} \quad \mathrm{Im}(z_1) = \mathrm{Im}(z_2)$$

Example 3.1

Write $(2+3i)^2(3+2i)$ in the form $a+ib$.

Solution: Consider

$$\begin{aligned}(2+3i)(2+3i)(3+2i) &= (4-9+12i)(3+2i)\\ &= (-5+12i)(3+2i)\\ &= (-15-24)+(-10+36)i\\ &= -39+26i\end{aligned}$$

Example 3.2

Find the real and imaginary parts of

$$z = (1+i)(5+2i)^2$$

Solution: Consider

$$\begin{aligned}z = (1+i)(5+2i)^2 &= (1+i)(25-4+20i)\\ &= (1+i)(21+20i)\\ &= 21-20+(21+20)i\\ &= 1+41i\end{aligned}$$

Therefore, $\mathrm{Re}(z) = 1$ and $\mathrm{Im}(z) = 41$.

Example 3.3

Find the real and imaginary parts of

$$z = \frac{(1+i)(2-3i)}{(1-i)(2+3i)}$$

Solution: Consider

$$\begin{aligned}z = \frac{(1+i)(2-3i)}{(1-i)(2+3i)} &= \frac{(2+3)+(2-3)i}{(2+3)+(-2+3)i}\\ &= \frac{5-i}{5+i} = \frac{(5-i)^2}{(5+i)(5-i)}\\ &= \frac{25-1-10i}{25+1}\\ &= \frac{24}{26}+\left(\frac{-10}{26}\right)i = \frac{12}{13}+\left(\frac{-5}{13}\right)i\end{aligned}$$

Therefore $\mathrm{Re}(z) = 12/13$ and $\mathrm{Im}(z) = -5/13$.

Example 3.4

Cube roots of unity

Compute all the complex numbers z such that $z^3 = 1$.

Solution: Let $z = a+ib$. Then

$$\begin{aligned}&z^3 = 1 \Rightarrow (a+ib)^2(a+ib) = 1\\ &\Rightarrow (a^2-b^2+2abi)(a+ib) = 1\\ &\Rightarrow (a^2-b^2)a-2ab^2+(2a^2b+a^2b-b^3)i = 1\\ &\Rightarrow (a^3-3ab^2)+(3a^2b-b^3)i = 1+0i\\ &\Rightarrow a^3-3ab^2 = 1 \quad \text{and} \quad 3a^2b-b^3 = 0\\ &\Rightarrow a(a^2-3b^2) = 1 \quad \text{and} \quad b(3a^2-b^2) = 0\\ &\Rightarrow (b = 0 \text{ and } a = 1) \quad \text{or} \quad [b^2 = 3a^2 \text{ and } a(a^2-3b^2) = 1]\\ &\Rightarrow z = 1 \quad \text{or} \quad (-8a^3 = 1 \text{ and } b = \pm\sqrt{3a^2})\\ &\Rightarrow z = 1 \quad \text{or} \quad \left(a = \frac{-1}{2} \text{ and } b = \pm\frac{\sqrt{3}}{2}\right)\\ &\Rightarrow z = 1; \quad \text{or} \quad z = \frac{-1}{2}+\frac{\sqrt{3}}{2}i; \quad \text{or} \quad z = \frac{-1}{2}-\left(\frac{\sqrt{3}}{2}\right)i\end{aligned}$$

Therefore

$$1, \quad \frac{-1+\sqrt{3}i}{2} \quad \text{and} \quad \frac{-1-\sqrt{3}i}{2}$$

are all the complex numbers z for which $z^3 = 1$.

Aliter:

$$z^3 - 1 = 0 \Leftrightarrow (z-1)(z^2+z+1) = 0$$
$$\Leftrightarrow z = 1 \quad \text{or} \quad z^2+z+1=0$$
$$\Leftrightarrow z = 1 \quad \text{or} \quad z = \frac{-1 \pm i\sqrt{3}}{2}$$

Thus, cube roots of unity are

$$1, \quad \frac{-1 \pm i\sqrt{3}}{2}$$

Now

$$\frac{-1+i\sqrt{3}}{2} \quad \text{and} \quad \frac{-1-i\sqrt{3}}{2}$$

are having the property that each is the square of the other. If we denote one of them as w, then the other will be w^2 and, further, $1 + w + w^2 = 0$.

Example 3.5

Express the complex number

$$z = \frac{3+i}{(1+i)(1-2i)}$$

in the algebraic form.

Solution: Consider

$$z = \frac{3+i}{(1+i)(1-2i)}$$
$$= \frac{3+i}{1+2+i-2i}$$
$$= \frac{3+i}{3-i} = \frac{(3+i)^2}{(3-i)(3+i)}$$
$$= \frac{9-1+6i}{9+1} = \frac{8+6i}{10} = \frac{4+3i}{5}$$
$$= \frac{4}{5} + i\frac{3}{5}$$

QUICK LOOK 1

Let us summarize and record the arithmetical operations on the complex numbers in algebraic form.

1. $(a+ib)+(c+id) = (a+c)+i(b+d)$
2. $(a+ib)-(c+id) = (a-c)+i(b-d)$
3. $(a+ib)\cdot(c+id) = (ac-bd)+i(ad+bc)$
4. $\dfrac{1}{a+ib} = \dfrac{a}{a^2+b^2} - i\dfrac{b}{a^2+b^2}$
5. $\dfrac{a+ib}{c+id} = \dfrac{1}{c^2+d^2}(c-id)(a+ib)$
$$= \frac{1}{c^2+d^2}[(ac+bd)+i(bc-ad)]$$

DEFINITION 3.6 A complex number z is called ***purely real*** if $\text{Im}(z) = 0$ and is called ***purely imaginary*** if $\text{Re}(z) = 0$.

Note: A complex number is both purely real and purely imaginary if and only if it is $0 (= 0 + i0)$.

Examples

(1) If x is a positive real number such that $(x+i)^2$ is purely imaginary, then

$$0 = \text{Re}(x+i)^2 = \text{Re}[x^2 - 1 + 2xi] = x^2 - 1$$

and hence $x = 1$ (since $x > 0$).

(2) If x is a real number such that $(2x+i)^2$ is purely real, then

$$0 = \text{Im}(2x+i)^2 = \text{Im}[4x^2 - 1 + 4xi] = 4x$$

and hence $x = 0$.

QUICK LOOK 2

Let us turn our attention to all the integral powers of i. Recall that i [= (0, 1)] is a complex number such that $i^2 = -1$. Now,

$$i^0 = 1, i^1 = i, i^2 = -1, i^3 = -i, i^4 = 1$$

Also,

$$i^{-1}\left(=\frac{1}{i}\right) = -i, i^{-2} = -1, i^{-3} = i, i^{-4} = 1, \ldots$$

Infact, for any integer n,

$$i^n = \begin{cases} 1 & \text{if } n \text{ is a multiple of } 4 \\ i & \text{if } n-1 \text{ is a multiple of } 4 \\ -1 & \text{if } n-2 \text{ is a multiple of } 4 \\ -i & \text{if } n-3 \text{ is a multiple of } 4 \end{cases}$$

THEOREM 3.3 The sum of any four complex numbers which are consecutive powers of i is zero.

PROOF Let z_1, z_2, z_3, z_4 be any four consecutive powers of i. Then, there is an integer n such that

$$z_1 = i^n, \quad z_2 = i^{n+1}, \quad z_3 = i^{n+2} \quad \text{and} \quad z_4 = i^{n+3}$$

Among the powers of i, $1, i, -1, -i$ occur cyclically and hence $z_1 + z_2 + z_3 + z_4 = 0$. ■

Examples

(1) $i^{2009} = i^{4(502)+1} = (i^4)^{502} \cdot i^1 = 1^{502} \cdot i = i$

(2) $i^{1947} + i^{1948} + i^{1949} + i^{1950} = i^3 + i^4 + i^5 + i^6$
$= -i + 1 + i - 1 = 0$

(3) $\sum_{n=1}^{2010} i^n = i + i^2 + \sum_{n=3}^{2010} i^n = i - 1 + 0 = -1 + i$

(4) $\sum_{n=1003}^{3005} i^n = \sum_{n=1}^{3005} i^n - \sum_{n=1}^{1002} i^n = i - (i + i^2) = 1$

DEFINITION 3.7 For any complex number $z = a + ib$ (a and b are real numbers), the ***conjugate* of *z*** is defined as

$$\bar{z} = a - ib$$

In the following theorem, whose proof is a straight forward verification, we list several properties of the conjugates of complex numbers.

QUICK LOOK 3

The following hold for any complex numbers z, z_1 and z_2.

1. $\overline{(\bar{z})} = z$
2. $\operatorname{Re}(z) = \dfrac{z + \bar{z}}{2}$
3. $\operatorname{Im}(z) = \dfrac{z - \bar{z}}{2i}$
4. $z = \bar{z} \Leftrightarrow z$ is purely real
5. $z = -\bar{z} \Leftrightarrow z$ is purely imaginary
6. $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$
7. $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$
8. $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$
9. If $z_2 \neq 0$, $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$
10. $z \cdot \bar{z}$ is a non-negative real number
11. $z\bar{z} = 0 \Leftrightarrow z = 0 \Leftrightarrow \bar{z} = 0$
12. $z_1\bar{z}_2 + \bar{z}_1 z_2 = 2\operatorname{Re}(\bar{z}_1 z_2) = 2\operatorname{Re}(z_1\bar{z}_2)$
13. $z_1\bar{z}_2 - \bar{z}_1 z_2 = -2i\operatorname{Im}(\bar{z}_1 z_2) = 2i\operatorname{Im}(z_1\bar{z}_2)$
14. For any polynomial $f(x)$ with real coefficients, $\overline{f(z)} = f(\bar{z})$

THEOREM 3.4 For any complex numbers z and w, with $w \neq 0$, there exists a complex number z_1 such that

$$wz_1 = z$$

This z_1 is unique and is denoted by z/w.

PROOF Let $z = a + ib$ and $w = c + id$, where a, b, c and d are real numbers such that $c^2 + d^2 > 0$. Put

$$z_1 = \frac{1}{c^2 + d^2} z\bar{w}$$

Then

$$wz_1 = w \cdot \frac{1}{c^2 + d^2} z\bar{w} = [(c + id)(c - id)z] \frac{1}{c^2 + d^2} = z$$

Also, for any complex number z_2,

$$wz_2 = z \Rightarrow \bar{w}wz_2 = \bar{w}z$$

$$\Rightarrow z_2 = \frac{1}{c^2 + d^2} \bar{w}z = z_1$$

Example 3.6

Find a complex number z such that $(2 + 3i)z = 3 - i$.

Solution: Take

$$z = \frac{1}{2^2 + 3^2}(2 - 3i)(3 - i)$$

Then

$$(2 + 3i)z = \frac{1}{2^2 + 3^2}(2 - 3i)(2 + 3i)(3 - i)$$

$$= \frac{2^2 + 3^2}{2^2 + 3^2}(3 - i) = 3 - i$$

Example 3.7

Express $\dfrac{4 + 3i}{2 + i}$ in the form $a + ib$.

Solution: Consider

$$\frac{4 + 3i}{2 + i} = \frac{(4 + 3i)}{(2 + i)} \frac{(2 - i)}{(2 - i)}$$

$$= \frac{1}{2^2 + 1^2}(8 + 3 + 6i - 4i)$$

$$= \frac{11}{5} + \frac{2}{5}i$$

3.3 | Geometric Interpretation

We have introduced the concept of a complex number as an ordered pair of real numbers that can be viewed as a point in the plane with respect to a given coordinate system. Infact, given a coordinate system in the plane, there is a one-to-one correspondence between the complex numbers and the points in the plane. This makes it possible to consider a complex number $a + ib$ as the point (a, b) in the coordinate plane. For this reason, the plane is called ***ARGAND'S plane*** **or** ***complex plane***. The abscissa axis is called the ***real axis*** **or** ***the axis of real numbers***, containing the points of the form $(a, 0)$, where a is a real number. The ordinate axis is called the ***imaginary axis*** **or** ***axis of imaginaries***, containing the points of the form $(0, b)$, where b is a real number.

For any complex number $z = a + ib$, it is often convenient to represent z by the vector $\overrightarrow{OM}$, where M is the point (a, b) in the plane and O is the origin. Also, every vector in the plane begining at the origin $O(0, 0)$ and terminating at the point $M(a, b)$ can be associated with the complex number $a + ib$. The origin $O(0, 0)$ is associated with the zero vector (Figure 3.1).

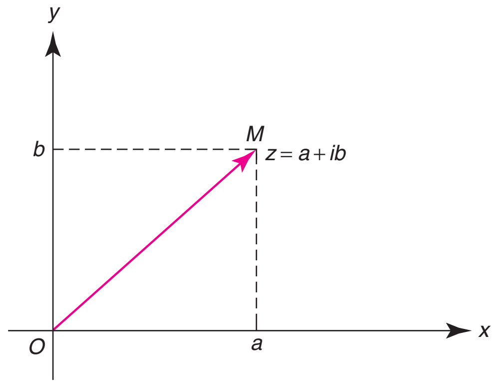


FIGURE 3.1 Graphical representation of a complex number of the form $z = a + ib$.

Representation of complex numbers as vectors facilitates a simple geometrical interpretation of operations on complex numbers. First, let us consider the addition of complex numbers. Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ be two complex numbers represented by the points M_1 and M_2 in the plane as shown in Figure 3.2.

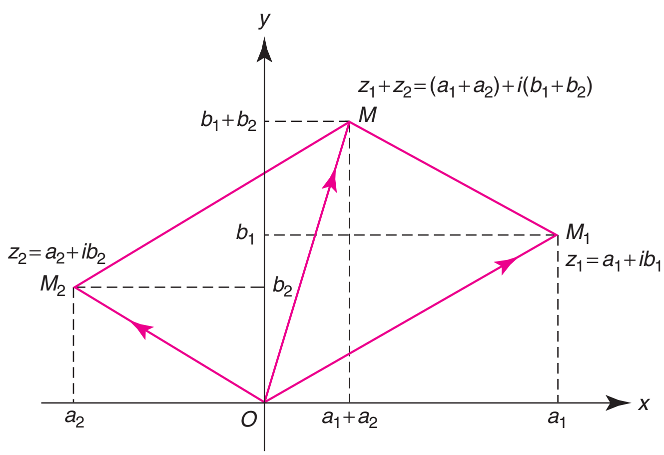


FIGURE 3.2 Geometrical interpretation of operations on complex numbers.

When z_1 and z_2 are added, their real and imaginary parts are added up (see Figure 3.2). When adding up vectors $\overrightarrow{OM_1}$ and $\overrightarrow{OM_2}$ corresponding to z_1 and z_2, their coordinates are added. Therefore, with the correspondence which we have established between complex numbers and vectors, the sum $z_1 + z_2$ of the numbers z_1 and z_2 will be associated with the vector $\overrightarrow{OM}$ which is equal to the sum of the vectors $\overrightarrow{OM_1}$ and $\overrightarrow{OM_2}$. Thus, a sum of complex numbers can be interpreted in terms of geometry as a vector equal to the sum of the vectors corresponding to the complex numbers (in other words, it also corresponds to the fourth vertex of the parallelogram constructed with $\overrightarrow{OM_1}$ and $\overrightarrow{OM_2}$ as adjacent sides).

For any complex number $z = a + ib$, the length of the vector $\overrightarrow{OM}$ corresponding to z has special importance. This is same as the distance of the point (a, b) from the origin O in the plane. This is termed as **modulus of z** and is denoted by $|z|$. The concept of the modulus of a complex number plays a vital role in the analysis of complex numbers. By the Pythagorean Theorem, it follows that the modulus of $a + ib$ is $\sqrt{a^2 + b^2}$. The following is a formal definition of the modulus of a complex number.

Modulus of z

DEFINITION 3.8 Let $z = a + ib$ be a complex number, where a and b are real numbers. The ***modulus* of z** is defined as $\sqrt{a^2 + b^2}$, the non-negative square root of $a^2 + b^2$ and is denoted by $|z|$. That is,

$$|z|^2 = a^2 + b^2 = [\mathrm{Re}(z)]^2 + [\mathrm{Im}(z)]^2$$

Try it out It can be easily seen that $z\bar{z} = (a+ib)(a-ib) = a^2 + b^2 = |z|^2$

In the following theorem, we list various properties of the modulus of a complex number and the proofs of these are straight forward routine verifications.

QUICK LOOK 4

The following hold for any complex numbers z, z_1 and z_2:

1. $|z|$ is a real number and $|z| \ge 0$
2. $|z| = 0$ if and only if $z = 0$
3. $|z| = |-z| = |\bar{z}| = |-\bar{z}|$
4. $|z_1 z_2| = |z_1||z_2|$
5. $|z|^2 = z\bar{z}$
6. $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$, if $z_2 \ne 0$
7. $|z_1 \pm z_2| \le |z_1| + |z_2|$

   Note that $|z_1 + z_2| = |z_1| + |z_2|$ if and only if the points z_1, z_2 are collinear with the origin and lie on the same side of the origin.
8. $|z^n| = |z^n|$ for all integers n
9. $|z_1 + z_2|^2 = (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) = |z_1|^2 + |z_2|^2 + (z_2\bar{z}_1 + \bar{z}_2 z_1)$

   $= |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}(z_1\bar{z}_2)$
10. $|z_1 - z_2|^2 = (z_1 - z_2)(\bar{z}_1 - \bar{z}_2) = |z_1|^2 + |z_2|^2 + (z_2\bar{z}_1 + \bar{z}_2 z_1)$

    $= |z_1|^2 + |z_2|^2 - 2\,\mathrm{Re}(z_1\bar{z}_2)$
11. $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2[|z_1|^2 + |z_2|^2]$
12. $||z_1| - |z_2|| \le |z_1 \pm z_2| \le |z_1| + |z_2|$

    Note that $||z_1| - |z_2|| = |z_1 - z_2|$ if and only if z_1, z_2 are collinear with the origin on the same side of the origin.

Property 12 above says that $|z_1| + |z_2|$ is the greatest possible value of $|z_1 \pm z_2|$ and $||z_1| - |z_2||$ is the least possible value of $|z_1 \pm z_2|$.

Unimodular Complex Number

DEFINITION 3.9 A complex number z is said to be ***unimodular*** if its modulus is 1, that is, $|z| = 1$.

Note that, for any non-zero complex number z, $z/|z|$ is always unimodular and

$$z = |z| \cdot \frac{z}{|z|}$$

This implies that z can be expressed as

$$z = rw$$

where $0 < r \in \mathbb{R}$ and $|w| = 1$. Moreover, this expression is unique, since

$$|z| = |rw| = |r||w| = r \cdot 1 = r \quad \text{and} \quad w = \frac{1}{r} z = \frac{z}{|z|}$$

Example 3.8

If z_1 and z_2 are non-zero complex numbers such that $(z_1 - z_2)/(z_1 + z_2)$ is unimodular, then prove that iz_1/z_2 is a real number.

Solution: We are given that

$$\left|\frac{z_1 - z_2}{z_1 + z_2}\right| = 1$$

Therefore

$$\left|\frac{(z_1/z_2) - 1}{(z_1/z_2) + 1}\right| = 1$$

$$\left|\left(\frac{z_1}{z_2}\right) - 1\right|^2 = \left|\left(\frac{z}{z_2}\right) + 1\right|^2$$

By properties 9 and 10 of Quick Look 4, we have

$$\left|\frac{z_1}{z_2}\right|^2 + 1 - 2\,\mathrm{Re}\left(\frac{z_1}{z_2}\right) = \left|\frac{z_1}{z_2}\right|^2 + 1 + 2\,\mathrm{Re}\left(\frac{z_1}{z_2}\right)$$

Therefore

$$\mathrm{Re}\left(\frac{z_1}{z_2}\right) = 0 \quad \text{or} \quad \frac{z_1}{z_2}$$

is purely imaginary

This implies

$$\frac{z_1}{z_2} = ia$$

where a is a real number and

$$i\frac{z_1}{z_2} = -a$$

which is a real number.

The complex numbers z having the same modulus $|z| = r$ evidently correspond to the points of the complex plane located on the circle of radius r with center at the origin. If $r > 0$, then there are infinitely many complex numbers with the given modulus r. If $r = 0$, then there is only one complex number, namely $z = 0$, whose modulus is 0.

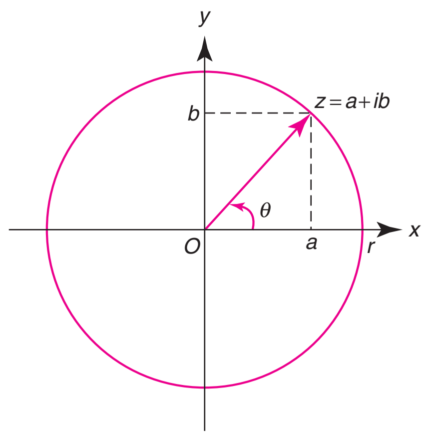


FIGURE 3.3 Geometrical determination of z using the angle θ and the modulus $\sqrt{a^2 + b^2}$.

From the geometrical point of view, it is evident that the complex number $z \neq 0$ is not completely determined by its modulus and depends on the direction also; for example, in Figure 3.3, z is determined by the angle θ and the modulus $\sqrt{a^2 + b^2}$. Next, we introduce another important concept which, together with the modulus, completely determines a complex number.

Argument of z

DEFINITION 3.10 Let $z \neq 0$ be a complex number and $\overrightarrow{OM}$ be the vector in the plane representing z. Then the ***argument of z*** is defined to be the magnitude of the angle between the positive direction of the real axis and the vector $\overrightarrow{OM}$, measured in counterclockwise sense. The angle will be considered *positive* if we measure counterclockwise and *negative* if we measure clockwise.

Note: For the complex number $z = 0$ the argument is not defined, and in this and only this case the number is specified exclusively by its modulus. Specification of the modulus and argument results in a unique representation of any non-zero complex number.

Unlike the modulus, the argument of a non-zero complex number is not defined uniquely. For example, the arguments of the complex number $z = a + ib$ shown in Figure 3.4 are the angles θ_1, θ_2 and θ_3. Note that

$$\theta_2 = 2\pi + \theta_1 \quad \text{and} \quad \theta_3 = \theta_1 - 2\pi$$

In general, θ is an argument of z if and only if $\theta = \theta_1 + 2n\pi$ for some integer n, where θ_1 is also an argument of z; that is, *any two arguments of a complex number differ by a number which is a multiple of* 2π. The set of all arguments of z will be denoted by arg z or arg$(a + ib)$. That is, if θ is an argument of z, then

$$\arg z = \{\theta + 2n\pi \mid n \text{ is an integer}\}$$

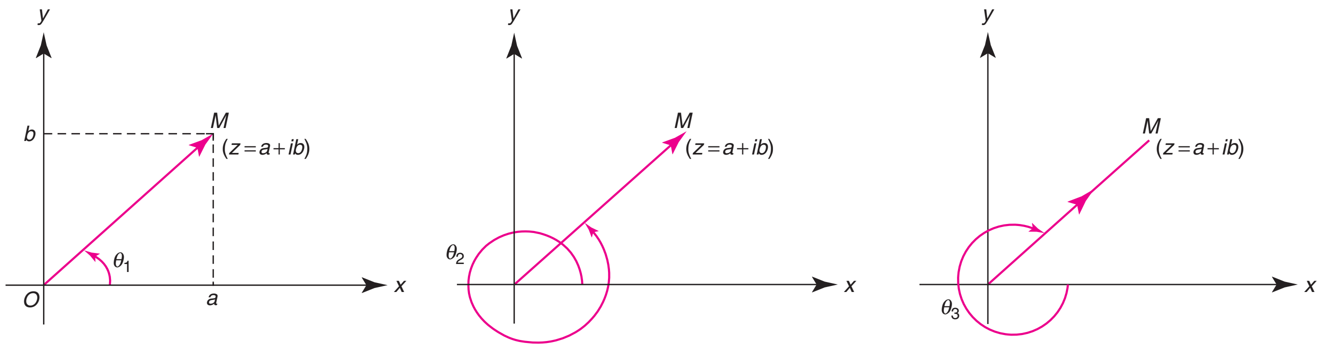


FIGURE 3.4 Different arguments of the complex number $z = a + ib$.

However, there is a unique θ such that $-\pi < \theta \le \pi$ and $\arg z = \{\theta + 2n\pi \mid n \text{ is an integer}\}$. This θ is called the *principal argument* of z and is denoted by Arg z (note that A here is uppercase). Note that

$$-\pi < \operatorname{Arg} z \le \pi$$

Also arg z and Arg z are related to each other by the relation

$$\arg z = \{\operatorname{Arg} z + 2n\pi \mid n \text{ is an integer}\}$$

Frequently, we denote arg z by Arg $z + 2n\pi$, where Arg z is the principal argument of z.

Example 3.9

Find the arguments of the complex numbers $z_1 = -i$, $z_2 = 1$ and $z_3 = -1 + i$.

Solution: From Figure 3.5, we have

$$\theta_1 = \frac{-\pi}{2}, \theta_2 = 0 \quad \text{and} \quad \theta_3 = \frac{3\pi}{4}$$

Therefore

$$\operatorname{Arg}(-i) = \frac{-\pi}{2} \quad \text{and} \quad \arg(-i) = \frac{-\pi}{2} + 2n\pi$$

$$\operatorname{Arg}(1) = 0 \quad \text{and} \quad \arg(1) = 2n\pi$$

$$\operatorname{Arg}(-1+i) = \frac{3\pi}{4} \quad \text{and} \quad \arg(-1+i) = \frac{3\pi}{4} + 2n\pi$$

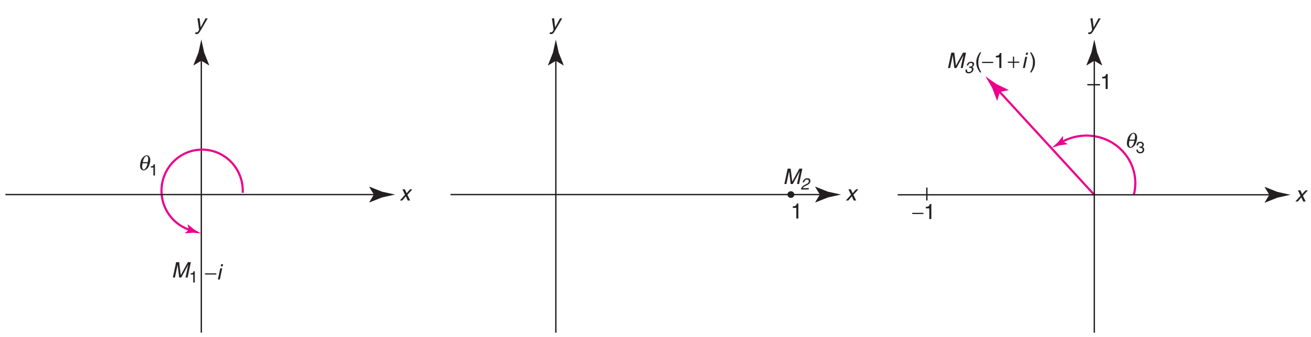


FIGURE 3.5 Example 3.9.

The real and imaginary parts of the complex number $z = a + ib$ can be expressed in terms of the modulus $|z| = r$ and argument θ as follows:

$$a = r\cos\theta \quad \text{and} \quad b = r\sin\theta$$

(see Figure 3.6) and hence

$$z = r(\cos\theta + i\sin\theta)$$

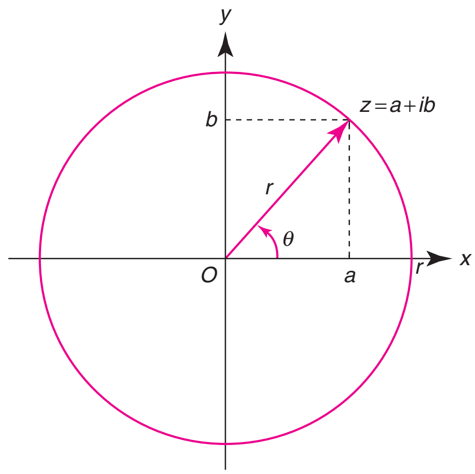


FIGURE 3.6 Geometrical interpretation of z in polar form.

Therefore, the arguments θ of a complex number $a + ib$ can be easily found from the following system of equations:

$$\cos\theta = \frac{a}{\sqrt{a^2+b^2}} \quad \text{and} \quad \sin\theta = \frac{b}{\sqrt{a^2+b^2}} \tag{3.1}$$

Example 3.10

Find the arguments of the complex number $z = -1 - i\sqrt{3}$.

Solution: In this case, we have $a = -1$ and $b = -\sqrt{3}$. Equation (3.1) takes the form

$$\cos\theta = \frac{-1}{2} \quad \text{and} \quad \sin\theta = \frac{-\sqrt{3}}{2}$$

Solving these we find that

$$\operatorname{Arg} z = \frac{-2\pi}{3}$$

and hence

$$\arg z = \frac{-2\pi}{3} + 2n\pi, \quad n \in \mathbb{Z}$$

The arguments of a complex number can be found by another method. It can be seen from formula (3.1) that each of the arguments satisfies the equation

$$\tan\theta = \frac{b}{a}$$

This equation is not equivalent to the system of equations (3.1). It has more solutions, but the selection of the required solutions (the arguments of the complex number) does not present any difficulties since it is always clear from the algebraic notation of the complex number in what quadrant of the complex plane it is located. This is elaborated in the following.

Key Points

Let $z = a + ib$ and $\theta = \operatorname{Arg} z$, the principal argument of z. Note that z is necessarily non-zero for the arg z to be defined.

1. If $a = 0$ and $b > 0$, then

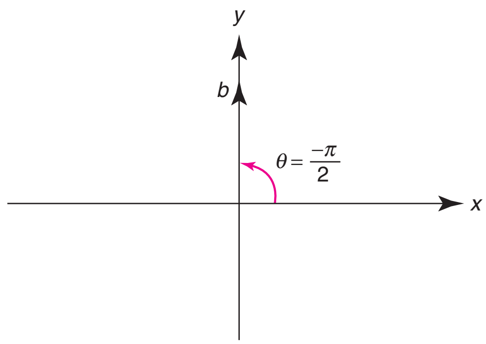
$$\operatorname{Arg} z = \frac{\pi}{2} \quad \text{and} \quad \arg z = \frac{\pi}{2} + 2n\pi, \quad n \in \mathbb{Z}$$

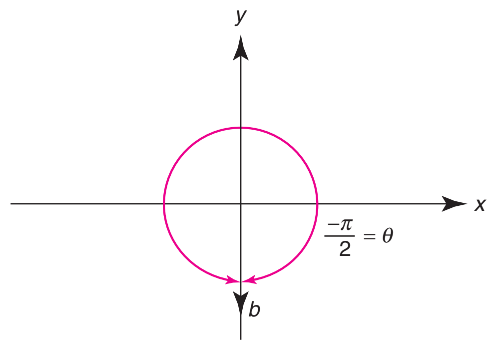
If $a = 0$ and $b < 0$, then

$$\operatorname{Arg} z = \frac{-\pi}{2} \quad \text{and} \quad \arg z = \frac{-\pi}{2} + 2n\pi, \quad n \in \mathbb{Z}$$

If $b = 0$ then $z = a$ lies on the x-axis and hence

$$\text{Arg } z = 0 \text{ or } \pi \quad \text{and} \quad \arg z = 2n\pi \text{ or } (2n+1)\pi, \quad n \in \mathbb{Z}$$



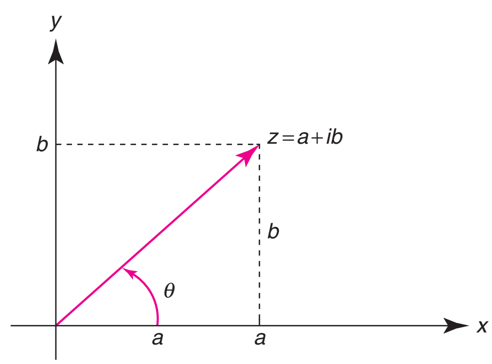


2. Let (a, b) belong to the first quadrant of the complex plane, that is, $a > 0$ and $b > 0$. Then the principal argument of z is given by

$$\text{Arg } z = \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

where $\tan\theta = b/a$. This is an acute angle $0 < \theta < \pi/2$ and positive. Therefore,

$$\arg z = 2n\pi + \tan^{-1}\left(\frac{b}{a}\right), \quad n \in \mathbb{Z}$$

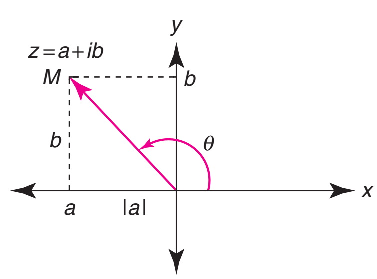


3. Let (a, b) belong to the second quadrant of the complex plane, that is, $a < 0$ and $b < 0$. Then the principal argument of z is given by

$$\text{Arg } z = \theta = \pi - \tan^{-1}\left(\frac{b}{|a|}\right)$$

This is an obtuse angle and is positive. Therefore,

$$\arg z = (2n+1)\pi - \tan^{-1}\left(\frac{-b}{a}\right), \quad n \in \mathbb{Z}$$

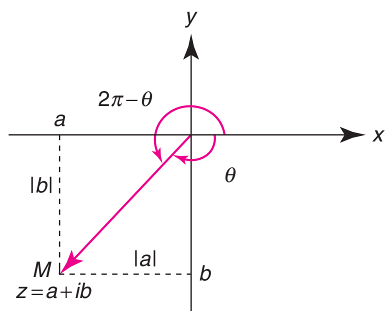


4. Let (a, b) lie in the third quadrant of the complex plane, that is, $a < 0$ and $b < 0$. Then the principal argument of z is given by

$$\text{Arg } z = \theta = -\pi + \tan^{-1}\left(\frac{b}{a}\right)$$

This is an obtuse angle and negative. Therefore

$$\arg z = (2n-1)\pi + \tan^{-1}\left(\frac{b}{a}\right), \quad n \in \mathbb{Z}$$

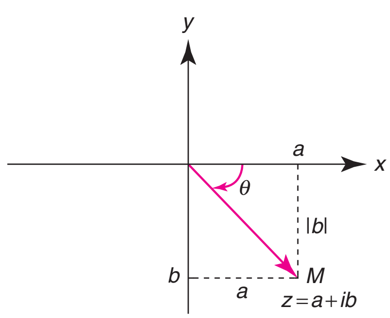


5. Let (a, b) lie in the fourth quadrant of the complex plane, that is, $a > 0$ and $b < 0$. Then the principal argument of z is given by

$$\text{Arg } z = \theta = -\tan^{-1}\left(\frac{|b|}{a}\right)$$

This is an acute angle and negative. Therefore

$$\arg z = 2n\pi - \tan^{-1}\left(\frac{|b|}{a}\right) = 2n\pi - \tan^{-1}\left(\frac{-b}{a}\right), \quad n \in \mathbb{Z}$$



Note: Arg z is the smallest angle of rotation of $\overrightarrow{OX}$ (positive x-axis) to fall on the vector $\overrightarrow{OM}$ $[M = (a, b)]$. Arg $z \gtrless 0$ according to whether the rotation of $\overrightarrow{OX}$ is anticlockwise or clockwise, respectively.

Example 3.11

Find the arguments of the complex number $z = -\sqrt{3} + i$.

Solution: In this case $z = a + ib$, where $a = -\sqrt{3}$ and $b = 1$. Therefore z lies in the second quadrant of the complex plane and hence the principal argument is

$$\text{Arg } z = \pi - \tan^{-1}\left(\frac{b}{|a|}\right) = \pi - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

Therefore

$$\arg(-\sqrt{3} + i) = \frac{5\pi}{6} + 2n\pi, \quad n \in \mathbb{Z}$$

Next we will discuss the geometrical constructions of difference, product and quotient of two complex numbers z_1 and z_2.

Construction of $z_2 - z_1$

Let us construct the vector $z_2 - z_1$ as the sum of the vectors z_2 and $-z_1$ (Figure 3.7). By the definition of the modulus, the real number $|z_2 - z_1|$ is the length of the vector $z_2 - z_1$; that is, the length of the vector $\overrightarrow{OM}$, where M, M_1, M_2 and N_1 represent the points in the complex plane corresponding to the complex numbers $z_2 - z_1, z_1, z_2$ and $-z_1$, respectively. The congruence of the triangles OMN_1 and M_1M_2O yields $|\overrightarrow{OM}| = |\overrightarrow{M_1M_2}|$. Therefore the length of the vector $z_2 - z_1$ is equal to the distance between the points z_1 and z_2. Thus we can say that *the modulus of the difference of two complex numbers is the distance between the points of the complex plane corresponding to those complex numbers.* This important geometrical interpretation of the modulus of the difference between two complex numbers makes it possible to use simple geometrical facts in solving certain problems. See examples given in Section 3.3.

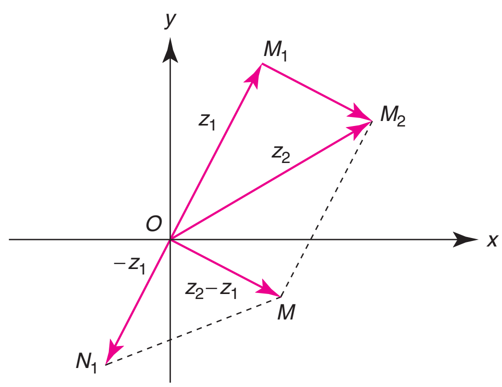


FIGURE 3.7 Construction the vector $z_2 - z_1$ as the sum of the vectors z_2 and $-z_1$.

Before going to illustrate the construction of the product and quotient of complex numbers, we present the following definition:

DEFINITION 3.11 Two triangles ABC and $A'B'C'$ are said to be ***directly similar*** if $\angle A = \angle A'$, $\angle B = \angle B'$ and $\angle C = \angle C'$ and the ratios of the sides opposite to equal angles are equal.

Note that directly similar triangles are similar and not vice-versa. For example, if ΔABC and $\Delta A'B'C'$ are directly similar, then ΔABC and $\Delta B'A'C'$ are not directly similar (unless they are equilateral triangle).

Construction of $z_1 z_2$ and z_1/z_2 ($z_2 \neq 0$)

Step 1: Let z_1 and z_2 be complex numbers and P and Q the points representing them, respectively. Let O be the origin so that the vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ represent z_1 and z_2, respectively. Let A be the point $(1, 0)$. Join A and P, and on the base OQ, construct the triangle OQR directly similar to the triangle OAP (Figure 3.8). Then

$$\angle QOR = \angle AOP, \angle OQR = \angle OAP, \angle QRO = \angle APO$$

and further,

$$\frac{OR}{OP} = \frac{OQ}{OA} = \frac{QR}{AP}$$

Therefore

$$OR = OP \cdot OQ \quad (\because OA = 1) \tag{3.2}$$

Let $\angle XOP = \theta_1$ and $\angle XOQ = \theta_2$. Then

$$\begin{aligned} \angle XOR &= \angle XOQ + \angle QOR \\ &= \angle XOQ + \angle AOP \\ &= \theta_2 + \theta_1 = \theta_1 + \theta_2 \end{aligned} \tag{3.3}$$

From Eqs. (3.2) and (3.3),

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

where $r_1 = OP$ and $r_2 = OQ$. Therefore R represents $z_1 \cdot z_2$.

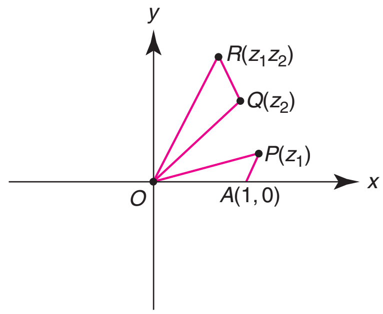


FIGURE 3.8 Step 1.

Step 2: Draw a triangle OPR directly similar to the triangle OQA. By the above construction, if R is represented by z, then $z \cdot z_2 = z_1$ (Figure 3.9). Notice that $\angle QOP = \arg(z_1/z_2)$ is the angle through which $\overrightarrow{OQ}$ must be rotated in order that it may lie along $\overrightarrow{OP}$ and $\arg(z_1/z_2)$ is positive or negative according as the rotation of $\overrightarrow{OQ}$ is anticlockwise or clockwise.

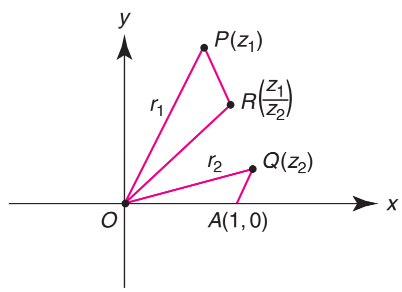


FIGURE 3.9 Step 2.

In the following theorem, an important consequence of $\arg(z_1/z_2)$ is derived. This can help the reader in solving many problems in the geometry of complex numbers.

THEOREM 3.5 Let z_1, z_2 and z_3 be three complex numbers represented by P, Q and R, respectively. If α is the angle $\angle PRQ$, then

$$\frac{z_2 - z_3}{z_1 - z_3} = \frac{RQ}{RP}(\cos\alpha + i\sin\alpha)$$

PROOF Let the points A and B represent $z_1 - z_3$ and $z_2 - z_3$, respectively, so that $RP = OA$, $RQ = OB$ and $PQ = AB$ (Figure 3.10). Therefore ΔPQR and ΔABO are congruent and hence $\angle AOB = \alpha$. By Step 2 above,

$$\alpha = \arg\left(\frac{z_2 - z_3}{z_1 - z_3}\right)$$

Therefore

$$\frac{z_2 - z_3}{z_1 - z_3} = \frac{|z_2 - z_3|}{|z_1 - z_3|}(\cos\alpha + i\sin\alpha)$$

$$= \frac{RQ}{RP}(\cos\alpha + i\sin\alpha)$$

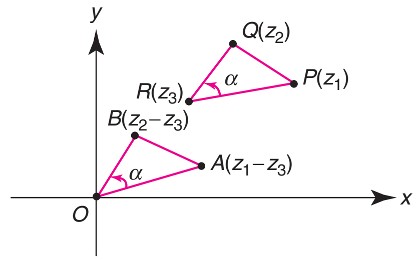
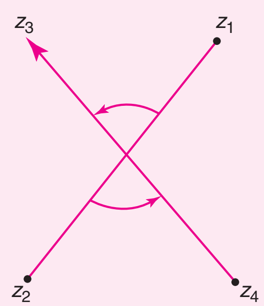


FIGURE 3.10 Theorem 3.5.

QUICK LOOK 5

1. $\arg\left(\dfrac{z_2 - z_3}{z_1 - z_3}\right)$ is the angle of rotation of the vector $\overrightarrow{RP}$ to fall along $\overrightarrow{RQ}$.



2. For any four points z_1, z_2, z_3 and z_4, the angle of inclination of the line joining z_1 to z_2 with the line joining z_3 to z_4 is
$$\arg\left(\frac{z_3 - z_4}{z_1 - z_2}\right)$$
3. The lines joining z_1 to z_2 and z_3 to z_4 are at right angles if and only if
$$\arg\left(\frac{z_3 - z_4}{z_1 - z_2}\right) = \pm\frac{\pi}{2}$$
and hence
$$\frac{z_3 - z_4}{z_1 - z_2} = \pm\lambda i$$
where $\lambda > 0$.

Example 3.12

Determine the sets of complex numbers defined by each of the following conditions.

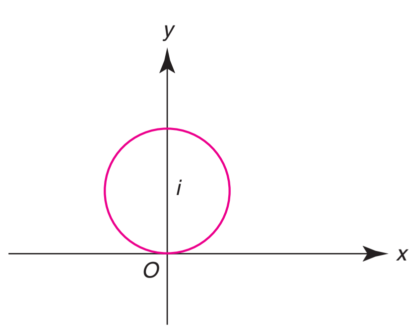
(1) $|z - i| = 1$

(2) $|2 + z| < |2 - z|$

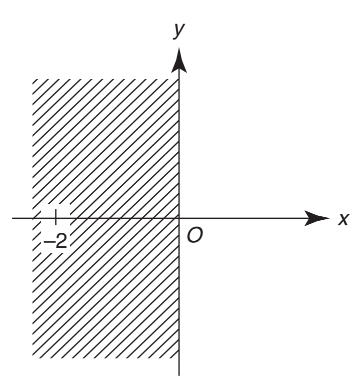
(3) $2 \le |z - 1 + 2i| < 3$

Solution:

(1) $|z - i| = 1$ is satisfied by those and only those points of the complex plane which are at a distance equal to 1 from the point i. Therefore, the set of complex numbers z satisfying the condition $|z - i| = 1$ is precisely the circle of unit radius with center at the point i (see the figure below).



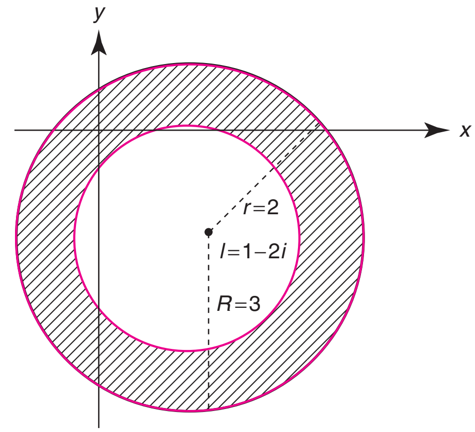
(2) We can give a different formulation of the problem, using the geometrical interpretation of the modulus of the difference between two complex numbers. We are asked to determine the set of points in the complex plane that are located closer to the point $z = -2$ than to the point $z = 2$. It is clear that this property is possessed by all the points of the plane that lie to the left of the imaginary axis and only by those points. In the figure given below, the shaded portion of the complex plane represents the set of points satisfying $|2 + z| < |2 - z|$.



(3) The given condition is

$$2 \le |z-(1-2i)| < 3$$

A complex number z satisfies the given condition if and only if its distance from the point $1-2i$ is greater than or equal to 2 but less than 3. Such points lie in the interior and on the inner boundary of the ring formed by two concentric circles with centers at the point $1-2i$ and the radii $r=2$ and $R=3$. The required set is indicated by the shaded portion of the figure at the right side.



Next, we will turn our attention to general equations of certain geometrical figures in the complex plane, in terms of a complex variable.

THEOREM 3.6 The general equation of a straight line in the complex plane is

$$\bar{l}z + l\bar{z} + m = 0$$

where l is a non-zero complex number and m is a real number.

PROOF Let $l = a+ib$ be a non-zero complex number and m a real number. Consider the equation

$$\bar{l}z + l\bar{z} + m = 0$$

Let $z = x+iy$ be an arbitrary point on this curve. Then

$$(\overline{a+ib})(x+iy) + (a+ib)(\overline{x+iy}) + m = 0$$

Therefore

$$(a-ib)(x+iy) + (a+ib)(x-iy) + m = 0$$

Solving we get

$$2ax + 2by + m = 0, \quad a \ne 0 \quad \text{or} \quad b \ne 0 \text{ (since } l \ne 0)$$

This represents a straight line in the plane. Conversely, if $px+qy+r=0$ is a straight line, where p, q, r are reals and $p \ne 0$ or $q \ne 0$ and if $z = x+iy$ is a point on this line, then

$$p\left(\frac{z+\bar{z}}{2}\right) + q\left(\frac{z-\bar{z}}{2i}\right) + r = 0$$

Therefore

$$pz + p\bar{z} - qiz + qi\bar{z} + 2r = 0$$

$$(p-qi)z + (p+qi)\bar{z} + 2r = 0$$

By taking $l = p+qi$ and $m = 2r$, the above equation takes the form

$$\bar{l}z + l\bar{z} + m = 0$$

Note that $l \ne 0$, since $p \ne 0$ or $q \ne 0$. ■

THEOREM 3.7 In the complex plane the equation of the line joining the points z_1 and z_2 is

$$\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$$

PROOF Let the points z_1 and z_2 be A and B, respectively. Then $P(z)$ is a point on the line AB if and only if A, P and B are collinear which implies

$$\arg\left(\frac{z_1 - z}{z_2 - z}\right) = 0 \quad \text{or} \quad \pi$$

$$\Leftrightarrow \frac{z_1 - z}{z_2 - z} \text{ is pure real}$$

$$\Leftrightarrow \frac{z_1 - z}{z_2 - z} = \frac{\bar{z}_1 - \bar{z}}{\bar{z}_2 - \bar{z}}$$

$$\Leftrightarrow (z_1 - z)(\bar{z}_2 - \bar{z}) = (z_2 - z)(\bar{z}_1 - \bar{z})$$

$$\Leftrightarrow \begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$$

■

QUICK LOOK 6

1. The complex number $(z_1 - z_2)/(\bar{z}_1 - \bar{z}_2)$ is called the *complex slope* of the line joining z_1 and z_2.
2. For any two points z_1 and z_2 on the straight line $\bar{l}z + l\bar{z} + m = 0$ (where l is a non-zero complex number and m is a real number), the complex number $(z_1 - z_2)/(\bar{z}_1 - \bar{z}_2)$ is equal to $-l/\bar{l}$ and hence $-l/\bar{l}$ is the complex slope of the line $\bar{l}z + l\bar{z} + m = 0$.

THEOREM 3.8 The equation of the perpendicular bisector of the line segment joining the points z_1 and z_2 is

$$(\bar{z}_1 - \bar{z}_2)z + (z_1 - z_2)\bar{z} + z_2\bar{z}_2 - z_1\bar{z}_1 = 0$$

PROOF Let $A(z_1)$ and $B(z_2)$ be the given points and L be the perpendicular bisector of the line segment AB. Then $P(z)$ is point on L. This implies

$$PA = PB$$

$$\Leftrightarrow |z - z_1| = |z - z_2|$$

$$\Leftrightarrow (z - z_1)(\bar{z} - \bar{z}_1) = (z - z_2)(\bar{z} - \bar{z}_2)$$

$$\Leftrightarrow (z_1 - z_2)\bar{z} + (\bar{z}_1 - \bar{z}_2)z + z_2\bar{z}_2 - z_1\bar{z}_1 = 0$$

■

In the following theorem we obtain a necessary and sufficient condition for two points in the complex plane to be images of each other in a given straight line in the same plane.

THEOREM 3.9 Two points z_1 and z_2 are images of each other in the line $\bar{l}z + l\bar{z} + m = 0$ $(0 \neq l \in \mathbb{C}$ and $m \in \mathbb{R})$ if and only if $\bar{l}z_1 + l\bar{z}_2 + m = 0$.

PROOF Suppose that z_1 and z_2 are images of each other in the line $\bar{l}z + l\bar{z} + m = 0$. Then this line is the perpendicular bisector of the line segment joining z_1 and z_2. By Theorem 3.7, the equation of the perpendicular bisector is

$$(\bar{z}_1 - \bar{z}_2)z + (z_1 - z_2)\bar{z} + z_2\bar{z}_2 - z_1\bar{z}_1 = 0$$

Therefore

$$\frac{\bar{l}}{\bar{z}_1 - \bar{z}_2} = \frac{l}{z_1 - z_2} = \frac{m}{z_2\bar{z}_2 - z_1\bar{z}_1} = k \quad \text{(say)}$$

Now,

$$\begin{aligned}\bar{l}z_1 + l\bar{z}_2 + m &= k(\bar{z}_1 - \bar{z}_2)z_1 + k(z_1 - z_2)\bar{z}_2 + (z_2\bar{z}_2 - z_1\bar{z}_1)k \\ &= k[\bar{z}_1 z_1 - \bar{z}_2 z_1 + z_1\bar{z}_2 - z_2\bar{z}_2 + z_2\bar{z}_2 - z_1\bar{z}_1] \\ &= k(0) = 0\end{aligned}$$

Conversely, suppose that $\bar{l}z_1 + l\bar{z}_2 + m = 0$. Let z be any point on the given line. Then

$$\bar{l}z + l\bar{z} + m = 0$$

and therefore

$$\bar{l}(z - z_1) + l(\bar{z} - \bar{z}_2) = 0$$

which implies that

$$|\bar{l}(z - z_1)| = |-l(\bar{z} - \bar{z}_2)|$$

and hence

$$|z - z_1| = |\bar{z} - \bar{z}_2| = |z - z_2|$$

That is, z is equidistant from both the points z_1 and z_2. Therefore the line $\bar{l}z + l\bar{z} + m = 0$ is the perpendicular bisector of the line segment joining z_1 and z_2. ■

THEOREM 3.10 The perpendicular distance of the straight line $\bar{l}z + l\bar{z} + m = 0$ $(0 \neq l \in \mathbb{C}$ and $m \in \mathbb{R})$ from a given point z_0 is

$$\left|\frac{\bar{l}z_0 + l\bar{z}_0 + m}{2l}\right|$$

PROOF Let $z = x + iy$, so that the equation of the given line is

$$(\bar{l} + l)x + i(\bar{l} - l)y + m = 0$$

which is a first degree equation in x and y with real coefficients. Therefore, the distance of the line from the point $z_0 = a + ib$ is

$$\left|\frac{(\bar{l} + l)a + i(\bar{l} - l)b + m}{\sqrt{(\bar{l} + l)^2 - (\bar{l} - l)^2}}\right| = \left|\frac{\bar{l}(a + ib) + l(a - ib) + m}{\sqrt{4l\bar{l}}}\right|$$

$$= \left|\frac{\bar{l}z_0 + \bar{z}_0 + m}{2l}\right|$$ ■

THEOREM 3.11 The general equation of a circle in the complex plane is

$$z\bar{z} + \bar{b}z + b\bar{z} + c = 0$$

where b is a complex number and c is a real number.

PROOF Let z_0 be a fixed point in the complex plane and r a non-negative real number. Then the equation

$$|z - z_0| = r$$

represents the locus of the point z whose distance from the point z_0 is the constant r. We know that this locus is a circle with centre at z_0 and radius r. This equation is equivalent to

$$(z - z_0)(\bar{z} - \bar{z}_0) = r^2$$

That is, $z\bar{z}+(-\bar{z}_0)z+(-z_0)\bar{z}+(z_0\bar{z}_0-r^2)=0$ which is of the form $z\bar{z}+\bar{b}z+b\bar{z}+c=0$, where $b=-z_0$ and $c=z\bar{z}_0-r^2$. On the other hand, any equation $z\bar{z}+\bar{b}z+b\bar{z}+c=0$ can be written as

$$(z+b)(\bar{z}+\bar{b})=b\bar{b}-c$$

That is,

$$|z+b|=\sqrt{b\bar{b}-c}$$

which represents a circle with center at $-b$ and radius $\sqrt{b\bar{b}-c}$. Note that $b\bar{b}$ and c are real numbers and $b\bar{b}-c>0$ or $=0$ or <0. ■

QUICK LOOK 7

1. Note that the circle $z\bar{z}+\bar{b}z+b\bar{z}+c=0$ is real or point circle or imaginary circle according as $b\bar{b}-c$ is a positive real number or $b\bar{b}=c$ or negative real number, respectively.
2. If $A(z_1)$ and $B(z_2)$ are points in the complex plane and $P(z)$ is a point on the line joining $A(z_1)$ and $B(z_2)$ dividing the line segment AB in the ratio $m:n\,(m+n\neq 0)$, then

$$z=\frac{mz_2+nz_1}{m+n}$$

3. If $A(z_1), B(z_2)$ and $C(z_3)$ are the vertices of a triangle, then the complex number $(z_1+z_2+z_3)/3$ represents the centroid of the triangle ABC.

Example 3.13

Find the center and radius of the circle

$$z\bar{z}-(2+3i)\bar{z}-(2-3i)z-3=0$$

Solution: This equation is of the form

$$z\bar{z}+\bar{b}z+b\bar{z}+c=0$$

where $b=-(2+3i)$ and $c=-3$. Therefore $-b(=2+3i)$ is the center of the circle and $\sqrt{b\bar{b}-c}$ $[=\sqrt{(2+3i)(2-3i)+3}$ $=\sqrt{16}=4]$ is the radius.

Example 3.14

If $2+i$ and $4+3i$ represent the extremities A and C, respectively, of a diagonal of a square $ABCD$, described in counterclock sense, then find the other two vertices B and D.

Solution: Let E be the intersection of the diagonals. Then E is represented by

$$\frac{(2+i)+(4+3i)}{1+1}=3+2i$$

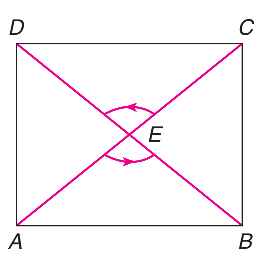


FIGURE 3.11 Example 3.14.

In Figure 3.11 ΔEAB is right angled at E. If z represents B, then

$$\frac{z-(3+2i)}{(2+i)-(3+2i)}=\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}=i$$

and therefore, $z=i(-1-i)+3+2i=4+i$. Similarly, from ΔECD, if z' represents D, then

$$\frac{z'-(3+2i)}{(4+3i)-(3+2i)}=i$$

and hence $z'=2+3i$.

Example 3.15

If z_1, z_2, z_3 and z_4 are the vertices of a square described in the counterclock sense, then express z_2 and z_4 in terms of z_1 and z_3, and z_1 and z_3 in terms of z_2 and z_4 (Figure 3.12).

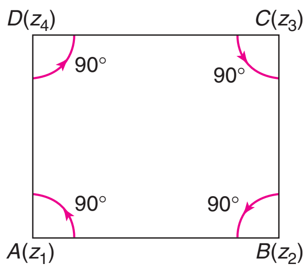


FIGURE 3.12 Example 3.15.

Solution: Rotate BC about B through 90° in anticlockwise sense. Then

$$\frac{z_1 - z_2}{z_3 - z_2} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$$

$$z_1 - z_2 = i(z_3 - z_2)$$

$$z_1 - iz_3 = (1-i)z_2$$

$$z_2 = \frac{1}{2}[(1+i)z_1 + (1-i)z_3]$$

Similarly

$$z_4 = \frac{1}{2}[(1-i)z_1 + (1+i)z_3]$$

$$z_3 = \frac{1}{2}[(1+i)z_2 + (1-i)z_4]$$

$$z_1 = \frac{1}{2}[(1-i)z_2 + (1+i)z_4]$$

Example 3.16

Let $\overline{l}_1 z + l_1 \overline{z} + m_1 = 0$ and $\overline{l}_2 z + l_2 \overline{z} + m_2 = 0$ be two straight lines in the complex plane. Then prove that

(1) the lines are parallel if and only if $l_1 \overline{l}_2 = l_2 \overline{l}_1$.

(2) the lines are perpendicular if and only if $l_1 \overline{l}_2 + l_2 \overline{l}_1 = 0$.

Solution: Writing $z = x + iy\,(x \text{ and } y \text{ real})$, the equations of the given straight lines are transformed into

$$(\overline{l}_1 + l_1)x + i(\overline{l}_1 - l_1)y + m_1 = 0$$

and
$$(\overline{l}_2 + l_2)x + i(\overline{l}_2 - l_2)y + m_2 = 0$$

which are first degree equations with real coefficients [recall that $\overline{z} + z$ and $i(\overline{z} - z)$ are always real numbers for all complex numbers z]. Therefore, we can use the conditions for parallelness and perpendicularity as in two-dimentional geometry.

Calculations are left for students as an exercise.

Example 3.17

Let $\overline{l}z + l\overline{z} + m = 0$ be a straight line in the complex plane and $P(z_0)$ be a point in the plane. Then prove that

(1) the equation of the line passing through $P(z_0)$ and parallel to the given line is

$$\overline{l}(z - z_0) + l(\overline{z} - \overline{z}_0) = 0$$

(2) the equation of the line passing through $P(z_0)$ and perpendicular to the given line is

$$\overline{l}(z - z_0) - l(\overline{z} - \overline{z}_0) = 0$$

Solution: Let $Q(z)$ be any point on the given line.

(1) We have

$$\frac{-l}{\overline{l}} = \text{slope of the line} = \frac{z - z_0}{\overline{z} - \overline{z}_0}$$

which gives the required equation.

(2) If $Q(z)$ is any line passing through $P(z_0)$ and is perpendicular to the given line, then

$$\frac{z - z_0}{\overline{z} - \overline{z}_0} = \frac{l}{\overline{l}} \quad \text{(see Example 3.16)}$$

which gives the required equation.

Example 3.18

Find the foot of the perpendicular drawn from a point $P(z_0)$ onto to a line $\overline{l}z + l\overline{z} + m = 0$.

Solution: The given line is

$$\overline{l}z + l\overline{z} + m = 0 \tag{3.4}$$

The line passing through $P(z_0)$ and perpendicular to the given line is

$$\bar{l}(z - z_0) - l(\bar{z} - \bar{z}_0) = 0 \qquad (3.5)$$

The foot of the perpendicular from $P(z_0)$ satisfies both Eqs. (3.4) and (3.5). Therefore, eliminating $\bar{z}$ from Eqs. (3.4) and (3.5), we have

$$z = \frac{\bar{l}z_0 - l\bar{z}_0 - m}{2\bar{l}}$$

which is the foot of the perpendicular.

Example 3.19

Find the radius and the center of the circle

$$z\bar{z} + (2 - 3i)z + (2 + 3i)\bar{z} + 4 = 0$$

Solution: If $b = 2 + 3i$, then the given equation is

$$z\bar{z} + \bar{b}z + b\bar{z} + 4 = 0$$

This equation represents a circle with center at $-b$ $(= -2 - 3i)$ and radius $\sqrt{b\bar{b} - 4}$ $(= \sqrt{4 + 9 - 4} = 3)$.

Example 3.20

Prove that the equation $|z + 1| = \sqrt{2}\,|z + 1|$ represents a circle and find its center and radius.

Solution: The given equation is equivalent to

$$(z + 1)(\bar{z} + 1) = 2(z - 1)(\bar{z} - 1)$$

That is,

$$z\bar{z} + (-3)z + (-3)\bar{z} + 1 = 0$$

which represents a circle with centre at 3 [= (3, 0)] and radius $\sqrt{3^2 - 1} = 2\sqrt{2}$.

3.4 | The Trigonometric Form

In the previous section, we have noted that the real and imaginary parts of a complex number $z = a + ib$ can be expressed in terms of the modulus $|z| = r$ and argument θ as

$$a = r\cos\theta \quad \text{and} \quad b = r\sin\theta$$

Therefore, any non-zero complex number z can be expressed as

$$z = r(\cos\theta + i\sin\theta)$$

where r is the modulus of z and θ is an argument of z. This expression of a complex number is called the *trigonometric notation* or *trigonometric form* or *polar form* of z.

Let us recall that the expression $z = a + ib$, where a and b are real numbers and $i^2 = -1$, is called the *algebraic form* of z. To pass from algebraic form to trigonometric form, it is sufficient to find the modulus of a complex number and one of its arguments. Let us consider certain examples.

Example 3.21

Express the following complex numbers in trigonometric form:

(1) $z_1 = -1 - i$

(2) $z_2 = -2$

(3) $z_3 = i$

Solution:

(1) $|z_1| = \sqrt{2}$ and $\operatorname{Arg} z_1 = -3\pi/4$ and hence

$$z_1 = \sqrt{2}\left[\cos\left(\frac{-3\pi}{4}\right) + i\sin\left(\frac{-3\pi}{4}\right)\right]$$

(2) $|z_2| = 2$ and $\operatorname{Arg} z_2 = \pi$ and hence

$$z_2 = 2(\cos\pi + i\sin\pi)$$

(3) $|z_3| = 1$ and $\operatorname{Arg} z_3 = \pi/2$ and hence

$$z_3 = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)$$

Example 3.22

Express the following complex numbers in trigonometric form:

(1) $z_1 = 2\cos\left(\frac{7\pi}{4}\right) - 2i\sin\left(\frac{\pi}{4}\right)$

(2) $z_2 = -\cos\left(\frac{\pi}{17}\right) + i\sin\left(\frac{\pi}{17}\right)$

Solution: Note that in these cases, we need not find the modulus and arguments, although it is easy to find these. Instead, we will make use of the facts that

$$\cos\left(\frac{7\pi}{4}\right) = \cos\left(2\pi - \frac{\pi}{4}\right) = \cos\left(\frac{-\pi}{4}\right)$$

and $$-\sin\left(\frac{\pi}{4}\right) = \sin\left(\frac{-\pi}{4}\right)$$

Now, we have

$$z_1 = 2\left[\cos\left(\frac{-\pi}{4}\right) + i\sin\left(\frac{-\pi}{4}\right)\right]$$

$$\begin{aligned} z_2 &= -\cos\left(\frac{\pi}{17}\right) + i\sin\left(\frac{\pi}{17}\right) \\ &= \cos\left(\pi - \frac{\pi}{17}\right) + i\sin\left(\pi - \frac{\pi}{17}\right) \\ &= \cos\left(\frac{16\pi}{17}\right) + i\sin\left(\frac{16\pi}{17}\right) \end{aligned}$$

The operations of multiplication and division of complex numbers can be easily performed by transforming the given complex numbers into trigonometric form. We have already noted that the modulus of the product (quotient) of any two complex numbers is the product (quotient) of their moduli.

Now, let us turn our attention to the arguments of products and quotients.

THEOREM 3.12 The following hold for any two non-zero complex numbers z_1 and z_2.

1. $z_1 = z_2 \Leftrightarrow |z_1| = |z_2|$ and $\arg z_1 = \arg z_2$
2. $\arg(z_1 z_2) = \operatorname{Arg} z_1 + \operatorname{Arg} z_2 + 2n\pi,\ n \in \mathbb{Z}$
3. $\arg\left(\frac{1}{z_2}\right) = -\operatorname{Arg} z_2 + 2n\pi,\ n \in \mathbb{Z}$
4. $\arg\left(\frac{z_1}{z_2}\right) = \operatorname{Arg} z_1 - \operatorname{Arg} z_2 + 2n\pi,\ n \in \mathbb{Z}$

PROOF First let us express the given non-zero complex numbers z_1 and z_2 in trigonometric form. Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $r_1 > 0, -\pi < \theta_1 \le \pi$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, $r_2 > 0, -\pi < \theta_2 \le \pi$. That is, $|z_1| = r_1, |z_2| = r_2$, $\operatorname{Arg} z_1 = \theta_1$ and $\operatorname{Arg} z_2 = \theta_2$.

1. This part is clear since $\arg z_1 = \operatorname{Arg} z_1 + 2n\pi, n \in \mathbb{Z}$.
2. Consider the product,

$$\begin{aligned} z_1 z_2 &= r_1(\cos\theta_1 + i\sin\theta_1) r_2(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2[(\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2) + i(\cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2)] \\ &= r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \end{aligned}$$

and therefore

$$|z_1 z_2| = r_1 r_2 \quad \text{and} \quad \arg(z_1 z_2) = \theta_1 + \theta_2 + 2n\pi = \operatorname{Arg} z_1 + \operatorname{Arg} z_2 + 2n\pi, \quad n \in \mathbb{Z}$$

3. This follows from the fact that

$$\frac{1}{\cos\theta + i\sin\theta} = \frac{\cos\theta - i\sin\theta}{(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)}$$
$$= \frac{\cos\theta - i\sin\theta}{\cos^2\theta + \sin^2\theta}$$
$$= \cos(-\theta) + i\sin(-\theta)$$

Therefore,

$$\arg\left(\frac{1}{z_2}\right) = -\operatorname{Arg} z_2 + 2n\pi, \quad n \in \mathbb{Z}$$

4. It follows from (2) and (3). ■

Example 3.23

Let

$$z_1 = \sqrt{2}\left[\cos\left(\frac{11\pi}{4}\right) + i\sin\left(\frac{11\pi}{4}\right)\right]$$

and $$z_2 = \sqrt{8}\left[\cos\left(\frac{3\pi}{8}\right) + i\sin\left(\frac{3\pi}{8}\right)\right]$$

Find $z_1 z_2$ and z_1/z_2.

Solution: First note that

$$\frac{11\pi}{4} = 2\pi + \frac{3\pi}{4}$$

Now, $|z_1| = \sqrt{2}$ and $|z_2| = \sqrt{8}$, therefore

$$\operatorname{Arg} z_1 = \frac{3\pi}{4} \quad \text{and} \quad \operatorname{Arg} z_2 = \frac{3\pi}{8}$$

Therefore, $|z_1 z_2| = |z_1||z_2| = \sqrt{2}\sqrt{8} = 4$. Now

$$\arg(z_1 z_2) = \operatorname{Arg} z_1 + \operatorname{Arg} z_2 + 2n\pi, \quad n \in \mathbb{Z}$$
$$= \frac{3\pi}{4} + \frac{3\pi}{8} + 2n\pi, \quad n \in \mathbb{Z}$$
$$= \frac{9\pi}{8} + 2n\pi, \quad n \in \mathbb{Z}$$
$$= \frac{-7\pi}{8} + 2(n+1)\pi, \quad n \in \mathbb{Z}$$
$$= \frac{-7\pi}{8} + 2m\pi, \quad m \in \mathbb{Z}$$

and hence

$$\operatorname{Arg}(z_1 z_2) = \frac{-7\pi}{8}$$

Therefore

$$z_1 z_2 = 4\left[\cos\left(\frac{-7\pi}{8}\right) + i\sin\left(\frac{-7\pi}{8}\right)\right]$$
$$= 4\left[\cos\left(\frac{7\pi}{8}\right) - i\sin\left(\frac{7\pi}{8}\right)\right]$$

Also,

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} = \frac{\sqrt{2}}{\sqrt{8}} = \frac{1}{2}$$

$$\arg\left(\frac{z_1}{z_2}\right) = \operatorname{Arg} z_1 - \operatorname{Arg} z_2 + 2n\pi, \quad n \in \mathbb{Z}$$
$$= \frac{3\pi}{4} - \frac{3\pi}{8} + 2n\pi, \quad n \in \mathbb{Z}$$
$$= \frac{3\pi}{8} + 2n\pi, \quad n \in \mathbb{Z}$$

Therefore,

$$\operatorname{Arg}\left(\frac{z_1}{z_2}\right) = \frac{3\pi}{8}$$

and hence

$$\frac{z_1}{z_2} = \frac{1}{2}\left[\cos\left(\frac{3\pi}{8}\right) + i\sin\left(\frac{3\pi}{8}\right)\right]$$

3.5 | De Moivre's Theorem

In the previous section, we have derived formulas for the product and quotient of two complex numbers in trigonometric form. The formula for the product of two complex numbers can be extended to the case of n factors by mathematical induction. As a special case, we have the following.

THEOREM 3.13 (DE MOIVRE'S THEOREM) For any real number θ and any positive integer n,

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

PROOF We prove this by induction on n. If $n = 1$, this is trivial. Now, let $n > 1$ and assume that

$$(\cos\theta + i\sin\theta)^{n-1} = \cos[(n-1)\theta] + i\sin[(n-1)\theta]$$

Now, consider

$$\begin{aligned}(\cos\theta + i\sin\theta)^n &= (\cos\theta + i\sin\theta)^{n-1}(\cos\theta + i\sin\theta)\\ &= [\cos\{(n-1)\theta\} + i\sin\{(n-1)\theta\}](\cos\theta + i\sin\theta)\\ &= [\cos\{(n-1)\theta\}\cos\theta - \sin\{(n-1)\theta\}\sin\theta]\\ &\quad + i[\cos\{(n-1)\theta\}\sin\theta + \cos\theta\sin\{(n-1)\theta\}]\\ &= \cos[(n-1)\theta + \theta] + i\sin[(n-1)\theta + \theta]\\ &= \cos(n\theta) + i\sin(n\theta)\end{aligned}$$

■

COROLLARY 3.2 For all real numbers θ and for all integers n,

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

PROOF For $n = 0$, this is obvious. Let $n < 0$. First observe that

$$\begin{aligned}\frac{1}{\cos\theta + i\sin\theta} &= \frac{\cos\theta - i\sin\theta}{(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)}\\ &= \frac{\cos(-\theta) + i\sin(-\theta)}{\cos^2\theta + \sin^2\theta}\\ &= \cos(-\theta) + i\sin(-\theta)\end{aligned}$$

Now,

$$\begin{aligned}(\cos\theta + i\sin\theta)^n &= [(\cos\theta + i\sin\theta)^{-n}]^{-1}\\ &= \frac{1}{(\cos\theta + i\sin\theta)^{-n}}\\ &= \frac{1}{\cos(-n\theta) + i\sin(-n\theta)}\\ &= \cos(n\theta) + i\sin(n\theta)\end{aligned}$$

since $-n > 0$ and by Theorem 3.13. ■

In the following we demonstrate the use of De Moivre's Theorem in expressing certain powers of complex numbers with natural exponents in algebraic form.

Example 3.24

Express the number $z = (i - \sqrt{3})^{13}$ in algebraic form.

Solution: First we write the given number in trigonometric form and then pass to the algebraic form. Let $w = i - \sqrt{3}$. Then $|w| = \sqrt{1+3} = 2$ and $\operatorname{Arg} w = 5\pi/6$. Therefore

$$w = 2\left[\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right]$$

and hence

$$z = w^{13} = 2^{13}\left[\cos\left(13\cdot\frac{5\pi}{6}\right) + i\sin\left(13\cdot\frac{5\pi}{6}\right)\right]$$

$$= 2^{13}\left[\cos\left(\frac{65\pi}{6}\right) + i\sin\left(\frac{65\pi}{6}\right)\right]$$

$$= 2^{13}\left[\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right]$$

Thus

$$(i - \sqrt{3})^{13} = 2^{13}\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = -2^{12}\sqrt{3} + 2^{12}i$$

Next, let us find the root of a given degree of a complex number.

Roots of Degree n

DEFINITION 3.12 If z and w are complex numbers and n a positive integer such that $z^n = w$, then z is called a ***root of degree n*** or ***nth root*** of the number w and is denoted by $\sqrt[n]{w}$. Roots of degree 2 or 3 are called ***square roots*** or ***cube roots***, respectively.

For example, i and $-i$ are both square roots of -1. In general, to extract a root of degree n of a complex number w, it is sufficient to solve the equation $z^n = w$. If $w = 0$, then the equation $z^n = w$ has exactly one solution, namely $z = 0$. The case $w \neq 0$ is dealt with in the following.

THEOREM 3.14 Let w be a non-zero complex number and n a positive integer. Then the equation $z^n = w$ has n solutions.

PROOF First we represent z and w in the trigonometric form. Let

$$z = r(\cos\theta + i\sin\theta) \quad \text{and} \quad w = s(\cos\alpha + i\sin\alpha)$$

The equation $z^n = w$ takes the form

$$r^n(\cos(n\theta) + i\sin(n\theta)) = s(\cos\alpha + i\sin\alpha)$$

Two complex numbers are equal if and only if their moduli are equal and argument differ by an integral multiple of 2π. Therefore,

$$r^n = s \quad \text{and} \quad n\theta = \alpha + 2m\pi, \quad m \in \mathbb{Z}$$

or

$$r = \sqrt[n]{s} \quad \text{and} \quad \theta = \frac{\alpha}{n} + \frac{2\pi}{n}m, \quad m \in \mathbb{Z}$$

Thus, all the solutions of the equation $z^n = w$ can be written as follows:

$$z_m = \sqrt[n]{s}\left[\cos\left(\frac{\alpha}{n} + \frac{2\pi}{n}m\right) + i\sin\left(\frac{\alpha}{n} + \frac{2\pi}{n}m\right)\right], \quad m \in \mathbb{Z}$$

It can be easily seen that z_m for $m = 0, 1, \ldots, n-1$ are different. For $m \geq n$, we cannot obtain any other complex numbers different from $z_0, z_1, \ldots, z_{n-1}$. For example, for $m = n$, we get

$$z_n = \sqrt[n]{s}\left[\cos\left(\frac{\alpha}{n} + 2\pi\right) + i\sin\left(\frac{\alpha}{n} + 2\pi\right)\right]$$
$$= \sqrt[n]{s}\left[\cos\left(\frac{\alpha}{n}\right) + i\sin\left(\frac{\alpha}{n}\right)\right] = z_0$$

It can be seen that $z_{n+k} = z_k$ for all $k \geq 0$. Thus, these are exactly n roots of degree n of the number w and they are all obtained from the formula

$$z_m = \sqrt[n]{s}\left[\cos\left(\frac{\alpha}{n} + \frac{2\pi}{n}m\right) + i\sin\left(\frac{\alpha}{n} + \frac{2\pi}{n}m\right)\right], \quad \text{for } m = 0, 1, 2, \ldots, n-1$$

It can be seen from the above formula that all the roots of degree n of the number w have one and the same moduli but distinct arguments differing from each other by $(2\pi/n)m$, where m is some integer.

QUICK LOOK 8

1. All the roots of degree n of the complex number w correspond to the points of the complex plane lying at the vertices of a regular n-gon inscribed in a circle of radius $\sqrt[n]{|w|}$ with centre at the point $z = 0$.
2. Usually the expression $\sqrt[n]{w}$ is to be understood as the set of all roots of degree n of w. For example, $\sqrt{-1}$ is understood to be the set consisting of two complex numbers i and $-i$. Sometimes, $\sqrt[n]{w}$ is understood as a root of degree n of w. In such cases, it must be indicated what value of the root is meant.

Theorem 3.13 paves a way to formulate and prove the most general version of the De Moivre's Theorem in the following. If z_0 is a solution of the equation $z^n = w$, then let us agree to write z_0 as $w^{1/n}$. Therefore $w^{1/n}$ has n values. In particular, if w is any complex number and $r = m/n$, where m and n are integers and $n > 0$, then $w^{1/n}$ has n values.

THEOREM 3.15 (DE MOIVRE'S THEOREM FOR RATIONAL INDEX) For any real number θ and any rational number r,

$$(\cos\theta + i\sin\theta)^r = \cos(r\theta) + i\sin(r\theta)$$

PROOF Let θ be a real number and $r = n/m$, where n and m are integers and $m > 0$. Then

$$[\cos(r\theta) + i\sin(r\theta)]^m = \left[\cos\left(\frac{n}{m}\theta\right) + i\sin\left(\frac{n}{m}\theta\right)\right]^m$$
$$= \cos(n\theta) + i\sin(n\theta) \quad \text{(by Theorem 3.13)}$$
$$= (\cos\theta + i\sin\theta)^n \quad \text{(by Corollary 3.2)}$$

Therefore $\cos(r\theta) + i\sin(r\theta)$ is the mth root of $(\cos\theta + i\sin\theta)^n$ or is a value of $[(\cos\theta + i\sin\theta)^n]^{1/m}$. Thus $\cos(r\theta) + i\sin(r\theta)$ is a value of $(\cos\theta + i\sin\theta)^r$.

Example 3.25

Find all the squares of the roots of the equation

$$x^{11} - x^7 + x^4 - 1 = 0$$

Solution: We have $x^{11} - x^7 + x^4 - 1 = x^7(x^4 - 1) + x^4 - 1 = (x^7 + 1)(x^4 - 1)$. If z is a root of $x^{11} - x^7 + x^4 - 1 = 0$, then z must be either a seventh root of -1 or a fourth root of unity; that is, $z = (-1)^{1/7}$ or $z = (1)^{1/4}$. Therefore, $z^2 = (-1)^{2/7}$ or $z^2 = (1)^{2/4} = (1)^{1/2} = 1$ or -1. That is

$$z^2 = 1 \quad \text{or} \quad -1 \quad \text{or} \quad (1)^{1/7}$$

This implies that z^2 is either a square root of 1 or a seventh root of 1.

Example 3.26

Find all the values of $\sqrt[6]{-64}$.

Solution: First, we should express $w = -64$ in trigonometric form:

$$w = -64 = 64(\cos\pi + i\sin\pi)$$

Now, if z_m are the values of $\sqrt[6]{-64}$, then

$$z_m = \sqrt[6]{64}\left[\cos\left(\frac{\pi}{6} + \frac{2\pi}{6}m\right) + i\sin\left(\frac{\pi}{6} + \frac{2\pi}{6}m\right)\right]$$

for $m = 0, 1, 2, 3, 4$ and 5. Therefore,

$$z_0 = 2\cos\left(\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = \sqrt{3} + i$$

$$z_1 = 2\left[\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right] = 2i$$

$$z_2 = 2\left[\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right] = -\sqrt{3} + i$$

$$z_3 = 2\left[\cos\left(\frac{7\pi}{6}\right) + i\sin\left(\frac{7\pi}{6}\right)\right] = -\sqrt{3} - i$$

$$z_4 = 2\left[\cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right)\right] = -2i$$

$$z_5 = 2\left[\cos\left(\frac{11\pi}{6}\right) + i\sin\left(\frac{11\pi}{6}\right)\right] = \sqrt{3} - i$$

These lie on the circle of radius 2 with center at $z = 0$ and form vertices of a regular hexagon.

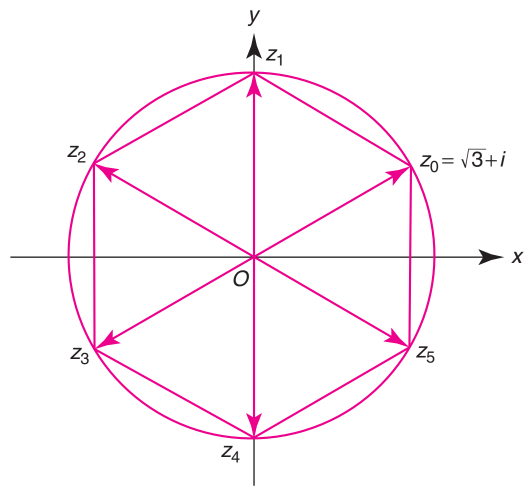


FIGURE 3.13 Example 3.27.

In the following, we express the square roots of a given complex number and nth roots of unity in algebraic form. These are straight forward verifications.

Square Root of a Complex Number

The square roots of $z = a + ib$ are given as

$$\pm\left[\sqrt{\frac{|z| + a}{2}} + i\sqrt{\frac{|z| - a}{2}}\right] \quad \text{if } b > 0 \tag{3.6a}$$

and

$$\pm\left[\sqrt{\frac{|z| + a}{2}} - i\sqrt{\frac{|z| - a}{2}}\right] \quad \text{if } b < 0 \tag{3.6b}$$

QUICK LOOK 9

1. The square roots of i are $\pm(1 + i/\sqrt{2})$
2. The square roots of $-i$ are $\pm(1 - i/\sqrt{2})$
3. The square roots of $-7 - 24i$ are

$$\pm\left[\sqrt{\frac{25 - 7}{2}} - i\sqrt{\frac{25 + 7}{2}}\right] = \pm(3 - 4i)$$

Cube Roots of Unity

The cube roots of unity (solutions of $z^3 = 1$) are

$$1, \quad \frac{-1 + i\sqrt{3}}{2} \quad \text{and} \quad \frac{-1 - i\sqrt{3}}{2}$$

Usually $(-1 + i\sqrt{3})/2$ is denoted by w. Note that $1, w, w^2$ are the cube roots of unity.

Properties of Cube Roots of Unity

Let $w \neq 1$ be a cube root of unity; that is

$$w = \frac{1}{2}(-1 \pm i\sqrt{3})$$

Then the following properties are satisfied by w.

QUICK LOOK 10

1. $1 + w + w^2 = 0$
2. $w^{3n} = 1, w^{3n+1} = w$ and $w^{3n+2} = w^2$ for any integer n
3. $\overline{w} = w^2$
4. $(\overline{w})^2 = w$
5. The values $1, w, w^2$ represent the vertices of an equilateral triangle inscribed in a circle of radius 1 with center at $z = 0$, one vertex being on positive real axis.
6. For any real numbers a, b and c,

$$a + bw + cw^2 = 0 \Leftrightarrow a = b = c$$

In the following we list certain important relations concerning the cube roots $1, w$ and w^2 of unity.

Relations Concerning the Cube Roots of Unity

Let $w \neq 1$ be a cube root of unity. The following relations hold good. Here x is any real or complex variable.

1. $1 + x + x^2 = (x - w)(x - w^2)$
2. $1 - x + x^2 = (x + w)(x + w^2)$
3. $x^2 + xy + y^2 = (x - yw)(x - yw^2)$
4. $x^2 - xy + y^2 = (x + yw)(x + yw^2)$
5. $x^3 + y^3 = (x + y)(x + yw)(x + yw^2)$
6. $x^3 - y^3 = (x - y)(x - yw)(x - yw^2)$
7. $x^2 + y^2 + z^2 - xy - yz - zx = (x + yw + zw^2)(x + yw^2 + zw)$
8. $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x + yw + zw^2)(x + yw^2 + zw)$

Example 3.27

If α, β and γ are roots of $x^3 - 3x^2 + 3x + 7 = 0$, then find the value of

$$\frac{\alpha - 1}{\beta - 1} + \frac{\beta - 1}{\gamma - 1} + \frac{\gamma - 1}{\alpha - 1}$$

in terms of a cube root of unity.

Solution: The given equation $x^3 - 3x^2 + 3x + 7 = 0$ can be expressed as

$$(x-1)^3 + 8 = 0$$

That is,

$$(x-1)^3 = (-2)^3$$

$$\left(\frac{x-1}{-2}\right)^3 = 1$$

$$\frac{x-1}{-2} = 1, w, w^2$$

which are the cube roots of unity. Therefore $-1, 1 - 2w, 1 - 2w^2$ are the roots of the given equation. Let $\alpha = -1$, $\beta = 1 - 2w$ and $\gamma = 1 - 2w^2$. Then $\alpha - 1 = -2$, $\beta - 1 = -2w$ and $\gamma - 1 = -2w^2$. Hence

$$\frac{\alpha - 1}{\beta - 1} + \frac{\beta - 1}{\gamma - 1} + \frac{\gamma - 1}{\alpha - 1} = \frac{-2}{-2w} + \frac{-2w}{-2w^2} + \frac{-2w^2}{-2}$$

$$= \frac{1}{w} + \frac{1}{w} + w^2$$

$$= w^2 + w^2 + w^2 = 3w^2$$

Properties of nth Roots of Unity

Let n be a positive integer and

$$\alpha = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$$

Then all the properties in "Quick Look 11" hold.

QUICK LOOK 11

1. $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$ are all the nth roots of unity and

$$\alpha^r = \cos\left(\frac{2\pi}{n}r\right) + i\sin\left(\frac{2\pi}{n}r\right) \quad \text{for } 0 \le r < n$$

2. $$1 + \alpha + \alpha^2 + \cdots + \alpha^{n-1} = \frac{1-\alpha^n}{1-\alpha} = \frac{1-[\cos(2\pi) + i\sin(2\pi)]}{1-\alpha}$$

and therefore

$$1 + \alpha + \alpha^2 + \cdots + \alpha^{n-1} = 0$$

3. The summation

$$\sum_{r=0}^{n-1} \alpha^r = 0$$

and hence

$$\sum_{r=0}^{n-1}\left[\cos\left(\frac{2\pi}{n}r\right) + i\sin\left(\frac{2\pi}{n}r\right)\right] = 0$$

and therefore

$$\sum_{r=0}^{n-1}\cos\left(\frac{2\pi}{n}r\right) = 0 = \sum_{r=0}^{n}\sin\left(\frac{2\pi}{n}r\right)$$

4. The terms $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$ represent the vertices of a regular n-gon inscribed in the unit circle with center at the origin, one vertex being on the positive real axis.

Example 3.28

If $1, w, w^2, \ldots, w^{n-1}$ are all the nth roots of unity, find the value of the product

$$(5-w)(5-w^2)\cdots(5-w^{n-1})$$

Solution: The polynomial $x^n - 1$ has n roots, namely $1, w, w^2, \ldots, w^{n-1}$ and hence

$$x^n - 1 = (x-1)(x-w)(x-w^2)\cdots(x-w^{n-1})$$

Therefore

$$\frac{x^n-1}{x-1} = (x-w)(x-w^2)\cdots(x-w^{n-1})$$

This is true for all numbers $x \ne 1$. Substituting $x = 5$, we get that

$$(5-w)(5-w^2)\cdots(5-w^{n-1}) = \frac{5^n-1}{4}$$

3.6 | Algebraic Equations

Most gratifying fact about complex numbers is that any polynomial (algebraic) equation with complex numbers as coefficients has a solution. We will discuss the same in this section.

DEFINITION 3.13 Let $f(z) = a_0 + a_1z + a_2z^2 + \cdots + a_nz^n$, $a_n \ne 0$ and $a_0, a_1, \ldots, a_n$ complex numbers. Then

$$f(z) = 0$$

is called an ***algebraic equation*** of degree n. Any algebraic equation of degree 2 is called a *quadratic* equation. A complex number z_0 is called a ***solution*** or ***root*** of the equation $f(z) = 0$ if $f(z_0) = 0$; that is,

$$a_0 + a_1z_0 + a_2z_0^2 + \cdots + a_nz_0^n = 0$$

QUICK LOOK 12

1. $2 + i + z = 0$ is an algebraic equation of degree 1 and $z_0 = -2 - i$ is the only root of this.
2. $z^2 - 1 = 0$ is an algebraic equation of degree 2 and $z_0 = 1$ and $z_1 = -1$ are the roots of the equation $z^2 - 1 = 0$.
3. $i + iz^2 + z^3 + z^5 = 0$ is an algebraic equation of degree 5 and $z_0 = i$ is a root of this equation.
4. $32(1 - i)z + iz^7 = 0$ is an algebraic equation of degree 7 and $z_0 = 0$ is a root of this equation. In addition to $z_0 = 0$, any root of the equation $z^6 = 32(1 + i)$ must be a root of the given equation and hence there must be six more roots for the given equation.

The general form of an algebraic equation of the first degree is

$$a_0 + a_1 z = 0, \quad a_1 \neq 0$$

Such an equation possesses exactly one solution $z_0 = -a_0/a_1$. An equation of the second degree is generally written as

$$a_0 + a_1 z + a_2 z^2 = 0, \quad a_2 \neq 0$$

To solve this, we transform the equation as follows:

$$a_2\left(z^2 + \frac{a_1}{a_2}z + \frac{a_0}{a_2}\right) = 0$$

$$a_2\left[\left(z + \frac{a_1}{2a_2}\right)^2 + \frac{a_0}{a_2} - \frac{a_1^2}{4a_2^2}\right] = 0$$

$$a_2\left[\left(z + \frac{a_1}{2a_2}\right)^2 - \frac{a_1^2 - 4a_0a_2}{4a_2^2}\right] = 0$$

and find the roots of

$$\left(z + \frac{a_1}{2a_2}\right)^2 = \frac{a_1^2 - 4a_0a_2}{4a_2}$$

as

$$z = \frac{-a_1}{2a_2} + \frac{\sqrt{a_1^2 - 4a_0a_2}}{2a_2}$$

that is,

$$z = \frac{-a_1 + \sqrt{D}}{2a_2}$$

where $D = a_1^2 - 4a_0a_2$. D is called the ***discriminant*** of the equation $a_0 + a_1 z + a_2 z^2 = 0$. $\sqrt{D}$ is to be understood as all the values of the square root of D. The formula

$$z = \frac{-a_1 + \sqrt{D}}{2a_2}$$

for the roots of a quadratic equation has the same form as in the case when the coefficients of the equation are real numbers and the solutions are thought in the set of real numbers. But in as much as in the set of complex numbers the operation of extracting a square root is meaningful for any complex number, the restriction $D > 0$ becomes superfluous. Moreover, the restriction loses sense since the discriminant D may prove to be not a real number, and the concepts of "greater than" and "less than" are not defined for such numbers. Thus, in the set of complex numbers, any quadratic equation is always solvable. If the discriminant D is zero, then the equation has one root. If $D \neq 0$, the equation has two roots that are given by the formula

$$z_0 = \frac{-a_1 + \sqrt{D}}{2a_2}$$

This is known as the ***standard formula*** for the roots of a quadratic equation.

Example 3.29

Solve the equations:

(1) $z^2 + 3z + 3 = 0$

(2) $z^2 - 8z - 3iz + 13 + 13i = 0$

Solution:

(1) By the formula for the roots of a quadratic equation, the roots of $z^2 + 3z + 3 = 0$ are given by

$$z = \frac{-3 + \sqrt{9 - 12}}{2} = \frac{-3 + \sqrt{-3}}{2}$$

Since $\sqrt{-3} = \pm i\sqrt{3}$, it follows that

$$z_1 = \frac{-3 + i\sqrt{3}}{2} \quad \text{and} \quad z_2 = \frac{-3 - i\sqrt{3}}{2}$$

are the solutions of the equation $z^2 + 3z + 3 = 0$.

(2) The given equation can be written as

$$(13 + 13i) + (-8 - 3i)z + z^2 = 0$$

By the standard formula for the roots of a quadratic equation, we get that

$$z = \frac{8 + 3i + \sqrt{(8 + 3i)^2 - 4(13 + 13i)}}{2} = \frac{8 + 3i + \sqrt{3 - 4i}}{2}$$

are the solutions of the given equation. To find all the values of $\sqrt{3 - 4i}$, we can use the formula given in Eqs. (3.6a) and (3.6b). But another technique is much simpler. Let us put

$$\sqrt{3 - 4i} = x + iy$$

Then $3 - 4i = x^2 - y^2 + i(2xy)$ and therefore

$$x^2 - y^2 = 3 \quad \text{and} \quad xy = -2$$

x and y being real numbers. This system of simultaneous equations has two real solutions, $x = 2, y = -1$ and $x = -2, y = 1$. Therefore

$$\sqrt{3 - 4i} = 2 - i \quad \text{or} \quad -2 + i$$

Thus,

$$z_1 = \frac{8 + 3i + 2 - i}{2} = 5 + i$$

and

$$z_2 = \frac{8 + 3i - 2 + i}{2} = 3 + 2i$$

are the solutions of the given quadratic equation.

Solving algebraic equations of degree $n > 2$ is much more difficult. However, the great German mathematician Carl Gauss proved the following celebrated theorem in 1799. In view of its importance and in honor of Gauss, the theorem is named after Gauss and is popularly known as the ***Fundamental Theorem of Algebra***. Its proof is beyond the scope of this book and hence not given here.

Fundamental Theorem of Algebra

Every algebraic equation has atleast one root in the set of complex numbers.

The following theorem is an important consequence of the fundamental theorem of algebra.

Theorem 3.16 Every algebraic equation of degree n has exactly n roots, including the repeatitions (multiplicities) of the roots, in the set of complex numbers.

Proof Let

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n, \quad a_n \neq 0$$

where $a_0, a_1, a_2, \ldots, a_n$ are all complex numbers. Then $f(z) = 0$ is an algebraic equation of degree n. It can be proved that, for any complex number w,

$$f(z) = (z - w)\, g(z)$$

for some polynomial $g(z)$ with complex coefficients if and only if w is a root of the equation $f(z) = 0$; that is, $f(w) = 0$. This, together with the fundamental theorem of algebra, gives us that

$$f(z) = a_n(z - z_1)^{r_1}(z - z_2)^{r_2} \cdots (z - z_k)^{r_k}$$

where $z_1, z_2, \ldots, z_k$ are distinct complex numbers and $r_1, r_2, \ldots, r_k$ are positive integers such that

$$r_1 + r_2 + \cdots + r_k = n$$

Therefore, if follows that $z_1, z_2, \ldots, z_k$ are all the distinct roots of the equation $f(z) = 0$. Here we say that z_i is a root of multiplicity r_i. If we agree to count the root of the equation as many times as is its multiplicity, then we get that the equation $f(z) = 0$ has $r_1 + r_2 + \cdots + r_k\ (= n)$ roots in the set of complex numbers. ■

Theorem 3.16 and the fundamental theorem of algebra are both typical theorems of existence. They both present a comprehensive solution of the problem on the existence of roots of an arbitrary algebraic equation; but, unfortunately they do not say how to find these roots. The root of the first-degree equation

$$a_0 + a_1 z = 0$$

is determined by the formula

$$z = -\frac{a_0}{a_1}$$

and the roots of the second-degree equation

$$a_0 + a_1 z + a_2 z^2 = 0$$

are determined by the formula

$$z = \frac{-a_1 + \sqrt{D}}{2a_2}$$

where D is the determinant defined by

$$D = a_1^2 - 4a_0 a_2$$

The analogous formulae for the roots of third- and fourth-degree equations are so cumbersome that they are avoided. There is no general method for finding the roots of algebraic equations of degree greater than 4. The absence of a general method does not prevent us, of course, from finding all the roots in certain special cases, depending on the specific nature of the equation. For example, in Theorem 3.14, we discussed a method to find all the roots of the equation

$$a_0 + a_n z^n = 0$$

The following theorem often helps us in solving algebraic equations with integral coefficients.

THEOREM 3.17 Let $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$, $a_n \neq 0$, where $a_0, a_1, a_2, \ldots, a_n$ are all integers. If k is an integer and is a root of $f(z) = 0$, then k is a divisor of a_0.

PROOF Let k be an integer and $f(k) = 0$. That is, $a_0 + a_1 k + a_2 k^2 + \cdots + a_n k^n = 0$, and hence $a_0 = k(-a_1 - a_2 k - \cdots - a_n k^{n-1})$. Since k and $a_1, a_2, \ldots, a_n$ are integers, so is $-a_1 - a_2 k - \cdots - a_n k^{n-1}$. Therefore k is a divisor of a_0. ■

Example 3.30

Solve the equation

$$z^3 - z - 6 = 0$$

Solution: Note that all the coefficients are integers. By considering the divisors of the constant term –6 and by using Theorem 3.17, we get that 2 is the only integral root of $z^3 - z - 6 = 0$. By the usual division of $z^3 - z - 6$ by $z - 2$, we get that

$$(z-2)(z^2+2z+3) = z^3 - z - 6$$

Therefore, the roots of $z^3 - z - 6 = 0$ are precisely the roots of $z^2 + 2z + 3 = 0$ and 2. The roots of $z^2 + 2z + 3 = 0$ are

$$\frac{-2 \pm \sqrt{4-12}}{2}$$

Thus, $z_1 = 2$, $z_2 = -1 + \sqrt{2}i$ and $z_3 = -1 - \sqrt{2}i$ are all the roots of the equation $z^3 - z - 6 = 0$.

Example 3.31

Solve the equation

$$72 - 36z - 26z^2 + 13z^3 + 2z^4 - z^5 = 0$$

Solution: Let $f(z) = 72 - 36z - 26z^2 + 13z^3 + 2z^4 - z^5$. Note that all the coefficients are integers. Consider the constant term 72. Testing the divisors of the constant term 72, we find that $z_1 = 2$ and $z_2 = -2$ are roots of the given equation. By dividing $f(z)$ with $(z-2)(z+2) = z^2 - 4$, we get that

$$f(z) = (z^2 - 4)(-18 + 9z + 2z^2 - z^3)$$

Again -3 and 3 are roots of $-18 + 9z + 2z^2 - z^3$ and $-18 + 9z + 2z^2 - z^3 = (z^2 - 9)(z - 2)$. Therefore,

$$\begin{aligned} f(z) &= (z-2)(z+2)(z-3)(z+3)(z-2) \\ &= (z-2)^2(z+2)(z-3)(z+3) \end{aligned}$$

Thus the roots of $f(z) = 0$ are $3, -3, -2$ and 2 and the root 2 is of multiplicity 2.

WORKED-OUT PROBLEMS

Single Correct Choice Type Questions

1. If

$$\frac{\sqrt{3}+i}{2} = \frac{a+i}{a-i}$$

and a is a real number, then a is

(A) $1/2 + \sqrt{3}$ (B) $1/2 - 4\sqrt{3}$
(C) $2 - \sqrt{3}$ (D) $1/2 - \sqrt{3}$

Solution: The equation

$$\frac{\sqrt{3}+i}{2} = \frac{a+i}{a-i}$$

implies that

$$(\sqrt{3}+i)(a-i) = 2a + 2i$$

that is, $a(\sqrt{3} - 2 + i) = (\sqrt{3}+2)i - 1$. Therefore

$$\begin{aligned} a &= \frac{(\sqrt{3}+2)i - 1}{\sqrt{3} - 2 + i} \\ &= \frac{[(\sqrt{3}+2)i - 1][(\sqrt{3}-2) - i]}{[(\sqrt{3}-2)+i][(\sqrt{3}-2)-i]} \\ &= \frac{(3-4)i - \sqrt{3} + 2 + i + \sqrt{3} + 2}{(\sqrt{3}-2)^2 + 1} \\ &= \frac{4}{8 - 4\sqrt{3}} = \frac{1}{2-\sqrt{3}} \end{aligned}$$

Answer: (D)

2. If z_1, z_2 are complex numbers such that $\text{Re}(z_1) = |z_1 - 2|$, $\text{Re}(z_2) = |z_2 - 2|$ and $\arg(z_1 - z_2) = \pi/3$, then $\text{Im}(z_1 - z_2) =$

(A) $2/\sqrt{3}$ (B) $4/\sqrt{3}$ (C) $2\sqrt{3}$ (D) $\sqrt{3}$

Solution: Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then

$$x_1^2 = (x_1 - 2)^2 + y_1^2 \quad \text{and} \quad x_2^2 = (x_2 - 2)^2 + y_2^2$$

Therefore

$$4x_1 = y_1^2 + 4 \quad \text{and} \quad 4x_2 = y_2^2 + 4$$

On subtraction we get

$$4(x_1 - x_2) = y_1^2 - y_2^2 = (y_1 + y_2)(y_1 - y_2)$$

Hence

$$y_1 + y_2 = \frac{4(x_1 - x_2)}{y_1 - y_2} \tag{3.7}$$

Also $\arg(z_1 - z_2) = \pi/3$. Therefore

$$\tan\frac{\pi}{3} = \frac{y_1 - y_2}{x_1 - x_2}$$

$$\sqrt{3} = \frac{y_1 - y_2}{x_1 - x_2} \tag{3.8}$$

From (3.7) and (3.8), we have

$$\operatorname{Im}(z_1 + z_2) = y_1 + y_2 = \frac{4}{\sqrt{3}}$$

Answer: (B)

3. The smallest positive integer n for which $[(1+i)/(1-i)]^n = 1$ is

(A) 2 (B) 4 (C) 6 (D) 7

Solution: We have

$$\frac{1+i}{1-i} = \frac{(1+i)^2}{2} = i \quad \text{and} \quad i^n = 1 \quad \text{for } n = 4, 8, 12, \ldots$$

Therefore, the smallest positive integer n for which

$$\left(\frac{1+i}{1-i}\right)^n = 1 \text{ is } 4$$

Answer: (B)

4. Let C be the set of all complex numbers and

$$R = \left\{(z_1, z_2) \in C \times C : \frac{z_1 - z_2}{z_1 + z_2} \text{ is real}\right\}$$

Then, on C, R is a

(A) reflexive relation (B) symmetric relation
(C) transitive relation (D) equivalence relation

Solution: Since $(0,0) \notin R$, R is not reflexive, we have

$$\begin{aligned}(z_1, z_2) \in R &\Rightarrow \frac{z_1 - z_2}{z_1 + z_2} \text{ is real}\\ &\Rightarrow \frac{z_2 - z_1}{z_1 + z_2} \text{ is real}\\ &\Rightarrow (z_2, z_1) \in R\end{aligned}$$

Therefore R is symmetric.

Since $(0, z) \in R$ and $(z, 0) \in R$, but $(0,0) \notin R$, therefore R is not transitive. Hence R is not an equivalence relation.

Answer: (B)

5. If $z = x + iy$ is such that $|z-4| < |z-2|$, then

(A) $x > 0, y > 0$
(B) $x < 0, y > 0$
(C) $x > 2, y > 3$
(D) $x > 3$ and y is any real number

Solution: We have

$$\begin{aligned}|z-4| < |z-2| &\Leftrightarrow |z-4|^2 < |z-2|^2\\ &\Leftrightarrow (x-4)^2 + y^2 < (x-2)^2 + y^2\\ &\Leftrightarrow 12 < 4x\\ &\Leftrightarrow 3 < x\end{aligned}$$

Answer: (D)

6. If

$$x + iy = \frac{3}{2 + \cos\theta + i\sin\theta}$$

then $x^2 + y^2 =$

(A) $4x - 3$ (B) $3x - 4$ (C) $4x + 3$ (D) $3x + 4$

Solution:

$$\begin{aligned}x + iy &= \frac{3(2 + \cos\theta - i\sin\theta)}{(2+\cos\theta)^2 + \sin^2\theta}\\ &= \frac{3(2+\cos\theta) + i(-3\sin\theta)}{5 + 4\cos\theta}\end{aligned}$$

Comparing the real and imaginary parts we get

$$x = \frac{3(2+\cos\theta)}{5+4\cos\theta}, \quad y = \frac{-3\sin\theta}{5+4\cos\theta}$$

Squaring and adding values of x and y, we get

$$\begin{aligned}x^2 + y^2 &= \frac{9(2+\cos\theta)^2 + 9\sin^2\theta}{(5+4\cos\theta)^2}\\ &= \frac{9(5+4\cos\theta)}{(5+4\cos\theta)^2} = \frac{9}{5+4\cos\theta}\end{aligned}$$

Also

$$4x - 3 = \frac{12(2+\cos\theta)}{5+4\cos\theta} - 3 = \frac{9}{5+4\cos\theta}$$

Therefore

$$x^2 + y^2 = 4x - 3$$

Answer: (A)

7. If

$$x + iy = \sqrt{\frac{3+i}{1+3i}}$$

then $(x^2 + y^2)^2$ equals

(A) 0 (B) 2 (C) 3 (D) 1

Solution:

$$x^2 - y^2 + 2ixy = \frac{3+i}{1+3i} = \frac{(3+i)(1-3i)}{1+9}$$

Comparing the real and imaginary parts we get

$$x^2 - y^2 = \frac{6}{10} \quad \text{and} \quad 2xy = \frac{-8}{10}$$

Now

$$(x^2+y^2)^2=(x^2-y^2)^2+4x^2y^2$$

$$=\left(\frac{6}{10}\right)^2+\left(\frac{-8}{10}\right)^2=\frac{9}{25}+\frac{16}{25}=1$$

Answer: (D)

8. If a is a positive real number, $z=a+2i$ and $z|z|-az+1=0$, then

(A) z is pure imaginary
(B) $a^2=2$
(C) $a^2=4$
(D) no such complex number exists

Solution:

$$z|z|-az+1=0$$

$$(a+2i)\sqrt{a^2+4}=a(a+2i)-1$$

$$=a^2-1+2ai$$

This implies

$$a\sqrt{a^2+4}=a^2-1 \quad\text{and}\quad 2\sqrt{a^2+4}=2a$$

which gives $a^2=a^2-1$, which is absurd.

Answer: (D)

9. If $|z_1+z_2|=|z_1|+|z_2|$, then one of the values of $\arg(z_2/z_1)$ is

(A) 0 (B) π (C) $\pi/2$ (D) 3π

Solution: If $|z_1+z_2|=|z_1|+|z_2|$, then z_1, z_2 and origin are collinear and z_1, z_2 lie on same side to origin and hence $\arg(z_2/z_1)=2n\pi$. Then 0 is one of the values of $\arg(z_2/z_1)$.

Answer: (A)

Alternate Method:

Let $z_1=r_1(\cos\theta_1+i\sin\theta_1)$ and $z_2=r_2(\cos\theta_2+i\sin\theta_2)$. Then $|z_1+z_2|=|z_1|+|z_2|$ implies

$$(r_1\cos\theta_1+r_2\cos\theta_2)^2+(r_1\sin\theta_1+r_2\sin\theta_2)^2=(r_1+r_2)^2$$

That is

$$\cos(\theta_1-\theta_2)=1$$

Therefore

$$\theta_1-\theta_2=2n\pi$$

10. If w is a cube root of unity and $w\neq 1$, then

$$\cos\left(\sum_{k=1}^{10}(k-w)(k-w^2)\frac{\pi}{450}\right)$$

is equal to

(A) 1 (B) –1 (C) 0 (D) 1/2

Solution: We have

$$(k-w)(k-w^2)=k^2+k+1$$

Therefore

$$\cos\left(\sum_{k=1}^{10}(k-w)(k-w^2)\frac{\pi}{450}\right)=\cos\left(\sum_{k=1}^{10}(k^2+k+1)\frac{\pi}{450}\right)$$

$$=\cos\left(450\cdot\frac{\pi}{450}\right)$$

$$=\cos\pi=-1$$

Answer: (B)

11. If $\alpha=-1+i\sqrt{3}$ and n is a positive integer which is not a multiple of 3, then

$$\alpha^{2n}+2^n\alpha^n+2^{2n}=$$

(A) 1 (B) –1 (C) 0 (D) α^2

Solution: We have

$$\alpha^{2n}+2^n\alpha^n+2^{2n}=2^{2n}\left[\left(\frac{\alpha}{2}\right)^{2n}+\left(\frac{\alpha}{2}\right)^n+1\right]$$

$$=2^{2n}(w^{2n}+w^n+1)\left[\because \frac{\alpha}{2}=\frac{-1}{2}+\frac{i\sqrt{3}}{2} \quad\text{and}\quad \left(\frac{\alpha}{2}\right)^3=1\right]$$

$$=2^{2n}(0)=0 \text{ (since 3 does not divide } n)$$

Answer: (C)

12. If $\arg(z)<0$, then $\arg(-z)-\arg(z)=$

(A) π (B) $-\pi$ (C) $\pi/2$ (D) $-\pi/2$

Solution: Let $\arg(z)=\theta<0$. Then $-\pi<\theta<0$ and therefore $0<\theta+\pi<\pi$. Hence

$$\arg(-z)=\pi+\theta$$

$$\arg(-z)-\arg(z)=\pi$$

Answer: (A)

13. Let $w\neq 1$ be a cube root of unity and

$$E=2(1+w)(1+w^2)+3(2w+1)(2w^2+1)$$
$$+4(3w+1)(3w^2+1)+\cdots$$
$$+(n+1)(nw+1)(nw^2+1)$$

Then E is equal to

(A) $\dfrac{n^2(n+1)^2}{4}$ (B) $\dfrac{n^2(n+1)^2}{4}+n$

(C) $\dfrac{n^2(n+1)^2}{4}-n$ (D) $\dfrac{n^2(n+1)^2}{4}-(n+1)$

Solution: We have

$$(k+1)(kw+1)(kw^2+1) = (k+1)(k^2-k+1) = k^3+1$$

Therefore,

$$E = \sum_{k=1}^{n}(k^3+1) = \sum_{k=1}^{n} k^3 + n = \frac{n^2(n+1)^2}{4} + n$$

Answer: (B)

14. If $|z-3+2i| \le 4$, then the absolute difference between the maximum and minimum values of $|z|$ is

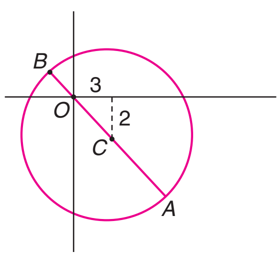
(A) $2\sqrt{11}$ (B) $3\sqrt{11}$ (C) $2\sqrt{13}$ (D) $3\sqrt{13}$

Solution: Let $C = 3-2i$ be the center of the circle $|z-3+2i| = 4$. Join the origin to C and let it meet the circle in A and B (see figure).

$$\begin{aligned}\text{Least value of } |z| &= OB \\ &= CB - OC \\ &= 4 - \sqrt{3^2+2^2} \\ &= 4 - \sqrt{13}\end{aligned}$$

$$\text{Maximum value of } |z| = OA = 4 + \sqrt{13}$$

The absolute difference between the maximum and minimum values of $|z|$ is $2\sqrt{13}$.



Answer: (C)

15. If z_1, z_2 and z_3 represent the vertices of a triangle whose circumcenter is at the origin, then the complex number representing the orthocenter of the triangle is

(A) $\dfrac{1}{z_1} + \dfrac{1}{z_2} + \dfrac{1}{z_3}$ (B) $z_1 + z_2 + z_3$

(C) $\dfrac{1}{z_1 z_2} + \dfrac{1}{z_2 z_3} + \dfrac{1}{z_3 z_1}$ (D) $z_1 z_2 + z_2 z_3 + z_3 z_1$

Solution: It is known that every complex number can be represented by means of a vector in the Argand's plane. If A and B represent the complex numbers z_1 and z_2, respectively, then the vector $\overline{AB}$ represents the complex number $z_2 - z_1$. (These matters will be discussed in detail later in Volume II) Correspondingly, if $\vec{a}, \vec{b}, \vec{c}$ are the position vectors of the points $A(z_1), B(z_2), C(z_3)$, then the orthocenter of the triangle ABC is represented by $\vec{a} + \vec{b} + \vec{c}$ and hence by $z_1 + z_2 + z_3$. Note that the origin is the circumcenter.

Answer: (B)

16. If z_1, z_2 and z_3 are the vertices of an equilateral triangle and z_0 be its orthocenter, then $z_1^2 + z_2^2 + z_3^2 = kz_0^2$, where k is equal to

(A) 3 (B) 2 (C) 6 (D) 9

Solution: In an equilateral triangle, the circumcenter, the centroid and the orthocenter are one and the same point. Therefore

$$\begin{aligned} z_0 &= \frac{z_1+z_2+z_3}{3} \\ 9z_0^2 &= z_1^2 + z_2^2 + z_3^2 + z(z_1 z_2 + z_2 z_3 + z_3 z_1) \\ &= 3(z_1^2 + z_2^2 + z_3^2) \end{aligned}$$

[since $z_1^2 + z_2^2 + z_3^2 = \sum z_1 z_2$ (by Problem 7 of Multiple Correct Choice Type Questions in Worked-Out Problems section)]. Therefore

$$z_1^2 + z_2^2 + z_3^2 = 3z_0^2 \quad \text{and} \quad k = 3$$

Answer: (A)

17. Let $z_1 = 10 + 6i$ and $z_2 = 4 + 6i$. If z is any complex number such that the argument of $(z - z_1)/(z - z_2)$ is $\pi/4$, then $|z - 7 - 9i|$ is equal to

(A) $2\sqrt{3}$ (B) $3\sqrt{2}$ (C) 3 (D) 2

Solution: Let $z = x + iy$, $x, y \in \mathbb{R}$. Then $z - z_1 = (x-10) + i(y-6)$ and $z - z_2 = (x-4) + i(y-6)$. Therefore

$$\begin{aligned} \frac{z-z_1}{z-z_2} &= \frac{(x-10)+i(y-6)}{(x-4)+i(y-6)} \\ &= \frac{[(x-10)+i(y-6)][(x-4)-i(y-6)]}{(x-4)^2+(y-6)^2} \end{aligned}$$

Therefore

$$\text{Real part of } \frac{z-z_1}{z-z_2} = \frac{(x-10)(x-4)+(y-6)^2}{(x-4)^2+(y-6)^2}$$

$$\begin{aligned}\text{Imaginary part of } \frac{z-z_1}{z-z_2} &= \frac{(x-4)(y-6)-(x-10)(y-6)}{(x-4)^2+(y-6)^2} \\ &= \frac{6(y-6)}{(x-4)^2+(y-6)^2}\end{aligned}$$

Now,

$$\frac{\pi}{4} = \arg\left(\frac{z-z_1}{z-z_2}\right) = \tan^{-1}\left[\frac{6(y-6)}{(x-10)(x-4)+(y-6)^2}\right]$$

Therefore

$$(x-10)(x-4)+(y-6)^2 = 6(y-6)$$
$$x^2+y^2-14x-18y+112=0$$

Now,

$$\begin{aligned}|z-7-9i|^2 &= (x-7)^2+(y-9)^2\\ &= x^2-14x+y^2-18y+130\\ &= -112+130=18\end{aligned}$$

Therefore

$$|z-7-9i| = \sqrt{18} = 3\sqrt{2}$$

Answer: (B)

18. If $x=\cos\alpha + i\sin\alpha$ and $y=\cos\beta + \mathrm{i}\sin\beta$, then$(x-y)/(x+y)$ is equal to

(A) $i\tan\left(\dfrac{\alpha-\beta}{2}\right)$ (B) $-i\tan\left(\dfrac{\alpha-\beta}{2}\right)$

(C) $i\tan\left(\dfrac{\alpha+\beta}{2}\right)$ (D) $-i\tan\left(\dfrac{\alpha+\beta}{2}\right)$

Solution: We know that

$$\begin{aligned}\frac{x-y}{x+y} &= \frac{(\cos\alpha-\cos\beta)+i(\sin\alpha-\sin\beta)}{(\cos\alpha+\cos\beta)+i(\sin\alpha+\sin\beta)}\\ &= \frac{-2\sin[(\alpha+\beta)/2]\sin[(\alpha-\beta)/2] + 2i\cos[(\alpha+\beta)/2]\sin[(\alpha-\beta)/2]}{2\cos[(\alpha+\beta)/2]\cos[(\alpha-\beta)/2] + 2i\sin[(\alpha+\beta)/2]\cos[(\alpha-\beta)/2]}\\ &= \frac{i\sin[(\alpha-\beta)/2]\{\cos[(\alpha+\beta)/2]+i\sin[(\alpha+\beta)/2]\}}{\cos[(\alpha-\beta)/2]\{\cos[(\alpha+\beta)/2]+i\sin[(\alpha+\beta)/2]\}}\\ &= i\tan\left(\frac{\alpha-\beta}{2}\right)\end{aligned}$$

Answer: (A)

19. If $|z_1-1|<1$, $|z_2-2|<2$ and $|z_3-3|<3$, then $|z_1+z_2+z_3|$

(A) is less than 6 (B) is greater than 6

(C) is less than 12 (D) lies between 6 and 12

Solution: We have

$$\begin{aligned}|z_1+z_2+z_3-6+6| &\le |z_1+z_2+z_3-6|+6\\ &= |(z_1-1)+(z_2-2)+(z_3-3)|+6\\ &\le |z_1-1|+|z_2-2|+|z_3-3|+6\\ &< 1+2+3+6=12\end{aligned}$$

Answer: (C)

20. If $x^2+x+1=0$ then the value of

$$\left(x+\frac{1}{x}\right)^2+\left(x^2+\frac{1}{x^2}\right)^2+\left(x^3+\frac{1}{x^3}\right)^2+\cdots+\left(x^{27}+\frac{1}{x^{27}}\right)^2 \text{ is}$$

(A) 27 (B) 18 (C) 54 (D) –27

Solution: $x^2+x+1=0 \Rightarrow x$ is a non-real cube root of unity. Let $x=w\ne 1$ be a cube root of unity. Then $w^3=1$ and $1+w+w^2=0$. The given equation, thus, becomes

$$\begin{aligned}&\left(w+\frac{1}{w}\right)^2+\left(w^2+\frac{1}{w^2}\right)^2+\left(w^3+\frac{1}{w^3}\right)^2+\cdots\left(w^{27}+\frac{1}{w^{27}}\right)^2\\ &= \left(\frac{w^2+1}{w}\right)^2+\left(\frac{w^4+1}{w}\right)^2+(1+1)^2+\left(w+\frac{1}{w}\right)^2+\cdots\\ &\quad +(1+1)^2\\ &= 9\left[\left(\frac{-w}{w}\right)^2+\left(\frac{-w^2}{w^2}\right)^2+(1+1)^2\right]\\ &= 9(1+1+4)=54\end{aligned}$$

Answer: (C)

21. If z is a complex number and $i=\sqrt{-1}$, then the minimum possible value of $|z|^2+|z-3|^2+|z-6i|^2$ is

(A) 15 (B) 30 (C) 20 (D) 45

Solution: Let $z=x+iy$. Then

$$\begin{aligned}|z^2|+|z-3|^2+|z-6i|^2 &= x^2+y^2+(y-3)^2+y^2+x^2+(y-6)^2\\ &= 3(x^2+y^2)-6x-12y+45\\ &= 3[(x-1)^2+(y-2)^2]+30\ge 30\end{aligned}$$

(equality holds when $z=1+2i$). Therefore, the minimum value is 30.

Answer: (B)

22. The curve in the complex plane given by the equation $\mathrm{Re}(1/z)=1/4$ is a

(A) vertical line intersecting with the x-axis at $(4, 0)$

(B) a circle with radius 2 and centre at $(2, 0)$

(C) circle with unit radius

(D) straight line not passing through the origin

Solution: Let $z=x+iy$, where x and y are reals. Then

$$\begin{aligned}\mathrm{Re}\left(\frac{1}{z}\right)=\frac{1}{4} &\Rightarrow \mathrm{Re}\left(\frac{x-iy}{x^2+y^2}\right)=\frac{1}{4}\\ &\Rightarrow \frac{x}{x^2+y^2}=\frac{1}{4}\\ &\Rightarrow x^2+y^2=4x\\ &\Rightarrow (x-2)^2+y^2=4=2^2\end{aligned}$$

This is the equation of the circle with radius 2 and center at $(2, 0)$.

Answer: (B)

23. The origin and the points represented by the roots of the equation $z^2+mz+n=0$ form the vertices of an equilateral triangle if and only if

(A) $m^2 = 3n$ (B) $n^2 = 3m$
(C) $3m^2 = n$ (D) $3n^2 = m$

Solution: The points z_1, z_2 and z_3 are the vertices of an equilateral triangle if and only if

$$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$$

(see Problem 7 of Multiple Correct Choice Type Questions in Worked-Out Problems section). Let z_1 and z_2 be the roots of $z^2 + mz + n = 0$. Therefore $z_1 + z_2 = -m$, $z_1 z_2 = n$. Now z_1, z_2 and the origin form an equilateral triangle if and only if

$$z_1^2 + z_2^2 = z_1 z_2$$
$$\Leftrightarrow (z_1 + z_2)^2 = 3z_1 z_2$$
$$\Leftrightarrow (-m)^2 = 3n$$

Answer: (A)

24. Let $z = x + iy$, where x and y are real. The points (x, y) in the plane, for which $(z+i)/(z-i)$ is purely imaginary, lie on

(A) a straight line
(B) a circle
(C) a curve whose equation is of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ a \neq 1,\ b \neq 1$$

(D) a curve whose equation is of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Solution: We have

$$\frac{z+i}{z-i} = \frac{x + i(y+1)}{x + i(y-1)} = \frac{[x + i(y+1)][x - i(y-1)]}{x^2 + (y-1)^2}$$

This is pure imaginary if and only if

$$\mathrm{Re}\left(\frac{z+i}{z-i}\right) = 0$$
$$\Leftrightarrow \frac{x^2 + (y^2 - 1)}{x^2 + (y-1)^2} = 0$$
$$\Leftrightarrow x^2 + y^2 = 1$$

Therefore (x, y) lie on the circle $|z| = 1$.

Answer: (B)

25. Let z_1 and z_2 be given by

$$z_1 = \left(\frac{2 + i\sqrt{5}}{2 - i\sqrt{5}}\right)^{10} \quad \text{and} \quad z_2 = \left(\frac{2 - i\sqrt{5}}{2 + i\sqrt{5}}\right)^{10}$$

Then $|z_1 + z_2|$ is equal to

(A) $2\cos\left(20\cos^{-1}\dfrac{2}{3}\right)$ (B) $2\sin\left(10\cos^{-1}\dfrac{2}{3}\right)$

(C) $2\cos\left(10\cos^{-1}\dfrac{2}{3}\right)$ (D) $2\sin\left(20\cos^{-1}\dfrac{2}{3}\right)$

Solution: Adding the two we get

$$z_1 + z_2 = \frac{(2 + i\sqrt{5})^{20} + (2 - i\sqrt{5})^{20}}{9^{10}}$$

Suppose $2 + i\sqrt{5} = r(\cos\theta + i\sin\theta)$, so that $r = \sqrt{2^2 + 5} = 3$, $r\cos\theta = 2$ and $r\sin\theta = \sqrt{5}$. Therefore

$$\cos\theta = \frac{2}{3} \quad \text{and} \quad \sin\theta = \frac{\sqrt{5}}{3}$$

In this case

$$z_1 + z_2 = \frac{1}{9^{10}}[r^{20}\{\cos(20\theta) + i\sin(20\theta) + \cos(20\theta) - i\sin(20\theta)\}]$$
$$= \frac{r^{20}}{9^{10}} 2\cos(20\theta) = \frac{3^{20}}{9^{10}} 2\cos\left[20\cos^{-1}\left(\frac{2}{3}\right)\right]$$
$$= 2\cos\left[20\cos^{-1}\left(\frac{2}{3}\right)\right]$$

Answer: (A)

26. If $(1 + z)^n = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$, where $a_0, a_1, a_2, \ldots, a_n$, are real, then

$$(a_0 - a_2 + a_4 - a_6 + \cdots)^2 + (a_1 - a_3 + a_5 - a_7 + \cdots)^2 =$$

(A) 2^n (B) $a_0^2 + a_1^2 + a_2^2 + \cdots + a_n^2$
(C) 2^{n^2} (D) $2n^2$

Solution: Substitute $z = i$ on both sides. Then

$$(1 + i)^n = (a_0 - a_2 + a_4 - a_6 + \cdots) + i(a_1 - a_3 + a_5 - a_7 + \cdots)$$

Therefore

$$|1 + i|^{2n} = (a_0 - a_2 + a_4 - a_6 + \cdots)^2 + (a_4 - a_3 + a_5 - a_7 + \cdots)^2$$
$$2^n = (a_0 - a_2 + a_4 - a_6 + \cdots)^2 + (a_1 - a_3 + a_5 - a_7 + \cdots)^2$$

Answer: (A)

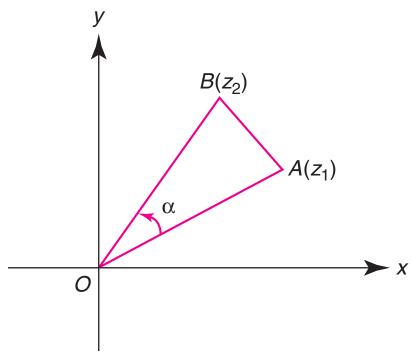
27. Let z_1 and z_2 be roots of the equation $z^2 + pz + q = 0$, where p, q may be complex numbers. Let A and B represent z_1 and z_2 in the complex plane. If $\angle AOB = \alpha \neq 0$ and $OA = OB$, where O is the origin, then

(A) $p^2 = 4q\cos^2\left(\dfrac{\alpha}{2}\right)$ (B) $p^2 = 4q\sin^2\left(\dfrac{\alpha}{2}\right)$

(C) $p^2 = -4q\cos^2\left(\dfrac{\alpha}{2}\right)$ (D) $q^2 = 4p\sin^2\left(\dfrac{\alpha}{2}\right)$

Solution: z_1 and z_2 are roots of $z^2 + pz + q = 0$. This implies $z_1 + z_2 = -p$ and $z_1 z_2 = q$. Now

$$\frac{z_2-0}{z_1-0}=\frac{OB}{OA}(\cos\alpha+i\sin\alpha)$$



Therefore

$$\frac{z_2}{z_1}=\cos\alpha+i\sin\alpha$$

$$\frac{z_2-z_1}{z_1}=-1+\cos\alpha+i\sin\alpha$$

This gives

$$(z_2-z_1)^2=z_1^2\left[-2\sin^2\frac{\alpha}{2}+2i\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\right]^2$$

$$=z_1^2\left(2i\sin\frac{\alpha}{2}\right)^2\left[\cos\frac{\alpha}{2}+i\sin\frac{\alpha}{2}\right]^2$$

$$=-4z_1^2\sin^2\frac{\alpha}{2}(\cos\alpha+i\sin\alpha)$$

$$=-4z_1^2\frac{z_2}{z_1}\sin^2\frac{\alpha}{2}=-4q\sin^2\frac{\alpha}{2}$$

Hence,

$$p^2=(z_1+z_2)^2=(z_1-z_2)^2+4z_1z_2$$

$$=-4q\sin^2\frac{\alpha}{2}+4q$$

$$=4q\left(1-\sin^2\frac{\alpha}{2}\right)=4q\cos^2\left(\frac{\alpha}{2}\right)$$

Answer: (A)

28. The continued product of all the four values of the complex number $(1+i)^{3/4}$ is

(A) $2^3(1+i)$ (B) $2(1-i)$
(C) $2(1+i)$ (D) $2^3(1-i)$

Solution: Let

$$z=1+i=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$$

Therefore

$$z^{3/4}=2^{3/8}\left[\cos\left(2k\pi+\frac{\pi}{4}\right)\frac{3}{4}+i\sin\left(2k\pi+\frac{\pi}{4}\right)\frac{3}{4}\right]$$

for $k=0,1,2,3$. The product of the values of this is equal to

$$2^{3/2}\left[\operatorname{cis}\left(\frac{\pi}{4}+\frac{9\pi}{4}+\frac{17\pi}{4}+\frac{25\pi}{4}\right)\frac{3}{4}\right]=2^{3/2}\operatorname{cis}\left(\frac{52\pi}{4}\cdot\frac{3}{4}\right)$$

$$=2^{3/2}\operatorname{cis}\frac{39\pi}{4}$$

$$=2^{3/2}\operatorname{cis}\left(9\pi+\frac{3\pi}{4}\right)$$

$$=2^{3/2}\operatorname{cis}\left(10\pi-\frac{\pi}{4}\right)$$

$$=2^{3/2}\left[\cos\left(10\pi-\frac{\pi}{4}\right)+i\sin\left(10\pi-\frac{\pi}{4}\right)\right]$$

$$=2^{3/2}\left[\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right]$$

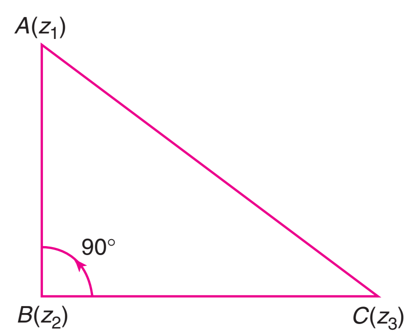
$$=2^{3/2}\left(\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)$$

$$=2(1-i)$$

Answer: (B)

29. If z_1, z_2 and z_3 are the vertices of a right-angled isosceles triangle, right-angled at the vertex z_2 (see figure), then $z_1^2+2z_2^2+z_3^2=kz_2(z_1+z_3)$, where the value of k is

(A) 0 (B) 1 (C) –2 (D) 2



Solution: Let A, B and C represent z_1, z_2 and z_3, respectively, described in counterclock sense. Therefore

$$\frac{z_1-z_2}{z_3-z_2}=\frac{BA}{BC}\operatorname{cis}\left(\frac{\pi}{2}\right)=i$$

$$(z_1-z_2)^2=-(z_3-z_2)^2$$

$$z_1^2 + z_2^2 - 2z_1z_2 = -z_3^2 - z_2^2 + 2z_2z_3$$
$$z_1^2 + 2z_2^2 + z_3^2 = 2z_2(z_1 + z_3)$$

This gives $k = 2$.

Answer: (D)

30. Let z_1, z_2 and z_3 be vertices of a triangle and $|z_1| = a$, $|z_2| = b$ and $|z_3| = c$ such that

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$$

Then

$$\arg\left(\frac{z_3}{z_2}\right) = k \arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)$$

where k is

(A) 0 (B) 1 (C) 2 (D) 3

Solution: We have

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$$

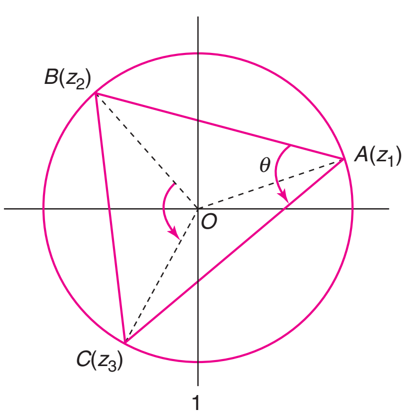
$$\Rightarrow 3abc - a^3 - b^3 - c^3 = 0$$

$$\Rightarrow a^3 + b^3 + c^3 - 3abc = 0$$

$$\Rightarrow (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) = 0$$

$$\Rightarrow (a + b + c)\frac{1}{2}((a - b)^2 + (b - c)^2 + (c - a)^2) = 0$$

Therefore $(a - b)^2 = 0 = (b - c)^2 = (c - a)^2$ and hence $a = b = c$ (since a, b, c are positive). This implies that z_1, z_2 and z_3 represent points on a circle with center at the origin. Suppose A, B and C represent z_1, z_2 and z_3, respectively, described in counterclock sense (see figure). If $\angle BAC = \theta$, then $\angle BOC = 2\theta$. In such case



$$\arg\left(\frac{z_3}{z_2}\right) = 2\theta = 2\arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)$$

Therefore $k = 2$.

Answer: (C)

31. Let $z = (\sqrt{3}/2) - (i/2)$. Then the smallest positive integer n such that $(z^{95} + i^{67})^{94} = z^n$ is

(A) 12 (B) 10 (C) 9 (D) 8

Solution: From the hypothesis we have

$$z = \frac{\sqrt{3}}{2} - \frac{i}{2} = i\left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right) = iw$$

where $w = (-1/2) - (i\sqrt{3}/2)$ which is a cube root of unity. Now, $z^{95} = (iw)^{95} = -iw^2$ (since $w^3 = 1$) and $i^{67} = i^3 = -i$. Therefore,

$$z^{95} + i^{67} = -i(1 + w^2) = (-i)(-w) = iw$$
$$(z^{95} + i^{67})^{94} = (iw)^{94} = i^2 w = -w$$

Now

$$-w = z^n = (iw)^n$$

$$\Rightarrow i^n \cdot w^{n-1} = -1$$

$$\Rightarrow n = 2, 6, 10, 14, \ldots \quad \text{and} \quad n - 1 = 3, 6, 9, \ldots$$

Therefore $n = 10$ is the required least positive integer.

Answer: (B)

32. The number of complex numbers z satisfying the conditions $|(z/\bar{z}) + (\bar{z}/z)| = 1$, $|z| = 1$ and $\arg z \in (0, 2\pi)$ is

(A) 1 (B) 2 (C) 4 (D) 8

Solution: It is given that $|z| = 1$ which implies that $z = \cos\theta + i\sin\theta, 0 \le \theta < 2\pi$:

$$\left|\frac{z}{\bar{z}} + \frac{\bar{z}}{z}\right| = 1$$

$$\Rightarrow 2|\cos 2\theta| = 1$$

$$\Rightarrow \cos 2\theta = \frac{1}{2} \quad \text{or} \quad \cos 2\theta = \frac{-1}{2}$$

Now

$$\cos 2\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

and

$$\cos 2\theta = -\frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$$

Answer: (D)

Multiple Correct Answer Type Questions

1. The complex number z that satisfies simultaneously the equations is

$$\left|\frac{z-4i}{z-2i}\right| = 1 \quad \text{and} \quad \left|\frac{z-8+3i}{z+3i}\right| = \frac{3}{5}$$

(A) $3+8i$ (B) $8+3i$ (C) $3+17i$ (D) $17+3i$

Solution:

$$\left|\frac{z-4i}{z-2i}\right| = 1 \Rightarrow |z-4i| = |z-2i|$$

Therefore, the point representing z in the Argand's plane is equidistant from the points (0, 2) and (0, 4). Hence, z lies on the line $y = 3$ and so

$$z = x + yi = x + 3i$$

Substituting $z = x + 3i$ in the second equation, we get that

$$\left|\frac{x+3i-8+3i}{x+3i+3i}\right| = \frac{3}{5}$$

$$\left|\frac{x-8+6i}{x+6i}\right| = \frac{3}{5}$$

Therefore

$$25[(x-8)^2+36] = 9(x^2+36)$$

$$16x^2 - 400x + 2176 = 0$$

$$x^2 - 25x + 136 = 0$$

$$(x-8)(x-17) = 0$$

$$x = 8, 17$$

Hence

$$z = 8+3i, 17+3i$$

Answers: (B), (D)

2. If z_1 and z_2 are complex numbers such that $|z_1+z_2|^2 = |z_1|^2 + |z_2|^2$, then

(A) $z_1\bar{z}_2$ is pure imaginary (B) $\bar{z}_1 z_2 + z_1\bar{z}_2 = 0$

(C) $\text{Arg}\left(\dfrac{z_1}{z_2}\right) = \pm\dfrac{\pi}{2}$ (D) $\text{Arg}\left(\dfrac{z_1}{z_2}\right) = \pm\dfrac{\pi}{2}$

Solution:

$$|z_1+z_2|^2 = |z_1|^2 + |z_2|^2$$

$$(z_1+z_2)(\bar{z}_1+\bar{z}_2) = z_1\bar{z}_1 + z_2\bar{z}_2$$

$$z_1\bar{z}_2 + z_2\bar{z}_1 = 0$$

$$\frac{z_1}{z_2} = -\left(\frac{\bar{z}_1}{\bar{z}_2}\right)$$

$z_1\,\bar{z}_2$ and z_1/z_2 are pure imaginary.

Answers: (A), (B), (C), (D)

3. If z_1 and z_2 are two complex numbers, then

(A) $2(|z_1|^2+|z_2|^2) = |z_1+z_2|^2 + |z_1-z_2|^2$

(B) $|z_1+\sqrt{z_1^2-z_2^2}| + |z_1-\sqrt{z_1^2-z_2^2}| = |z_1+z_2| + |z_1-z_2|$

(C) $\left|\dfrac{z_1+z_2}{2}+\sqrt{z_1z_2}\right| + \left|\dfrac{z_1+z_2}{2}-\sqrt{z_1\,z_2}\right| = |z_1|+|z_2|$

(D) $|z_1+z_2|^2 - |z_1-z_2|^2 = 2(z_1\bar{z}_2 + \bar{z}_1z_2)$

Solution: $|z_1+z_2|^2 = (z_1+z_2)(\bar{z}_1+\bar{z}_2)$

$$= |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + \bar{z}_1z_2$$

and $|z_1-z_2|^2 = (z_1-z_2)(\bar{z}_1-\bar{z}_2)$

Therefore

$$|z_1+z_2|^2 + |z_1-z_2|^2 = 2(|z_1|^2+|z_2|^2) \quad \text{(A is true)}$$

$$|z_1+z_2|^2 - |z_1-z_2|^2 = 2(z_1\bar{z}_2+\bar{z}_1z_2) \quad \text{(D is true)}$$

Now

$$(|z_1+\sqrt{z_1^2-z_2^2}| + |z_1-\sqrt{z_1^2-z_2^2}|)^2$$

$$= |z_1+\sqrt{z_1^2-z_2^2}|^2 + |z_1-\sqrt{z_1^2-z_2^2}|^2 + 2|z_1^2-(z_1^2-z_2^2)|$$

$$= 2(|z_1|^2 + |z_1^2-z_2^2|) + 2|z_2|^2$$

$$= 2(|z_1|^2+|z_2|^2) + 2|z_1^2-z_2^2|$$

$$= |z_1+z_2|^2 + |z_1-z_2|^2 + 2|z_1+z_2||z_1-z_2|$$

$$= (|z_1+z_2| + |z_1-z_2|)^2$$

Therefore

$$|z_1+\sqrt{z_1^2-z_2^2}| + |z_1-\sqrt{z_1^2-z_2^2}| = |z_1+z_2| + |z_1-z_2|$$

Hence (B) is true. Also

$$\left|\frac{z_1+z_2}{2}+\sqrt{z_1z_2}\right| + \left|\frac{z_1+z_2}{2}-\sqrt{z_1z_2}\right|$$

$$= \frac{1}{2}|\sqrt{z_1}+\sqrt{z_2}|^2 + \frac{1}{2}|\sqrt{z_1}-\sqrt{z_2}|^2$$

$$= \frac{1}{2}[2|\sqrt{z_1}|^2 + 2|\sqrt{z_2}|^2]$$

$$= |z_1| + |z_2|$$

Therefore (C) is true.

Answers: (A), (B), (C), (D)

4. If x and y are real numbers and

$$\frac{(1+i)x-2i}{3+i}+\frac{(2-3i)y+i}{3-i}=i$$

then

(A) $x=3$ (B) $y=1$ (C) $y=-1$ (D) $x=-3$

Solution: From the given equation, we get that

$$(3-i)[(1+i)x-2i]+(3+i)[(2-3i)y+i]=10i$$

Therefore

$$4x-2+i(2x-6)+9y-1+i(3-7y)=10i$$
$$4x+9y-3+(2x-7y-13)i=0$$
$$4x+9y=3 \quad \text{and} \quad 2x-7y=13$$

The two equations give $x=3$ and $y=-1$.

Answers: (A), (C)

5. The complex number(s) satisfying the equations

$$\left|\frac{z-12}{z-8i}\right|=\frac{5}{3} \quad \text{and} \quad \left|\frac{z-4}{z-8}\right|=1 \quad \text{is (are)}$$

(A) $6-8i$ (B) $6+17i$ (C) $6+8i$ (D) $6-17i$

Solution: Let $z=x+iy$

$$\left|\frac{z-4}{z-8}\right|=1$$
$$(x-4)^2+y^2=(x-8)^2+y^2$$
$$x=6$$

Therefore

$$z=6+iy$$

Now

$$\left|\frac{z-12}{z-8i}\right|=\frac{5}{3}$$
$$9(36+y^2)=25[36+(y-8)^2]$$
$$(y-8)(y-17)=0$$
$$y=8,17$$

Therefore

$$z=6+8i,\ 6+17i$$

Answers: (B), (C)

6. If x is a real number such that $0\le x\le 2\pi$ and

$$\frac{[\sin(x/2)+\cos(x/2)]+i\tan x}{1+2i\sin(x/2)}$$

is real, then the possible value(s) of x is (are)

(A) 0 (B) 2π (C) $\pi/4$ (D) $5\pi/4$

Solution: Let

$$\frac{[\sin(x/2)+\cos(x/2)]+i\tan x}{1+2i\sin(x/2)}$$

Then

$$z=\frac{[\sin(x/2)+\cos(x/2)+i\tan x][1-2i\sin(x/2)]}{1+4\sin^2(x/2)}$$

Suppose that z is real. Then $\operatorname{Im}(z)=0$. Therefore

$$\tan x-2\sin\frac{x}{2}\left[\sin\left(\frac{x}{2}\right)+\cos\left(\frac{x}{2}\right)\right]=0$$
$$\sin x-2\sin\frac{x}{2}\cos x\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)=0$$
$$\sin x-\sin x\cos x-2\sin^2\left(\frac{x}{2}\right)\cos x=0$$
$$\sin x(1-\cos x)-(1-\cos x)\cos x=0$$
$$(1-\cos x)(\sin x-\cos x)=0$$
$$\cos x=1 \quad \text{or} \quad \tan x=1$$

Therefore

$$x=2n\pi,\ x=n\pi+\frac{\pi}{4}, \quad n \text{ is an integer}$$

Since $0\le x\le 2\pi$, $x=0,\pi/4,2\pi,5\pi/4$.

Answers: (A), (B), (C), (D)

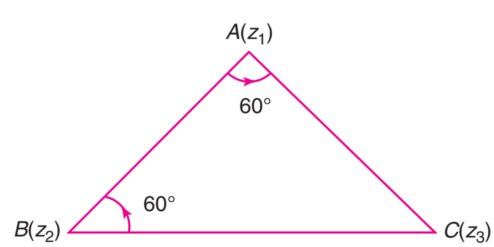
7. If z_1, z_2 and z_3 represent the vertices A, B and C, respectively, of a triangle (see figure), then the triangle ABC is equilateral if and only if

(A) $z_1^2+z_2^2+z_3^2=z_1z_2+z_2z_3+z_3z_1$

(B) $\dfrac{1}{z_1-z_2}+\dfrac{1}{z_2-z_3}+\dfrac{1}{z_3-z_1}=0$

(C) $|z_1+z_2+z_3|=\sqrt{3}$

(D) $|z_1z_2+z_2z_3+z_3z_1|=\sqrt{3}$



Solution: Suppose that triangle ABC is equilateral. Then

$$\frac{z_3-z_1}{z_2-z_1}=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3} \quad \text{and} \quad \frac{z_1-z_2}{z_3-z_2}=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$$

Therefore

$$(z_3 - z_1)(z_3 - z_2) = (z_2 - z_1)(z_1 - z_2)$$
$$z_3^2 - z_3z_2 - z_1z_3 + z_1z_2 = z_2z_1 - z_2^2 - z_1^2 + z_1z_2$$
$$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$$

Conversely, suppose that

$$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$$

Then,

$$z_1(z_1 - z_2) + z_2(z_2 - z_3) + z_3(z_3 - z_1) = 0$$

Therefore

$$z_1(z_1 - z_2) + z_2(z_2 - z_1 + z_1 - z_3) + z_3(z_3 - z_1) = 0$$
$$(z_1 - z_2)^2 - (z_2 - z_3)(z_3 - z_1) = 0$$

That is

$$(z_1 - z_2)^2 = (z_2 - z_3)(z_3 - z_1)$$
$$(z_1 - z_2)^3 = (z_1 - z_2)(z_2 - z_3)(z_3 - z_1)$$

Similarly,

$$(z_2 - z_3)^3 = (z_1 - z_2)(z_2 - z_3)(z_3 - z_1)$$

and $$(z_3 - z_1)^3 = (z_1 - z_2)(z_2 - z_3)(z_3 - z_1)$$

Therefore

$$(z_1 - z_2)^3 = (z_2 - z_3)^3 = (z_3 - z_1)^3$$
$$|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1|$$

Therefore $AB = BC = CA$. That is ΔABC is equilateral.

Answer: (A)

We will prove that (B) is also correct. Suppose that ΔABC is equilateral. Then

$$|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1| = k \quad \text{(say)}$$

Let $\alpha = z_1 - z_2$, $\beta = z_2 - z_3$ and $\gamma = z_3 - z_1$. Then $\alpha + \beta + \gamma = 0$ and hence $\bar{\alpha} + \bar{\beta} + \bar{\gamma} = 0$. That is

$$\frac{k^2}{\alpha} + \frac{k^2}{\beta} + \frac{k^2}{\gamma} = 0 \quad (\text{since } \alpha\bar{\alpha} = |\alpha|^2 = k^2)$$

Therefore

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 0$$
$$\frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} = 0$$

Conversely, suppose that

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 0$$

Then

$$\frac{\alpha + \beta}{\alpha\beta} = -\frac{1}{\gamma}$$

Therefore

$$-\gamma^2 = -\alpha\beta \quad (\text{since } \alpha + \beta = -\gamma)$$
$$\gamma^3 = \alpha\beta\gamma$$

Similarly

$$\beta^3 = \alpha\beta\gamma = \alpha^3$$

This gives $\alpha^3 = \beta^3 = \gamma^3$ and therefore $|\alpha| = |\beta| = |\gamma|$. That is,

$$|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1|$$

Therefore ΔABC is equilateral.

Answers: (A) and (B)

8. If $c \geq 0$, then the equation $|z|^2 - 2iz + 2c(1 + i) = 0$ (z is complex) has

(A) infinitely many solutions if $c < \sqrt{2} - 1$
(B) has unique solution if $c = \sqrt{2} - 1$
(C) finite number of solutions if $c > \sqrt{2} - 1$
(D) no solutions if $c > \sqrt{2} - 1$

Solution: Let $z = x + iy$. Then

$$(x^2 + y^2) - 2i(x + iy) + 2c(1 + i) = 0$$

Therefore

$$x^2 + y^2 + 2y + i(2c - 2x) + 2c = 0$$
$$x^2 + y^2 + 2y + 2c = 0 \tag{3.9}$$

and $$2c - 2x = 0 \quad \text{or} \quad x = c \tag{3.10}$$

Substituting $x = c$ in Eq. (3.9), we get that

$$c^2 + y^2 + 2y + 2c = 0 \tag{3.11}$$

Equation (3.11) has solutions if $4 - 4(c^2 + 2c) \geq 0$, that is $1 - c^2 - 2c \geq 0$. Therefore

$$(c + 1)^2 \leq 2 \quad \text{or} \quad -\sqrt{2} \leq c + 1 \leq \sqrt{2}$$
$$-\sqrt{2} - 1 \leq c \leq \sqrt{2} - 1$$

It is given that $c \geq 0$. Therefore $0 \leq c \leq \sqrt{2} - 1$.

(i) If $c < \sqrt{2} - 1$, then $z = c + (-1 \pm \sqrt{1 - 2c - c^2})i$.
(ii) If $c = \sqrt{2} - 1$, then $z = (\sqrt{2} - 1) - i$.
(iii) If $c > \sqrt{2} - 1$, the equation has no solutions.

Answers: (B), (D)

9. If z_1, z_2, z_3 are complex numbers such that

$$|z_1| = |z_2| = |z_3| = 1 \quad \text{and} \quad \frac{z_1^2}{z_2z_3} + \frac{z_2^2}{z_3z_1} + \frac{z_3^2}{z_1z_2} = -1$$

then the value of $|z_1 + z_2 + z_3|$ can be

(A) 0 (B) 1 (C) 2 (D) 3/2

Solution: Let $z = z_1 + z_2 + z_3$. Then

$$\bar{z} = \bar{z}_1 + \bar{z}_2 + \bar{z}_3 = \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} = \frac{z_1z_2 + z_2z_3 + z_3z_1}{z_1z_2z_3}$$

Therefore $\bar{z}b = z_1z_2 + z_2z_3 + z_3z_1$, where $b = z_1z_2z_3$. Hence

$$\frac{z_1^2}{z_2z_3} + \frac{z_2^2}{z_3z_1} + \frac{z_3^2}{z_1z_2} = -1 \Rightarrow z_1^3 + z_2^3 + z_3^3 = -z_1z_2z_3$$

$$\Rightarrow z_1^3 + z_2^3 + z_3^3 - 3z_1z_2z_3 = -4b$$

Now

$$(z_1 + z_2 + z_3)[(z_1 + z_2 + z_3)^2 - 3(z_1z_2 + z_2z_3 + z_3z_1)] = -4b$$

That is

$$z(z^2 - 3\bar{z}b) = -4b$$

Therefore

$$z^3 - 3|z|^2 b + 4b = 0$$

$$z^3 = (3|z|^2 - 4)b$$

$$|z|^3 = |3|z|^2 - 4| \quad (\text{since } |b| = |z_1z_2z_3| = 1)$$

Case 1: Suppose that $3|z|^2 \geq 4$. Then

$$|z|^3 = 3|z|^2 - 4$$

$$|z|^3 - 3|z|^2 + 4 = 0$$

$$(|z| - 2)(|z|^2 - |z| - 2) = 0$$

$$(|z| - 2)(|z| - 2)(|z| + 1) = 0$$

$$|z| = 2$$

Case 2: Suppose that $3|z|^2 < 4$. Then

$$|z|^3 = |3|z|^2 - 4| = 4 - 3|z|^2$$

$$|z|^3 + 3|z|^2 - 4 = 0$$

$$(|z| - 1)(|z|^2 + 4|z| + 4) = 0$$

$$(|z| - 1)(|z| + 2)^2 = 0$$

$$|z| = 1$$

Answers: (B) and (C)

Note that, in case 2, z, z_1, z_2, and z_3 lie on the circle with radius 1 and center at the origin. Therefore, origin is the circumcenter of the triangle with z_1, z_2 and z_3 as vertices. Hence, $z_1 + z_2 + z_3\ (= z)$ represents the orthocenter. Thus z_1, z_2 and z_3 form a right-angled triangle because the distance between the orthocenter and circumcenter is equal to the radius of the circumcircle. Hence two of z_1, z_2, and z_3 are the reflections of each other, through the center of the circle. Since z_1, z_2, z_3 satisfy the condition $\sum z_1^2/z_2z_3 = -1$, it implies that two are real and the third is the reflection of them in the origin.

10. If

$$\arg(z^{3/8}) = (1/2)\arg(z^2 + \bar{z}z^{1/2})$$

then which of the following is (are) true?

(A) $|z| = 1$ (B) z is real

(C) z is pure imaginary (D) $z^{1/2} = 1$

Solution: The given relation is

$$2\arg(z^{3/8}) = \arg(z^2 + \bar{z}z^{1/2})$$

$$\Rightarrow \arg\left(\frac{z^{3/4}}{z^2 + \bar{z}z^{1/2}}\right) = 0$$

$$\Rightarrow \frac{z^{3/4}}{z^2 + \bar{z}z^{1/2}} \quad \text{is purely real}$$

$$\Rightarrow \frac{z^2 + \bar{z}z^{1/2}}{z^{3/4}} \quad \text{is purely real}$$

$$\Rightarrow z^{5/4} + \bar{z}z^{-1/4} \quad \text{is purely real}$$

$$\Rightarrow \overline{z^{5/4} + \bar{z}z^{-1/4}} = z^{5/4} + \bar{z}z^{-1/4}$$

$$\Rightarrow ((\bar{z})^{5/4} + z(\bar{z})^{-1/4}) = z^{5/4} + \bar{z}z^{-1/4}$$

$$\Rightarrow (\bar{z})^{5/4} - z^{5/4} = \bar{z}z^{-1/4} - z(\bar{z})^{-1/4} = \frac{(\bar{z})^{5/4} - z^{5/4}}{(z\bar{z})^{1/4}}$$

$$\Rightarrow [(\bar{z})^{5/4} - z^{5/4}]\left[1 - \frac{1}{(z\bar{z})^{1/4}}\right] = 0$$

$$\Rightarrow \bar{z} = z \quad \text{or} \quad \frac{1}{|z|^2} = 1$$

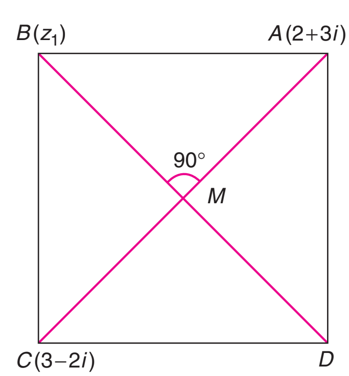
$$\Rightarrow \bar{z} = z \quad \text{or} \quad |z| = 1$$

Answers: (A) and (B)

11. The vertices A and C of a square $ABCD$ (see figure) are $2 + 3i$ and $3 - 2i$, respectively. If z_1 and z_2 represent the other two vertices B and D respectively, then

(A) $z_1 = 0$ (B) $z_2 = 5 - i$

(C) $z_1 = 1 + i$ (D) $z_2 = 5 + i$



Solution: Let M be the center of the square. Then

$$M = \frac{5}{2} + \frac{i}{2}$$

Let z_1 denote the point B. Then $\angle CMD = 90°$. Therefore

$$\frac{z_1 - (5+i)/2}{2 + 3i - (5+i)/2} = i$$

$$z_1 = \frac{5+i}{2} + i\left(2 + 3i - \frac{5+i}{2}\right)$$

$$= \frac{5+i}{2} + i\left(\frac{-1+5i}{2}\right)$$

$$= \frac{5+i-i-5}{2} = 0$$

Therefore,

$$D = 3 - 2i + 2 + 3i - 0$$

$$z_2 = 5 + i$$

Answers: (A) and (D)

12. For any complex number $z = x + iy$, define

$$(z) = |x| + |y|$$

If z_1 and z_2 are any complex numbers, then

(A) $(z_1 + z_2) \le (z_1) + (z_2)$

(B) $(z_1 + z_2) = (z_1) + (z_2)$

(C) $(z_1 + z_2) \ge (z_1) + (z_2)$

(D) $|(z_1 + z_2)| \le |(z_1)| + |(z_2)|$

Solution: Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$. Now

$$\begin{aligned}(z_1 + z_2) &= |x_1 + x_2| + |y_1 + y_2| \\ &\le |x_1| + |x_2| + |y_1| + |y_2| \\ &= (z_1) + (z_2)\end{aligned}$$

$$\begin{aligned}|(z_1 + z_2)| &= ||x_1 + x_2| + |y_1 + y_2|| \\ &= |x_1 + x_2| + |y_1 + y_2| \\ &\le |x_1| + |x_2| + |y_1| + |y_2| \\ &= |(z_1)| + |(z_2)|\end{aligned}$$

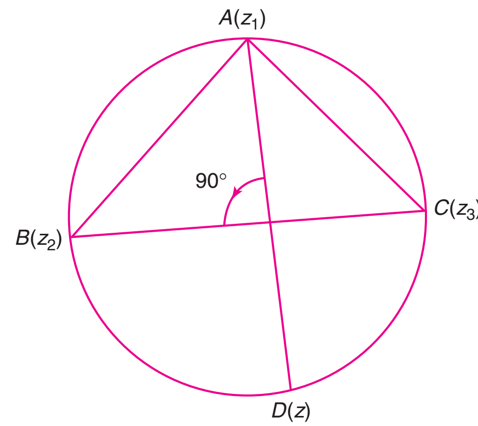
Answers: (A) and (D)

13. Let z_1, z_2 and z_3 be complex numbers representing three points A, B and C, respectively, on the unit circle $|z| = 1$ (see figure). Let the altitude through A meet the circle in $D(z)$. Then

(A) $z = \dfrac{-z_2 z_3}{z_1}$ (B) $z = \dfrac{-z_1}{z_2 z_3}$

(C) D is the reflection of the orthocenter in the side BC

(D) If H is the orthocenter, then HD is perpendicular to the side BD



Solution: AD is perpendicular to BC and therefore

$$\arg\left(\frac{z - z_1}{z_3 - z_2}\right) = \pm\frac{\pi}{2}$$

This implies that $(z - z_1)/(z_3 - z_2)$ is pure imaginary. Therefore

$$\frac{\bar{z} - \bar{z}_1}{\bar{z}_3 - \bar{z}_2} = -\left(\frac{z - z_1}{z_3 - z_2}\right)$$

$$\frac{(1/z) - (1/z_1)}{(1/z_3) - (1/z_2)} = -\left(\frac{z - z_1}{z_3 - z_2}\right)$$

$$\left(\frac{z_1 - z}{z_2 - z_3}\right)\left(\frac{z_2 z_3}{z z_1}\right) = -\left(\frac{z - z_1}{z_3 - z_2}\right)$$

$$\frac{z_2 z_3}{z z_1} = -1 \quad \text{or} \quad z = \frac{-z_2 z_3}{z_1}$$

This implies (A) is correct.

Also, since the orthocenter H is $z_1 + z_2 + z_3$, we have

$$BH = |z_1 + z_2 + z_3 - z_2| = |z_1 + z_3|$$

and $$BD = \left|z_2 + \frac{z_2 z_3}{z_1}\right| = \frac{|z_2|}{|z_1|}|z_1 + z_3| = |z_1 + z_3|$$

$$(\text{since } |z_1| = 1 = |z_2|)$$

Therefore, B is equidistant from H and D. Similarly, C is equidistant from H and D. This gives that BC is the perpendicular bisector of HD and so H, D are reflections of each other through the side BC.

Answers: (A) and (C)

14. Let a, b be real numbers such that $|b| \le 2a^2$. Let

$$X = \{z : |z-a| = \sqrt{2a^2+b}\}$$
$$Y = \{z : |z+a| = \sqrt{2a^2-b}\}$$
$$S = \{z : |z^2-a^2| = |2az+b|\}$$

Then which of the following is (are) true?

(A) X is a subset of S (B) Y is a subset of S
(C) $S = X \cup Y$ (D) $S = X \cap Y$

Solution: Let $z \in S$. Therefore $|z^2-a^2| = |2az+b|$. This relation is equivalent to

$$|z^2-a^2|^2 = |2az+b|^2$$
$$(z^2-a^2)(\bar{z}^2-a^2) = (2az+b)(2a\bar{z}+b)$$
$$|z|^4 - a^2(z^2+\bar{z}^2) + a^4 = 4a^2|z|^2 + 2ab(z+\bar{z}) + b^2$$
$$|z|^4 - a^2[(z+\bar{z})^2 - 2|z|^2] + a^4 = 4a^2|z|^2 + 2ab(z+\bar{z}) + b^2$$
$$|z|^4 - 2a^2|z|^2 + a^4 = a^2(z+\bar{z})^2 + 2ab(z+\bar{z}) + b^2$$

Hence, $(|z|^2 - a^2)^2 = [a(z+\bar{z})+b]^2$. Therefore

$$|z|^2 - a^2 = \pm[a(z+\bar{z})+b]$$

Therefore

$$|z|^2 - a^2 - a(z+\bar{z}) - b = 0$$

or
$$|z|^2 - a^2 + a(z+\bar{z}) + b = 0$$

This is equivalent to

$$(z-a)(\bar{z}-a) = 2a^2+b$$

or
$$(z+a)(\bar{z}+a) = 2a^2-b \qquad (3.12)$$

Hence,

$$|z-a| = \sqrt{2a^2+b} \quad \text{or} \quad |z+a| = \sqrt{2a^2-b}$$

Since $|b| \le 2a^2$, both $2a^2+b$ and $2a^2-b$ are non-negative. From Eq. (3.12), if we retrace the steps backwards, then we get z satisfying the relation

$$|z^2-a^2| = |2az+b|$$

Therefore

$$S = X \cup Y$$

Answers: (A), (B), (C)

15. Let z_1, z_2, z_3 be the complex numbers representing the vertices A, B, C of a triangle described in counterclocksense. Consider the following statements.

I. ΔABC is equilateral

II. $z_3 - z_1 = (z_2 - z_1)\left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right)$

III. $z_2 - z_1 = (z_3 - z_1)\left(\cos\dfrac{5\pi}{3} + i\sin\dfrac{5\pi}{3}\right)$

IV. $z_1 + z_2\left(\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}\right) + z_3\left(\cos\dfrac{4\pi}{3} + i\sin\dfrac{4\pi}{3}\right) = 0$

Then which one is correct:

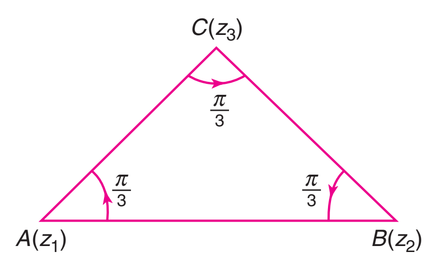
(A) I $\Rightarrow$ II (B) II $\Rightarrow$ III
(C) III $\Rightarrow$ IV (D) IV $\Rightarrow$ I

Solution:

I. Suppose ΔABC is equilateral (see figure). Rotating $\overrightarrow{AB}$ about A through the angle $\pi/3$ in anticlocksense, we get

$$\frac{z_3-z_1}{z_2-z_1} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$$

Therefore, I $\Rightarrow$ II. This implies (A) is true.



II. Assume that

$$z_3 - z_1 = (z_2 - z_1)\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

Therefore

$$\frac{z_3-z_1}{z_2-z_1} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$$

$$|z_3 - z_1| = |z_2 - z_1| \quad \text{and} \quad \angle BAC = \frac{\pi}{3}$$

This implies ΔABC is equilateral. Therefore, II $\Rightarrow$ I. Now rotate $\overline{AC}$ about A through angle $5\pi/3$ in anticlock sense so that

$$z_2 - z_1 = (z_3 - z_1)\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$$

This means II $\Leftrightarrow$ III.
Similarly we can see that III $\Leftrightarrow$ IV and IV $\Leftrightarrow$ I.

Answers: (A), (B), (C), (D)

Matrix-Match Type Questions

1. Match the items in Column I with those in Column II

Column I	*Column II*
(A) If $z = x + iy$, $z^{1/3} = a - ib$ and $\dfrac{x}{a} - \dfrac{y}{b} = \lambda(a^2 - b^2)$, then λ is	(p) 10
	(q) 14
(B) If $\lvert z - i\rvert < 1$, then the value of $\lvert z + 12 - 6i\rvert$ is less than	(r) 1
(C) If $\lvert z_1\rvert = 1$ and $\lvert z_2\rvert = 2$, then $\lvert z_1 + z_2\rvert^2 + \lvert z_1 - z_2\rvert^2$ is equal to	(s) 4
(D) If $z = 1 + i$, then $4(z^4 - 4z^3 + 7z^2 - 6z + 3)$ is equal to	(t) 5

Solution:

(A) $x + iy = z = (a - ib)^3 = a^3 - 3a^2 bi + 3a(ib)^2 - i^3 b^3$

$$= (a^3 - 3ab^2) + i(b^3 - 3a^2 b)$$

Comparing the real parts we get

$$x = a^3 - 3ab^2 = a(a^2 - 3b^2)$$

$$\frac{x}{a} = a^2 - 3b^2$$

Comparing the imaginary parts we get

$$y = b^3 - 3a^2 b = b(b^2 - 3a^2)$$

$$\frac{y}{b} = b^2 - 3a^2$$

Therefore

$$\frac{x}{a} - \frac{y}{b} = 4(a^2 - b^2)$$

$$\lambda = 4$$

Answer: (A) → (s)

(B) $\lvert z - 12 - 6i\rvert = \lvert (z - i) + (12 - 5i)\rvert$

$$\le \lvert z - i\rvert + \lvert 12 - 5i\rvert < 1 + 13 = 14$$

Answer: (B) → (q)

(C) $\lvert z_1 + z_2\rvert^2 + \lvert z_1 - z_2\rvert^2 = 2(\lvert z_1\rvert^2 + \lvert z_2\rvert^2) = 2(1 + 4) = 10$

Answer: (C) → (p)

(D) If $z = 1 + i$, then

$$(z - 1)^4 = i^4$$

Therefore

$$z^4 - 4z^3 + 6z^2 - 4z + 1 = 1$$

$$(z^4 - 4z^3 + 7z^2 - 6z + 3) - z^2 + 2z - 2 = 1$$

$$z^4 - 4z^3 + 7z^2 - 6z + 3 = z^2 - 2z + 3$$

$$= (z - 1)^2 + 2 = i^2 + 2 = 1$$

$$4(z^4 - 4z^3 + 7z^2 - 6z + 3) = 4$$

Answer: (D) → (s)

2. Match the items in Column I with those in Column II. In the following, $w \ne 1$ is a cube root of unity.

Column I	*Column II*
(A) The value of the determinant $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1-w^2 & w^2 \\ 1 & w^2 & w^4 \end{vmatrix}$ is	(p) $3w(1 - w)$
	(q) $3w(w - 1)$
(B) The value of $4 + 5w^{2002} + 3w^{2009}$ is	
	(r) $-i\sqrt{3}$
(C) The value of the determinant $\begin{vmatrix} 1 & 1+i+w^2 & w^2 \\ 1-i & -1 & w^2-1 \\ -i & -i+w^2+1 & -1 \end{vmatrix}$ is	(s) $i\sqrt{3}$
(D) $w^{2n} + w^n + 1$ (n is a positive integer and not a multiple of 3) is	(t) 0

Solution:

(A) $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1-w^2 & w^2 \\ 1 & w^2 & w^4 \end{vmatrix} = \begin{vmatrix} 3 & 1+w+w^2 & 1+w^2+w \\ 1 & w & w^2 \\ 1 & w^2 & w \end{vmatrix}$

$$= \begin{vmatrix} 3 & 0 & 0 \\ 1 & w & w^2 \\ 1 & w^2 & w \end{vmatrix}$$

$$= 3(w^2 - w^4) = 3w(w - 1)$$

Answer: (A) → (q)

(B) $4 + 5w^{2002} + 3w^{2009} = 4 + 5w + 3w^2 \quad (\because w^3 = 1)$

$$= 1 + 2w + 3(1 + w + w^2)$$

$$= 1 + 2w$$

Since

$$w = \frac{-1}{2} \pm \frac{i\sqrt{3}}{2}$$

we get

$$4 + 5w^{2002} + 3w^{2009} = 1 + 2\left(\frac{-1}{2} + \frac{i\sqrt{3}}{2}\right) \quad \text{or} \quad 1 + 2\left(\frac{-1}{2} - \frac{i\sqrt{3}}{2}\right)$$

$$= 1 - 1 + i\sqrt{3} \quad \text{or} \quad 1 - 1 - i\sqrt{3}$$
$$= \pm i\sqrt{3}$$

Answers: (B) → (r), (s)

(C)
$$\begin{vmatrix} 1 & 1+i+w^2 & w^2 \\ 1-i & -1 & w^2-1 \\ -i & -i+w-1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & -w+i & w^2 \\ 1-i & -1 & w^2-1 \\ -i & -i+w-1 & -1 \end{vmatrix}$$
$$= \begin{vmatrix} 1 & -w+i & -1 \\ 0 & 0 & 0 \\ -i & -i+w-1 & -1 \end{vmatrix} = 0$$

Answer: (C) → (t)

(D) Let $n > 0$ and $n \neq 3m$ for all integers m. Then $n = 3m + 1$ or $3m + 2$

$$n = 3m + 1 \Rightarrow w^{2n} + w^n + 1 = w^{6m+2} + w^{3n+1} + 1$$
$$= w^2 + w + 1 = 0$$
$$n = 3m + 2 \Rightarrow w^{2n} + w^n + 1 = w^{6m+4} + w^{3m+2} + 1$$
$$= w + w^2 + 1 = 0$$

Answer: (D) → (t)

3. Match the items in Column I with those in Column II. $w \neq 1$ is a cube root of unity.

Column I	*Column II*
(A) The value of $\frac{1}{3}(1-w)(1-w^2)(1-w^4)(1-w^8)$ is	(p) -128
	(q) 6
(B) $w(1+w-w^2)^7$ is equal to	(r) 0
(C) The least positive integer n such that $(1+w^2)^n = (1+w^4)^n$ is	(s) 128
(D) $\frac{1}{1+2w} + \frac{1}{2+w} - \frac{1}{1+w}$ is equal to	(t) 3

Solution: We have $w^3 = 1$ and $1 + w + w^2 = 0$.

(A) We have

$$\frac{1}{3}(1-w)(1-w^2)(1-w^4)(1-w^8) = \frac{1}{3}(1-w)^2(1-w^2)^2$$
$$= \frac{1}{3}[(1-w)(1-w^2)]^2$$
$$= \frac{1}{3}(1-w-w^2+w^3)^2$$
$$= \frac{1}{3}(1+1+1)^2 = 3$$

Answer: (A) → (t)

(B) We have

$$w(1+w-w^2)^7 = w(-w^2-w^2)^7$$
$$= w[2(-w^2)]^7$$
$$-2^7 w^{15} = -128$$

Answer: (B) → (p)

(C) We have

$$(1+w^2)^n = (1+w^4)^n$$
$$(1+w^2)^n = (1+w)^n$$
$$(-w)^n = (-w^2)^n$$
$$w^n = w^{2n}$$

The least such positive n is 3.

Answer: (C) → (t)

(D) We have

$$\frac{1}{1+2w} + \frac{1}{2+w} - \frac{1}{1+w} = \frac{1}{1+2w} + \frac{1+w-2-w}{(2+w)(1+w)}$$
$$= \frac{1}{1+2w} - \frac{1}{2+3w+w^2}$$
$$= \frac{1}{1+2w} - \frac{1}{1+2w} = 0$$

Answer: (D) → (r)

Comprehension-Type Questions

1. Passage: A complex number z is pure real if and only if $\bar{z} = z$ and is pure imaginary if and only if $\bar{z} = -z$. Answer the following questions:

(i) If x and y are real numbers and the complex number

$$\frac{(2+i)x - i}{4+i} + \frac{(1-i)y + 2i}{4i}$$

is pure real, the relation between x and y is

(A) $8x - 17y = 16$ (B) $8x + 17y = 16$
(C) $17x - 8y = 16$ (D) $17x - 8y = -16$

(ii) If

$$z = \frac{3 + 2i\sin\theta}{1 - 2i\sin\theta} \quad \left(0 < \theta \leq \frac{\pi}{2}\right)$$

is pure imaginary, then θ is equal to

(A) $\pi/4$ (B) $\pi/6$ (C) $\pi/3$ (D) $\pi/12$

(iii) If z_1 and z_2 are complex numbers such that

$$\left|\frac{z_1 - z_2}{z_1 + z_2}\right| = 1$$

then

(A) z_1/z_2 is pure real

(B) z_1/z_2 is pure imaginary

(C) z_1 is pure real

(D) z_1 and z_2 are pure imaginary

Solution:

(i) Let

$$z = \frac{(2+i)x - i}{4+i} + \frac{(1-i)y + 2i}{4i}$$

$$= \frac{2x + (x-1)i}{4+i} + \frac{y + (2-y)i}{4i}$$

$$= \frac{(2x + (x-1)i)(4-i)}{17} + \frac{-iy + (2-y)}{4}$$

$$= \frac{8x + x - 1 + i(4x - 4 - 2x)}{17} + \frac{(2-y) - iy}{4}$$

$$= \frac{9x - 1 + i(2x-4)}{17} + \frac{2 - y - iy}{4}$$

Now

$$z \text{ is real} \Leftrightarrow \bar{z} = z$$

$$\Leftrightarrow \operatorname{Im} z = 0$$

$$\Leftrightarrow \frac{2x-4}{17} - \frac{y}{4} = 0$$

$$\Leftrightarrow 8x - 16 = 17y$$

$$\Leftrightarrow 8x - 17y = 16$$

Answer: (A)

(ii) $$z = \frac{3 + 2i\sin\theta}{1 - 2i\sin\theta}$$

$$= \frac{(3 + 2i\sin\theta)(1 + 2i\sin\theta)}{1 + 4\sin^2\theta}$$

$$= \frac{(3 - 4\sin^2\theta) + i(8\sin\theta)}{1 + 4\sin^2\theta}$$

Now,

$$z \text{ is pure imaginary} \Leftrightarrow \bar{z} = -z$$

$$\Leftrightarrow \operatorname{Re}(z) = 0$$

$$\Leftrightarrow \frac{3 - 4\sin^2\theta}{1 + 4\sin^2\theta} = 0$$

$$\Leftrightarrow \sin^2\theta = \frac{3}{4}$$

$$\Leftrightarrow \sin\theta = \pm\frac{\sqrt{3}}{2}$$

$$\Leftrightarrow \theta = \frac{\pi}{3} \quad \left(\text{since } 0 < \theta \le \frac{\pi}{2}\right)$$

Answer: (C)

(iii) $|z_1 - z_2| = |z_1 + z_2|$

$$\Rightarrow (z_1 - z_2)(\bar{z}_1 - \bar{z}_2) = (z_1 + z_2)(\bar{z}_1 + \bar{z}_2)$$

$$\Rightarrow z_1\bar{z}_2 = -\bar{z}_1 z_2$$

$$\Rightarrow \frac{z_1}{z_2} = -\frac{\bar{z}_1}{\bar{z}_2} = -\left(\overline{\frac{z_1}{z_2}}\right)$$

$$\Rightarrow \frac{z_1}{z_2} \text{ is pure imaginary}$$

Answer: (B)

2. Passage: Consider $z = a + ib$ and $\bar{z} = a - ib$, where a and b are real numbers, are conjugates of each other. Answer the following three questions:

(i) If the complex numbers $-3 + i(x^2y)$ and $x^2 + y + 4i$, where x and y are real, are conjugate to each other, then the number of ordered pairs (x, y) is

(A) 1 (B) 2 (C) 3 (D) 4

(ii) Let $z = x^2 - 7x - 9yi$ such that $\bar{z} = y^2i + 20i - 12$, then the number of ordered pairs (x, y) is

(A) 1 (B) 2 (C) 3 (D) 4

(iii) The number of real values of x such that $\sin x + i\cos 2x$ and $\cos x - i\sin 2x$ are conjugate to each other is

(A) 1 (B) 2 (C) >2 (D) 0

Solution:

(i) $-3 + ix^2y = x^2 + y - 4i$ implies

$$x^2 + y = -3 \quad \text{and} \quad x^2y = -4 \tag{3.13}$$

Therefore

$$x^2 - \frac{4}{x^2} = -3$$

$$x^4 + 3x^2 - 4 = 0$$

$$(x^2 + 4)(x^2 - 1) = 0$$

This gives $x^2 = 1$ (since $x^2 \ne -4$). Therefore $x = \pm 1$ and $y = -4$. Hence the ordered pairs are $(1, -4)$ and $(-1, -4)$.

Answer: (B)

(ii) We have

$$z = x^2 - 7x - 9yi$$

$$\Rightarrow \bar{z} = x^2 - 7x + 9yi$$

$$\Rightarrow x^2 - 7x + 9yi = y^2i + 20i - 12$$

This implies that

$$x^2 - 7x = -12 \tag{3.14}$$

and

$$9y = y^2 + 20 \tag{3.15}$$

Solving Eq. (3.14) we get

$$x = 3, 4$$

Solving Eq. (3.15) we get

$$y^2 - 9y + 20 = 0 \Rightarrow y = 4, 5$$

Therefore, the required ordered pairs are (3, 4), (3, 5), (4, 4) and (4, 5).

Answer: (D)

(iii) $\sin x + i\cos 2x = \cos x + i\sin 2x$

$\Rightarrow \sin x = \cos x$ and $\cos 2x = \sin 2x$

$\Rightarrow 2\cos^2 x - 1 = \cos 2x = \sin 2x = 2\sin x\cos x = 2\cos^2 x$

$\Rightarrow -1 = 0$, which is absurd

Therefore, there are no such real numbers x.

Answer: (D)

Assertion–Reasoning Type Questions

In the following set of questions, a Statement I is given and a corresponding Statement II is given just below it. Mark the correct answer as:

(A) Both I and II are true and II is a correct reason for I

(B) Both I and II are true and II is not a correct reason for I

(C) I is true, but II is false

(D) I is false, but II is true

1. **Statement I:** If $z_1 = 9 + 5i$, $z_2 = 3 + 5i$ and arg $[(z-z_1)/(z-z_2)] = \pi/4$, then the values of $|z - 6 - 8i|$ is $3\sqrt{2}$.

   Statement II: In a circle, the angle made by a chord at the center is double the angle subtended by the same chord on the circumference.

Solution: Let z be a point such that

$$\arg\left(\frac{z_1 - z}{z_2 - z}\right) = \frac{\pi}{4}$$

Let $z = x + iy$. Then

$$\arg\left(\frac{z_1 - z}{z_2 - z}\right) = \frac{\pi}{4} \Rightarrow (x-9)(x-3) + (y-5)^2 = 6y - 30$$

$$\Rightarrow x^2 + y^2 - 12x - 16y + 82 = 0$$

Now

$$\begin{aligned} |z - 6 - 8i|^2 &= (x-6)^2 + (y-8)^2 \\ &= x^2 + y^2 - 12x - 16y + 100 \\ &= (x^2 + y^2 - 12x - 16y + 82) + 18 \\ &= 0 + 18 \end{aligned}$$

Therefore

$$|z - 6 - 8i| = 3\sqrt{2}$$

Answer: (B)

SUMMARY

Complex Number

3.1 ***Complex number:*** Any ordered pair (a, b) where a and b are real numbers is called a complex number and the set of all complex numbers is denoted by $\mathbb{C}$ which is $\mathbb{R}\times\mathbb{R}$.

3.2 ***Real number as a complex number:*** If a is a real number, we write a for the ordered pair $(a, 0)$ so that every real numbered is considered to be a complex number.

3.3 ***Algebraic operations:***

(1) Addition: If $z_1 = (a, b)$ and $z_2 = (c, d)$, then

$$z_1 + z_2 = (a + c, b + d)$$

(2) If $z = (a, b)$, then $-z = (-a, -b)$.

(3) $z_1 = (a, b)$, $z_2 = (c, d)$, then $z_1 - z_2 = z_1 + (-z_2) = (a - c, b - d)$.

(4) If $z = (a, b)$, and λ is real, then $\lambda z = (\lambda a, \lambda b)$.

(5) Product: If $z_1 = (a, b)$, and $z_2 = (c, d)$, then $z_1 z_2 = (ac - bd, ad + bc)$.

3.4 ***Zero complex number and unit complex number:*** (0, 0) is called zero complex number and is denoted by 0. (1, 0) is called unit complex number and is denoted by 1.

3.5 ***Complex number i:*** The complex number (0, 1) is such that (0, 1) (0, 1) = (−1, 0) = −1. (0, 1) is denoted by i with the convention that $i^2 = -1$ or $i = \sqrt{-1}$. If n is any positive integer, then $i^n + i^{n+1} + i^{n+2} + i^{n+3} = 0$.

3.6 ***Quotient of complex numbers:*** Let $z_1 = (a, b)$ and $z_2 = (c, d) \neq (0, 0)$. Then the unique complex number z such that $z \cdot z_2 = z_1$ is called quotient of z_1 and z_2 and is denotes by z_1/z_2. In particular, if $z = (a, b) \neq (0, 0)$, then there exists $z' = (c, d)$ such that $zz' = (1, 0) = 1$ and

$$c = \frac{a}{a^2 + b^2}, \quad d = \frac{-b}{a^2 + b^2}$$

3.7 ***Representation of (a, b) as a + ib:***

$$z = (a, b) = (a, 0) + (0, 1)\,(b, 0) = a + ib$$

3.8 ***Real and imaginary parts:*** If $z = a + ib$ (a, b are real), then a is called real part of z denoted by $\mathrm{Re}(z)$ and b is called imaginary part denoted by $\mathrm{Im}(z)$.

3.9 ***Usual operations:***

(1) $z_1 = a + ib, z_2 = c + id$, then $z_1 + z_2 = (a + c) + i(b + d)$ and $z_1 - z_2 = (a - c) + i(b - d)$

(2) $z_1 z_2 = (ac - bd) + i(ad + bc)$

(3) If $z_2 \neq 0$, then

$$\frac{z_1}{z_2} = \frac{ac + bd}{c^2 + d^2} + i\frac{(bc - ad)}{c^2 + d^2}$$

(4) If $z = x + iy \neq 0$, then

$$\frac{1}{z} = \frac{x}{x^2 + y^2} + i\left(\frac{-y}{x^2 + y^2}\right)$$

3.10 ***Cube roots of unity:***

(1) Roots of the equation $z^3 = 1$ are called cube roots of unity and they are

$$1, \quad \frac{-1 + i\sqrt{3}}{2} \quad \text{and} \quad \frac{-1 - i\sqrt{3}}{2}$$

(2) If $\omega \neq 1$ is a cube root of unity, then ω^2 is also cube root of unity and hence $1, \omega$ and ω^2 are cube roots of unity having the relation $1 + \omega + \omega^2 = 0$.

(3) If ω is a non-real cube root of unity and n is any positive integer, then

$$1 + \omega^n + \omega^{2n} = \begin{cases} 3 & \text{if } n \text{ is a multiple of 3} \\ 0 & \text{otherwise} \end{cases}$$

3.11 ***Pure real and pure imaginary:*** A complex number z is called pure real if $\mathrm{Im}(z) = 0$ and pure imaginary if $\mathrm{Re}(z) = 0$.

3.12 ***Conjugate:*** For $z = a + ib$, the complex number $\overline{z} = a - ib$ is called conjugate of z.

3.13 ***Properties of $\overline{z}$:***

(1) $\overline{(\overline{z})} = z$

(2) $\dfrac{z + \overline{z}}{2} = \mathrm{Re}(z)$ and $\dfrac{z - \overline{z}}{2i} = \mathrm{Im}(z)$

(3) If $z = a + ib$, there $z\overline{z} = a^2 + b^2$

(4) z is pure real $\Leftrightarrow \overline{z} = z$

(5) z is pure imaginary $\Leftrightarrow \overline{z} = -z$

(6) $(\overline{z_1 \pm z_2}) = \overline{z}_1 \pm \overline{z}_2$

(7) $\overline{z_1 z_2} = \overline{z}_1\, \overline{z}_2$

(8) If $z_2 \neq 0$, then $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z}_1}{\overline{z}_2}$

(9) $z_1\overline{z}_2 + \overline{z}_1 z_2 = 2\,\mathrm{Re}(\overline{z}_1 z_2) = 2\,\mathrm{Re}(z_1\overline{z}_2)$

(10) $z_1\overline{z}_2 - \overline{z}_1 z_2 = 2i\,\mathrm{Im}(z_1\overline{z}_2) = -2i\,\mathrm{Im}(\overline{z}_1 z_2)$

3.14 ***Modulus and its properties:*** If $z = a + ib$, then $|z| = \sqrt{a^2 + b^2}$ and $|\overline{z}| = |z| = |-z|$. Let z_1 and z_2 be complex numbers. Then

(1) $|z_1 z_2| = |z_1||z_2|$

(2) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ when $z_2 \neq 0$

(3) $|z|^2 = z\overline{z}$ (very useful)

(4) $|z_1 + z_2| \leq |z_1| + |z_2|$ (equality holds if and only if z_1 and z_2 are collinear with origin and lie on the same side of the origin)

(5) $|z_1 - z_2| \geq |z_1| - |z_2|$ (equality holds if and only if z_1, z_2 are collinear with origin and lie on the same side of origin)

(6) $|z^n| = |z|^n$ for all positive integers.

(7) $|z_1 + z_2|^2 = |z_1|^2 + |z^2|^2 + (\overline{z}_1 z_2 + z_1\overline{z}_2)$

(8) $|z_1 - z_2|^2 = |z_1|^2 + |z^2|^2 - (\overline{z}_1 z_2 + z_1\overline{z}_2)$

(9) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$ (parallelogram law)

(10) $|z_1| + |z_2|$ is the greatest possible value of $|z_1 \pm z_2|$ and $||z_1| - |z_2||$ is the least possible value of $|z_1 \pm z_2|$

3.15 ***Unimodular complex number:*** If $|z| = 1$, then z is called unimodular complex number. If $z \neq 0$, then $z/|z|$ is always unimodular.

3.16 ***Geometric interpretation (Argand's plane):*** Consider a plane and introduce coordinate system in the plane. Now, we can view any complex number $z = a + ib$ as the point (a, b) in the plane and any point (a, b) in the plane as the complex number $a + ib$. Also if $z = a + ib$ and P is the point representing z, then one can view the vector $\overrightarrow{OP}$ and conversely if P is a point with coordinates (a, b), we can consider the complex number $z = a + ib$ and the vector $\overrightarrow{OP}$ representing it. Hence there is a one-to-one correspondence between the set of complex numbers, the points in the Argand's plane and the vectors in the Argand's plane.

3.17 ***Identification of complex number:*** We identify complex number $z = a + ib$ with the point $P(a, b)$ and with the vector $\overrightarrow{OP}$ in the Argand's plane where O is the origin.

3.18 ***More about Argand's plane:*** Let z, z_1, z_2 be complex numbers and A, P and Q represent them in the Argand's plane. Then

(1) $\bar{z}$ is presented by the reflection of the point A in the real axis (i.e., x-axis).

(2) $-z$ is represented by the reflection of A through the origin that is $(-a, -b)$ represents $-z$.

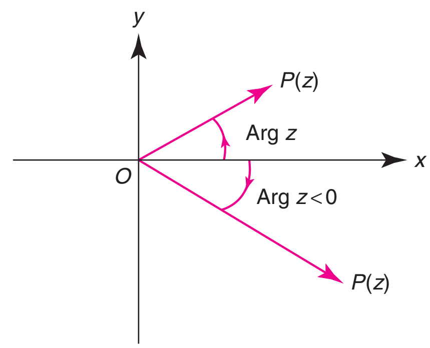
(3) $z_1 + z_2$ is the fourth vertex of the parallelogram constructed on $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ as adjacent sides.

(4) $z_1 - z_2$ is the fourth vertex of the parallelogram constructed with $\overrightarrow{OP}$ and $-\overrightarrow{OQ}$ (i.e., $\overrightarrow{QO}$) as adjacent sides.

3.19 ***Modulus and argument form (Trigonometric or Polar form):*** Every complex number z can be expressed as $r(\cos\theta + i\sin\theta)$ where $r = |z|$ and θ is the angle made by the vector $\overrightarrow{OP}$ with real axis and P represents z in the Argand plane. This θ is called argument of z and is denotes by arg z. If θ is an argument z, then $\theta + 2n\pi$ is also an argument of z.

3.20 ***Principal value of arg z (denoted by Arg z):*** If z is a complex number, then there exists, unique θ such that $-\pi < \theta \le \pi$ and $z = |z|\ (\cos\theta + i\sin\theta)$. This θ is called the principal value of arg z and is denoted by Arg z.

3.21 ***Geometrical meaning of Arg z and computing Arg z:*** Arg z is the shortest turn taken by the position x-axis about the origin to fall on the vector $\overrightarrow{OP}$ where P represents z. If the shortest turn is anticlockwise sense, then Arg z is positive, otherwise Arg $z < 0$. Further, let z_1 and z_2 be complex numbers.



(1) $z_1 = z_2 \Leftrightarrow |z_1| = |z_2|$ and Arg z_1 = Arg z_2.

(2) arg $(z_1 z_2)$ = Arg z_1 + Arg $z_2 + 2n\pi, n \in \mathbb{Z}$.

(3) arg (z_1/z_2) = Arg z_1 − Arg $z_2 + 2n\pi, n \in \mathbb{Z}$.

(4) arg $(1/z) = -$Arg$+ 2n\pi$, $n \in \mathbb{Z}$ for any complex number $z \ne 0$.

3.22 ***Geometrical meaning of arg (z_1/z_2):*** Let P and Q represent z_1 and z_2 in the Argands plane and 'O' is the origin. Thus arg (z_1/z_2) is the angle of rotation of OQ about origin to fall on the vector $\overrightarrow{OP}$. arg (z_1/z_2) is positive, if the rotation is anticlock sense otherwise it is negative.

3.23 ***Directly similar triangles:*** ΔABC and $\Delta A'B'C'$ are directly similar, if $\underline{|A} = A'$, $\underline{|B} = \underline{|B'}$, $\underline{|C} = \underline{|C'}$ and the sides about equal angles are proportional. That is, indirectly similar triangles, the angles at the vertices in the prescribed order are equal.

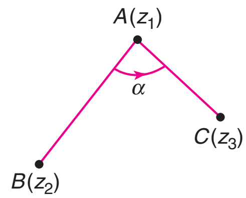
Let z_1, z_2, z_3 and z_1', z_2' and z_3' represent the vertices of two triangles. Then they are directly similar if and only if

$$\begin{vmatrix} z_1 & z_1' & 1 \\ z_2 & z_2' & 1 \\ z_3 & z_3' & 1 \end{vmatrix} = 0$$

QUICK LOOK

Directly similar triangles are similar also.

3.24 ***Most useful formula:*** Let A, B and C be three points representing the complex numbers z_1, z_2 and z_3 respectively and the points are described in counter clock sense and $\underline{|BAC} = \alpha$.



Then,

$$\frac{z_3 - z_1}{z_2 - z_1} = \left(\frac{CA}{BA}\right)(\cos\alpha + i\sin\alpha)$$

In particular the segments AB and AC are at right angles $\Leftrightarrow$ $\arg(z_3 - z_1/z_2 - z_1) = \pm\pi/2$ and in such a case $z_3 - z_1/z_2 - z_1$ is pure imaginary.

3.25 ***Equilateral triangles:*** Let A, B and C be the vertices of a triangle represented by z_1, z_2 and z_3 respectively. The following hold:

(1) ΔABC is equilateral $\Leftrightarrow$ $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$.

(2) ΔABC is equilateral

$$\Leftrightarrow \frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} = 0$$

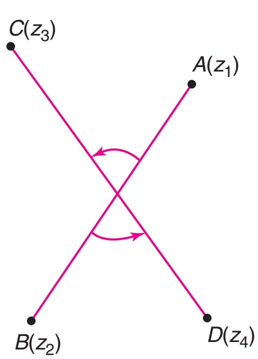
3.26 ***Orthocentre with reference to circumcentre:*** Let A, B, C be the vertices of a triangle whose circumcentre is at the origin. If z_1, z_2, z_3 represent A, B, C respectively, then the orthocentre of ΔABC is represented by $z_1 + z_2 + z_3$.

3.27 ***Angle between two segments:*** $A(z_1)$, $B(z_2)$, $C(z_3)$ and $D(z_4)$. There $\overline{AB}$ is inclined at an angle of $\arg(z_4 - z_3/z_2 - z_1)$ to $\overline{CD}$. The lines are at right angles if and only if

$$\arg\left(\frac{z_4 - z_3}{z_2 - z_1}\right) = \pm\frac{\pi}{2}$$

The points A, B, C are collinear

$$\Leftrightarrow \arg\left(\frac{z_2 - z_1}{z_3 - z_1}\right) = 0 \quad \text{or} \quad \pi$$



3.28 ***Line joining two points in the complex plane:*** The equation of the line joining the points $A(z_1)$ and $B(z_2)$ is

$$\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$$

The complex number $z_1 - z_2/\bar{z}_1 - \bar{z}_2$ is called complex slope of the line AB.

3.29 ***General equation of a straight line:*** If $l \neq 0$ is a complex number and m is a real number then the equation $\bar{l}z + l\bar{z} + m = 0$ represents a straight line in the complex plane. The real slope of this line is

$$\left(\frac{\bar{l} + l}{\bar{l} - l}\right)i$$

3.30 ***Condition for parallel and perpendicular lines:*** Let $\bar{l}_1 z + l_1\bar{z} + m_1 = 0, \bar{l}_2 z + l_2 z + m_2 = 0$ where m_1, m_2 are real be two straight lines. Then

(1) The two lines are parallel $\Leftrightarrow \bar{l}_1 l_2 = l_1 \bar{l}_2$.

(2) The two lines are perpendicular to each other if and only if $\bar{l}_1 l_2 + l_1 \bar{l}_2 = 0$.

3.31 Equation of the perpendicular bisector of the segment joining the points $A(z_1)$ and $B(z_2)$ is $|z - z_1| = |z - z_2|$. Equivalently

$$(\bar{z}_1 - \bar{z}_2)z + (z_1 - z_2)\bar{z} + z_2\bar{z}_2 - z_1\bar{z}_1 = 0$$

3.32 The points representing z_1 and z_2 in the Argand's plane are images of each other in the line $\bar{l}z + l\bar{z} + m = 0$ (m is real) if and only if $\bar{l}z_1 + l\bar{z}_2 + m = 0$.

3.33 ***Distance of a line from a point:*** The perpendicular distance drawn from a point $A(z_0)$ onto a straight line $\bar{l}z + l\bar{z} + m = 0$ (m is real) is

$$\frac{|\bar{l}z_0 + l\bar{z}_0 + m|}{2|l|}$$

3.34 Let $A(z_1), B(z_2)$ and $C(z_3)$ be the vertices of a triangle whose circum centre is the origin. If the altitude drawn from A onto the side BC, meets the circumcircle of ΔABC in D, then D is represented by the complex number $-z_2z_3/z_1$. Also note that D is the reflection of the orthocenter of ΔABC in the side BC.

Circle

3.35 ***Circle:*** Equation of the circle with centre at the point z_0 and radius r (> 0) is $|z - z_0| = r$.

QUICK LOOK

1. $|z - z_0| = r \Leftrightarrow |z - z_0|^2 = r^2$
$$\Leftrightarrow (z - z_0)(\bar{z} - \bar{z}_0) = r^2$$
$$\Leftrightarrow z\bar{z} - \bar{z}_0 z - z_0\bar{z} + z_0\bar{z}_0 - r^2 = 0$$

2. If $z_0 = 0$, then the equation of the circle with centre at origin and radius r is $|z| = r$.

3.36 ***General equation of a circle in the complex plane:*** If a is complex number and b is real, then the equation $z\bar{z} + \bar{a}z + a\bar{z} + b = 0$ represents circle with centre at the point $-a$ and radius $\sqrt{a\bar{a} - b}$. The circle is real circle or point circle or imaginary circle according as $a\bar{a} - b$ is positive or zero or negative.

EXERCISES

Single Correct Choice Type Questions

1. If $w \neq 1$ is a cube root of unity, then the value of the expression $(1 - w + w^2)$ $(1 - w^2 + w^4)$ $(1 - w^4 + w^8)$ $\cdots$ upto $2n$ factors is
(A) 2^n (B) 2^{2n} (C) 0 (D) 1

2. The value of $\sum_{k=1}^{10}[\sin(2k\pi/11) - i\cos(2k\pi/11)]$ is
(A) 1 (B) -1 (C) i (D) $-i$

3. If z is a complex number and n is a positive integer satisfying the equation $(1 + z)^n = (1 - z)^n$, then z lies on
(A) the line $x = 0$ (B) the line $x = 1/2$
(C) the line $y = 0$ (D) the line $x = -1/2$

4. Let a and b be complex numbers representing the points A and B, respectively, in the complex plane. If $(a/b) + (b/a) = 1$ and O is the origin, then ΔOAB is
(A) right angled (B) right-angled isosceles
(B) obtuse angled (D) equilateral

5. The complex numbers z_1, z_2, z_3 and z_4 represent the vertices of a parallelogram in this order, if
(A) $z_1 + z_2 = z_3 + z_4$ (B) $z_1 + z_3 = z_2 + z_4$
(C) $z_1 + z_4 = z_2 + z_3$ (D) $\dfrac{z_1 + z_3}{z_2 z_3} = \dfrac{z_1 + z_4}{z_1 z_4}$

6. The area of the region in the complex plane satisfying the inequality
$$\log_{\cos(\pi/6)}\left[\frac{|z-2|+5}{4|z-2|-4}\right] < 2$$
is
(A) 4π (B) 8π (C) 12π (D) 15π

7. If z is a non-zero complex number, then the equation $z^2 + |z|\,z + |z|^2 = 0$ has
(A) only two roots (B) only four roots
(C) no roots (D) infinite number of roots

8. In ΔABC, origin is the circumcenter, H is the orthocenter and D is the midpoint of the side BC. If P is any point on the circumcircle other than the vertices and T is the midpoint of PH, then the angle between AP and DT is
(A) $\pi/4$ (B) $\pi/3$ (C) $\pi/6$ (D) $\pi/2$

9. The number of solutions of the equation $z(\overline{z - 2i}) = 2(2 + i)$ is
(A) 4 (B) 3 (C) 2 (D) 0

10. If $0 < a, b < 1$, $z_1 = a + i$ and $z_2 = 1 + ib$ and if the origin, z_1 and z_2 represent the vertices of an equilateral triangle, then
(A) $a = \sqrt{3} - 1, b = \dfrac{\sqrt{3}}{2}$ (B) $a = 2 - \sqrt{3} = b$
(C) $a = \dfrac{1}{2}, b = \dfrac{3}{4}$ (D) $a = \dfrac{3}{4}, b = \dfrac{1}{2}$

11. If $|z + 1| = |z - 1|$ and $\arg(z - 1)/(z + 1) = \pi/4$, then z is equal to
(A) $(\sqrt{2} + 1) + i$ (B) $1 + i\sqrt{2}$
(C) $(1 \pm \sqrt{2})i$ (D) $(\sqrt{2} - 1)i$

12. If $z = (1 - t) + i\sqrt{t^2 + t + 2}$, where t is a real parameter, then z lies on the curve
(A) $x^2 + \left(y + \dfrac{3}{2}\right)^2 = \dfrac{7}{4}$ (B) $x^2 - \left(y + \dfrac{3}{2}\right)^2 = \dfrac{7}{4}$
(C) $y^2 - \left(x - \dfrac{3}{2}\right)^2 = \dfrac{7}{4}$ (D) $y^2 + \left(x - \dfrac{3}{2}\right)^2 = \dfrac{7}{4}$

13. z is a complex number and $z \neq i$. If $\arg(z+i)/(z-i) = \pi/2$, then z lies on the curve
(A) $x^2 + y^2 = 1$ (B) $x^2 - y^2 = 1$
(C) $xy = 1$ (D) $y = x + 1$

14. If z_1 and z_2 are complex nth roots of unity which subtend right angle at the origin, then n must be of the form
(A) $4K + 1$ (B) $4K + 2$
(C) $4K + 3$ (D) $4K$

15. $\sum_{m=1}^{32}(3m+2)\left(\sum_{n=1}^{10}\left(\sin\left(\frac{2n\pi}{11}\right) - i\cos\left(\frac{2n\pi}{11}\right)\right)\right)^m =$
(A) $4(1-i)$ (B) $12(1+i)$
(C) $12(1-i)$ (D) $48(1-i)$

16. The complex slope (see "**Quick Look 6**") of the line joining the two points $1-i$ and $2-5i$ is
(A) $\frac{1-4i}{1+4i}$ (B) $\frac{1+4i}{1-4i}$
(C) $\frac{1+2i}{1-2i}$ (D) $\frac{1-2i}{1+2i}$

17. If $|z-i| \leq 2$ and $z_0 = 5+3i$, then the maximum value of $|z_0 + iz|$ is
(A) 7 (B) $\sqrt{7}$ (C) 5 (D) 9

18. If $w \neq 1$ is a cube root of unity, $x = a + b$, $y = aw + bw^2$ and $z = aw^2 + bw$, then $x^3 + y^3 + z^3$ is equal to
(A) $3ab$ (B) 0 (C) $3a^3b^3$ (D) $3(a^3 + b^3)$

19. If a, b and c are integers not all simultaneously equal and $w \neq 1$ is a cube root of unity, then the minimum value of $|a + bw + cw^2|$ is
(A) 0 (B) 1 (C) $\sqrt{3}/2$ (D) 1/2

20. The center and the radius of the $z\bar{z} + (2-3i)z + (2+3i)\bar{z} + 4 = 0$ are
(A) $-2-3i$, 3 (B) $2-3i$, 3
(C) $2+3i$, 3 (D) $-2+3i$, 3

21. The distance of the point z_0 from the line $\bar{a}z + a\bar{z} + b = 0$ (b is real) is
(A) $\left|\frac{\bar{a}z_0 + a\bar{z}_0 + b}{2a}\right|$ (B) $\left|\frac{\bar{a}\,\bar{z}_0 + az_0 + b}{2a}\right|$
(C) $\left|\frac{\bar{a}z_0 + a\bar{z}_0 + b}{a}\right|$ (D) $\left|\frac{\bar{a}\,\bar{z}_0 + az_0 + b}{a}\right|$

22. Let a, b be non-zero complex numbers and z_1, z_2 be the roots of the equation $z^2 + az + b = 0$. If there exists $\lambda \geq 4$ such that $a^2 = \lambda b$, then the points z_1, z_2 and the origin
(A) form an equilateral triangle
(B) form a right-angled triangle, right angled at the origin
(C) are collinear
(D) form an obtuse-angled triangle

23. If $(w - \bar{w}z)/(1-z)$ is purely real, where $w = \alpha + i\beta$ and $z \neq 1$, then the set of values of z is
(A) $\{z : |z| = 1\}$ (B) $\{z : z = \bar{z}\}$
(C) $\{z : z \neq 1\}$ (D) $\{z : |z| = 1 \text{ and } z \neq 1\}$

24. If z_1, z_2 and z_3 are distinct complex numbers such that $|z_1| = |z_2| = |z_3| = 1$ and
$$\frac{z_1^2}{z_2z_3} + \frac{z_2^2}{z_3z_1} + \frac{z_3^2}{z_1z_2} = -1$$
then the value of $|z_1 + z_2 + z_3|$ can be
(A) 1/2 (B) 3 (C) 3/2 (D) 2

25. If z_1, z_2 and z_3 are the vertices of a right-angled isosceles triangle described in counter clock sense and right angled at z_3, then $(z_1 - z_2)^2$ is equal to
(A) $(z_1 - z_3)(z_3 - z_2)$ (B) $2(z_1 - z_3)(z_3 - z_2)$
(C) $3(z_1 - z_3)(z_3 - z_2)$ (D) $3(z_3 - z_1)(z_3 - z_2)$

Multiple Correct Choice Type Questions

In this section, each question has 4 choices (A), (B), (C), and (D) for its answer, out of which one or more is/are correct.

1. Let $z = 1 + \cos(10\pi/9) + i\sin(10\pi/9)$. Then
(A) $|z| = 2\cos\left(\frac{2\pi}{9}\right)$ (B) $\arg z = \frac{8\pi}{9}$
(C) $|z| = 2\cos\left(\frac{4\pi}{9}\right)$ (D) $\arg z = \frac{5\pi}{9}$

2. If $z = -(1+i)$, then
(A) $\arg z = \frac{\pi}{4}$ (B) $\arg z = \frac{5\pi}{4}$
(C) $\operatorname{Arg} z = \frac{-3\pi}{4}$ (D) $|z| = \sqrt{2}$

3. If z_1, z_2 and z_3 are the vertices of an equilateral triangle described in counterclock sense and $w \neq 1$ is a cube root of unity, then

(A) $z_1 - z_3 = (z_3 - z_2)w$

(B) $z_1 + z_2 w + z_3 w^2 = 0$

(C) $\dfrac{1}{z_1 - z_2} + \dfrac{1}{z_2 - z_3} + \dfrac{1}{z_3 - z_1} = 0$

(D) $z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$

4. Let $z_1 = 1 + i$, $z_2 = -1 - i$ and z_3 be complex numbers such that z_1, z_2 and z_3 form an equilateral triangle. Then z_3 is equal to

(A) $\sqrt{3}(1 + i)$ (B) $\sqrt{3}(1 - i)$

(C) $\sqrt{3}(i - 1)$ (D) $\sqrt{3}(-1 - i)$

5. If $\cos\alpha + \cos\beta + \cos\gamma = 0 = \sin\alpha + \sin\beta + \sin\gamma$, then

(A) $\cos(2\alpha) + \cos(2\beta) + \cos(2\gamma) = 0$

(B) $\sin(3\alpha) + \sin(3\beta) + \sin(3\gamma) = 3\sin(\alpha + \beta + \gamma)$

(C) $\cos(3\alpha) + \cos(3\beta) + \cos(3\gamma) = 3\cos(\alpha + \beta + \gamma)$

(D) $\sin(2\alpha) + \sin(2\beta) + \sin(2\gamma) = 0$

6. Let $a > 0$ and $|z + (1/z)| = a$ ($z \neq 0$ is a complex number). Then the maximum and minimum values of $|z|$ are

(A) $\dfrac{a + \sqrt{a^2 + 4}}{2}$ (B) $\dfrac{2a + \sqrt{a^2 + 4}}{2}$

(C) $\dfrac{\sqrt{a^2 + 4} - a}{2}$ (D) $\dfrac{\sqrt{a^2 + 4} - 2a}{2}$

7. $ABCD$ is a rhombus. Its diagonals AC and BD intersect at M and satisfy $BD = 2AC$. If the points D and M are represented by the complex numbers $1 + i$ and $2 - i$, respectively, then A is represented by

(A) $3 - i/2$ (B) $3 + i/2$ (C) $1 + 3i/2$ (D) $1 - 3i/2$

8. If the vertices of a square described in counter clock sense are represented by the complex numbers z_1, z_2, z_3 and z_4, then

(A) $z_2 = \dfrac{1}{2}(1 + i)z_1 + \dfrac{1}{2}(1 - i)z_3$

(B) $z_4 = \dfrac{1}{2}(1 - i)z_1 + \dfrac{1}{2}(1 + i)z_3$

(C) $z_3 = \dfrac{1}{2}(i - 1)z_2 + \dfrac{1}{2}(1 + i)z_4$

(D) $z_1 = \dfrac{1}{2}(i + 1)z_2 + \dfrac{1}{2}(1 - i)z_4$

9. Let p and q be positive integers having no positive common divisors except unity. Let $z_1, z_2, \ldots, z_q$ be the q values of $z^{p/q}$, where z is a fixed complex number. Then the product $z_1 z_2 \cdots z_q$ is equal to

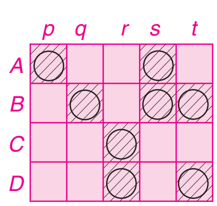
(A) z^p, if q is odd (B) $-z^p$, if q is even

(C) z^p, if q is even (D) $-z^p$, if q is odd

Matrix-Match Type Questions

In each of the following questions, statements are given in two columns, which have to be matched. The statements in Column I are labeled as **(A)**, **(B)**, **(C)** and **(D)**, while those in Column II are labeled as **(p)**, **(q)**, **(r)**, **(s)** and **(t)**. Any given statement in Column I can have correct matching with *one or more* statements in Column II. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following example.

Example: If the correct matches are (A) → (p), (s); (B) → (q), (s), (t); (C) → (r); (D) → (r), (t); that is if the matches are (A) → (p) and (s); (B) → (q), (s) and (t); (C) → (r); and (D) → (r), (t); then the correct darkening of bubbles will look as follows:



1. In Column I equations are given and in Column II the number of ordered pairs (x, y) satisfying the equations are given. Match them assuming that x and y are real numbers.

Column I	*Column II*
(A) $(x + 2y) + i(2x - 3y) = 5 - 4i$	(p) 1
(B) $(x + iy) + (7 - 5i) = 9 + 4i$	(q) 2
	(r) 3
(C) $x^2 - y^2 - i(2x + y) = 2i$	(s) 4
(D) $(2 + 3i)x^2 - (3 - 2i)y = 2x - 3y + 5i$	(t) 0

2. Match the items in Column I with those in Column II.

Column I	Column II
(A) The number of values of $\theta \in (-\pi, \pi)$ for which $\frac{3+2i\sin\theta}{1-2i\sin\theta}$ is purely real is	(p) 2
(B) The number of values of $\theta \in (-\pi, \pi)$ for which $\frac{3+2i\sin\theta}{1-2i\sin\theta}$ is purely imaginary is	(q) 3
(C) The number of solutions of the equation $(x^4+2xi)-(3x^2+iy) = (1+2yi)+(3-5i)$ where x and y are positive real is	(r) 4
	(s) 0
(D) The number of complex numbers z such that $\bar{z} = iz^2$ is	(t) 1

3. In Column I equations which are satisfied by complex number z are given. In Column II curves represented by equations with real coefficients are given. Match the items in Column I with those in Column II.

Column I	Column II
(A) If $\text{Re}\left(\frac{iz+1}{iz-1}\right) = 2$, then z lies on the curve	(p) $4x^2+4y^2+x-6y+2=0$
(B) $z_1 = 6+i, z_2 = 4-3i$ and z is a complex number such that $\arg\left(\frac{z-z_1}{z_2-z}\right) = \frac{\pi}{2}$, then z lies on	(q) $x^2+y^2+4y+3=0$
	(r) $3(x^2+y^2)-2x-4y=0$
(C) If $\text{Im}\left(\frac{2z+1}{1+iz}\right) = 2$, then z lies on	(s) $x^2+y^2-x+2y-1=0$
(D) If $\left\lvert\frac{2z-i}{z+1}\right\rvert = 1$, then z lies on	(t) $(x-5)^2+(y+1)^2=5$

Comprehension-Type Questions

1. Passage: If z_1, z_2 and z_3 are three complex numbers representing the points A, B and C, respectively, in the Argands plane and $\angle BAC = \alpha$, then

$$\frac{z_3-z_1}{z_2-z_1} = \left(\frac{AC}{AB}\right)(\cos\alpha + i\sin\alpha)$$

Answer the following three questions.

(i) The four points $2+i, 4+i, 4+3i$ and $2+3i$ represent the vertices of

(A) Square

(B) Rhombus but not a square

(C) Rectangle but not a square

(D) Trapezium which is not rhombus/square/rectangle

(ii) The roots of the equation $z^3-1=0$ represent the vertices of

(A) An obtuse-angled triangle

(B) Isosceles but not an equilateral triangle

(C) Equilateral triangle

(D) Right-angled isosceles triangle

(iii) If the roots of the equation

$$z^3+3a_1z^2+3a_2z+a_3=0$$

represent the vertices of an equilateral triangle, then

(A) $a_1^2 = a_3$ (B) $a_1^2 = a_2$

(C) $a_1^2 = a_2a_3$ (D) $a_1^3 = a_2a_3$

2. Passage: Let X, Y and Z be the three sets of complex numbers defined as follows:

$$X = \{z : \text{Im}(z) \geq 1\}$$
$$Y = \{z : |z-2-i| = 3\}$$
$$Z = \{z : \text{Re}(z(1-i)) = \sqrt{2}\}$$

Answer the following questions.

(i) The number of elements in the set $X \cap Y \cap Z$ is

(A) 0 (B) 1 (C) 3 (D) Infinite

(ii) Let z be any point in $X \cap Y \cap Z$. Then $|z+1-i|^2 + |z-5-i|^2$ lies between

(A) 25 and 29 (B) 30 and 34

(C) 35 and 39 (D) 40 and 44

(iii) Let z be any point in $X \cap Y \cap Z$ and w be any point satisfying $|w-2-i| < 3$. Then $|z| - |w| + 3$ lies between

(A) –6 and 3 (B) –3 and 6

(C) –6 and 6 (D) –3 and 9

Assertion–Reasoning Type Questions

In each of the following, two statements, I and II, are given and one of the following four alternatives has to be chosen.

(A) Both I and II are correct and II is a correct reasoning for I.
(B) Both I and II are correct but II is not a correct reasoning for I.
(C) I is true, but II is not true.
(D) I is not true, but II is true.

1. **Statement I:** If p_1 and p_2 are distinct prime numbers and a complex number $\alpha \neq 1$ satisfies the equation $z^{p_1+p_2} - z^{p_1} - z^{p_2} + 1 = 0$, then either $1 + \alpha + \alpha^2 + \cdots + \alpha^{p_1-1} = 0$ or $1 + \alpha + \alpha^2 + \cdots + \alpha^{p_2-1} = 0$ but not both.

Statement II: For any two distinct prime numbers p_1 and p_2, the two equations $z^{p_1} - 1 = 0$ and $z^{p_2} - 1 = 0$ cannot have common roots other than unity.

2. **Statement I:** If α is a complex number satisfying the equation $(z+1)^8 = z^8$, then $\text{Re}(z) = -1$.

Statement II: If z_1 and z_2 are fixed complex numbers and z is any complex number such that $|z - z_1| = |z - z_2|$, then z lies on the perpendicular bisector the segment joining z_1 and z_2.

3. Let

$$a = 1 + \frac{x^3}{\lfloor 3} + \frac{x^6}{\lfloor 6} + \cdots + \infty$$

$$b = \frac{x}{\lfloor 1} + \frac{x^4}{\lfloor 4} + \frac{x^7}{\lfloor 7} + \cdots + \infty$$

$$c = \frac{x^2}{\lfloor 2} + \frac{x^5}{\lfloor 5} + \frac{x^8}{\lfloor 8} + \cdots + \infty,$$

Statement I: $a^3 + b^3 + c^3 - 3abc = 1$.

Statement II: $a^3+b^3+c^3-3abc = (a+b+c)(a+bw+cw^2)$, $(a + bw^2 + cw)$ where $w \neq 1$ is a cube root of unity.

4. **Statement I:** Let $\bar{l}z + l\bar{z} + m = 0$ be a line in the complex plane, where $l \neq 0$ is a complex number and m is a real number. If two points z_1 and z_2 are reflections of each other in the line, then $\bar{l}z_1 + l\bar{z}_2 + m = 0$.

Statement II: Equation of the perpendicular bisector of the segment joining two points z_1 and z_2 in the complex plane is $z(\bar{z}_1 - \bar{z}_2) + \bar{z}(z_1 - z_2) - z_1\bar{z}_1 + z_2\bar{z}_2 = 0$.

5. **Statement I:** If a, b, c and u, v, w are complex numbers representing the vertices of two triangles such that $c = (1-\gamma)a + \gamma b$ and $w = (1-\gamma)u + \gamma v$, then the two triangles are similar.

Statement II: Complex numbers z_1, z_2, z_3 and z_1', z_2', z_3' represent the vertices of directly similar triangles if and only if the determinant

$$\begin{vmatrix} z_1 & z_1' & 1 \\ z_2 & z_2' & 1 \\ z_3 & z_3' & 1 \end{vmatrix} = 0$$

6. **Statement I:** If α and β are fixed complex numbers, then the equation $|(z-\alpha)/(z-\beta)| = K(\neq 1)$ represents a circle whose radius and center are $K|\alpha - \beta|/|1 - K^2|$ and $(\alpha - K^2\beta)/(1 - K^2)$.

Statement II: If a is a non-zero complex number and b is real such that $|a|^2 > b$, then the equation $z\bar{z} + a\bar{z} + \bar{a}z + b = 0$ represents a circle with center at $-a$ and radius $\sqrt{a\bar{a} - b}$.

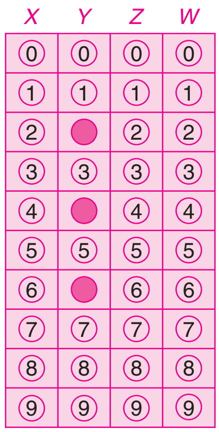
7. **Statement I:** Let A, B and C be vertices of a triangle described in counter clock sense and, respectively, be represented by z_1, z_2 and z_3. Then the area of ΔABC is $|\text{Im}(\bar{z}_1 z_2 + \bar{z}_2 z_3 + \bar{z}_3 z_1)/2|$.

Statement II: The area of ΔABC is equal to the absolute value of the number

$$\frac{i}{4}\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix}$$

Integer Answer Type Questions

The answer to each of the questions in this section is a non-negative integer. The appropriate bubbles below the respective question numbers have to be darkened. For example, as shown in the figure, if the correct answer to the question number Y is 246, then the bubbles under Y labeled as 2, 4, 6 are to be darkened.



1. The number of common roots of the equations $x^5 - x^3 + x^2 - 1 = 0$ and $x^4 - 1 = 0$ is ______.

2. The quadratic equation $z^2 + (a + ib)z + (c + id) = 0$ (a, b, c, d) are real and $(bd \neq 0)$ has equal roots. Then the value of ab/d is ______.

3. If the equation $z^2 + (a + ib)z + (c + id) = 0$ (a, b, c, d) are real and $(bd \neq 0)$ has real root, where k is real, then $d^2 - abd + bc$ is equal to ______.

4. If z_1 and z_2 are complex numbers such that $|z_2| \neq 1$ and $|(z_1 - 2z_2)/(2 - z_1\bar{z}_2)| = 1$, then $|z_1|$ is equal to ______.

5. If z_2/z_1 is pure imaginary and a and b are non-zero real numbers, then $|(az_1 + bz_2)/(az_1 - bz_2)|$ is equal to ______.

6. If the points $1 + 2i$ and $-1 + 4i$ are real reflections of each other in the line $z(1 + i) + \bar{z}(1 - i) + K = 0$, then the value of K is ______.

7. If the straight lines $\bar{a}_i z + a_i \bar{z} + b_i = 0 (i = 1, 2, 3)$, where b_i are real, are concurrent, then $\sum b_1(a_2\bar{a}_3 - \bar{a}_2 a_3)$ is equal to ______.

8. The number of points z in the complex plane satisfying both the equations $|z - 4 - 8i| = \sqrt{10}$ and $|z - 3 - 5i| + |z - 5 - 11i| = 4\sqrt{5}$ is ______.

9. If $z = x + iy$ satisfies the equation $z^2 + \bar{z}^2 = 2$ then $x^2 - y^2 = K$, where K is ______.

10. If the area of a triangle with vertices Z_1, Z_2 and Z_3 is the absolute value of the number

$$\lambda i \begin{vmatrix} Z_1 & \bar{Z}_1 & 1 \\ Z_2 & \bar{Z}_2 & 1 \\ Z_3 & \bar{Z}_3 & 1 \end{vmatrix}$$

then the value of $1/\lambda$ is equal to ______.

ANSWERS

Single Correct Choice Type Questions

1. (B)
2. (C)
3. (A)
4. (D)
5. (B)
6. (D)
7. (D)
8. (D)
9. (C)
10. (B)
11. (C)
12. (C)
13. (A)
14. (D)
15. (D)
16. (A)
17. (A)
18. (D)
19. (B)
20. (A)
21. (A)
22. (C)
23. (D)
24. (D)
25. (B)

Multiple Correct Choice Type Questions

1. (C), (D)
2. (B), (C), (D)
3. (A), (B), (C), (D)
4. (B), (C)
5. (A), (B), (C), (D)
6. (A), (C)
7. (A), (D)
8. (A), (B)
9. (A), (B)

Matrix-Match Type Questions

1. (A) $\rightarrow$ (p), (B) $\rightarrow$ (p), (C) $\rightarrow$ (q), (D) $\rightarrow$ (q)
2. (A) $\rightarrow$ (t), (B) $\rightarrow$ (r), (C) $\rightarrow$ (t), (D) $\rightarrow$ (r)
3. (A) $\rightarrow$ (q), (B) $\rightarrow$ (t), (C) $\rightarrow$ (p), (D) $\rightarrow$ (r)

Comprehension-Type Questions

1. (i) (A); (ii) (C); (iii) (B)
2. (i) (B); (ii) (C); (iii) (D)

Assertion–Reasoning Type Questions

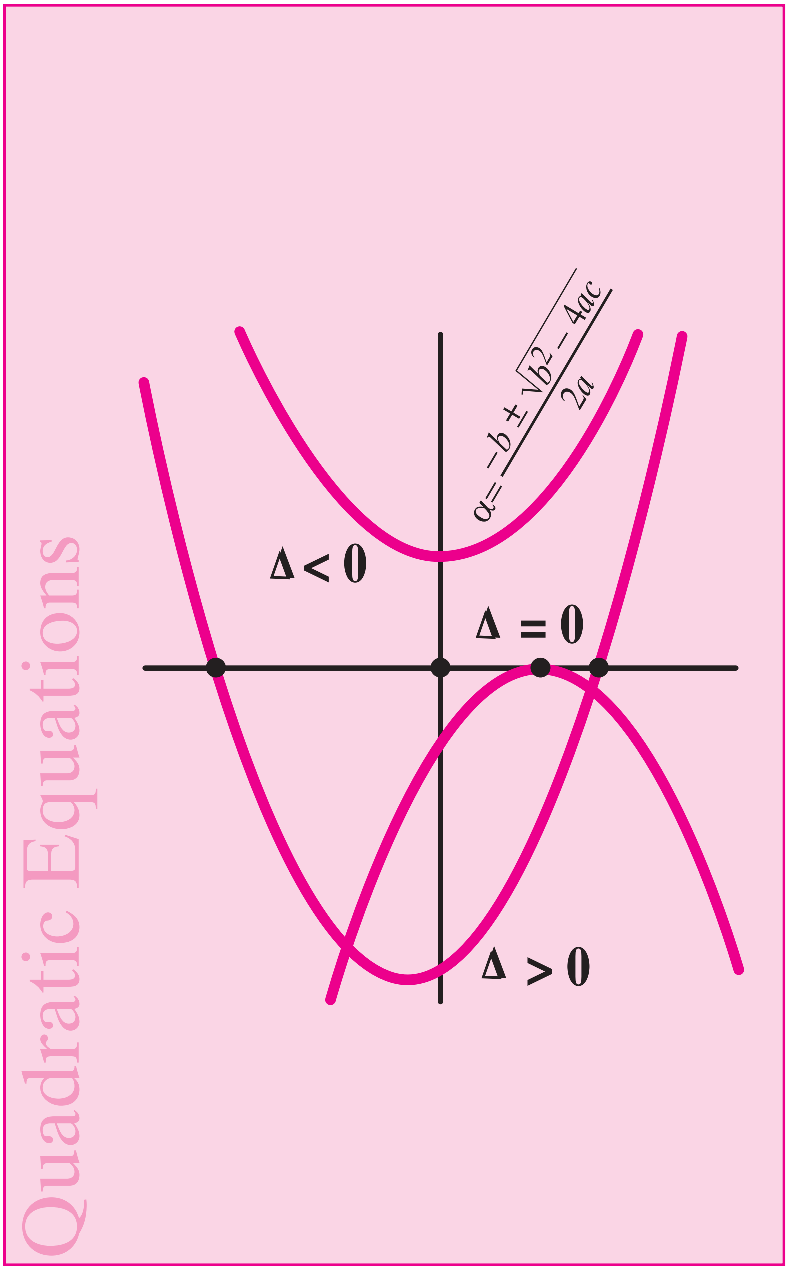
1. (A)
2. (D)
3. (A)
4. (A)
5. (A)
6. (A)
7. (A)

Integer Answer Type Questions

1. 2
2. 2
3. 0
4. 2
5. 1
6. 6
7. 0
8. 2
9. 1
10. 4

Quadratic Equations

4



Contents

A polynomial equation of the second degree having the general form

$$ax^2 + bx + c = 0$$

is called a **quadratic equation**. Here x represents a variable, and a, b, and c, constants, with $a \neq 0$. The constants a, b, and c are called, respectively, the quadratic coefficient, the linear coefficient and the constant term or the free term.

The term "quadratic" comes from *quadratus*, which is the Latin word for "square". Quadratic equations can be solved by factoring, completing the square, graphing, Newton's method, and using the **quadratic formula** (explained in the chapter).

In this chapter, we will discuss quadratic expressions and equations along with their roots. Numerous examples and worked-out problems would help the readers understand the concepts. Exercises at the end of the chapters would help evaluate your understanding.

4.1 | Quadratic Expressions and Equations

In this section, we discuss quadratic expressions and equations and their roots. Also, we derive various properties of the roots of quadratic equations and their relationships with the coefficients.

DEFINITION 4.1 A polynomial of the form $ax^2 + bx + c$, where a, b and c are real or complex numbers and $a \neq 0$, is called a ***quadratic expression*** in the variable x. In other words, a polynomial $f(x)$ of degree two over the set of complex numbers is called a *quadratic expression*. We often write $f(x) \equiv ax^2 + bx + c$ to denote a quadratic expression and this is known as the ***standard form***. In this case, a and b are called the coefficients of x^2 and x, respectively, and c is called the constant term. The term ax^2 is called the ***quadratic term*** and bx is called the ***linear term***.

DEFINITION 4.2 If $f(x) \equiv ax^2 + bx + c$ is a quadratic expression and α is a complex number, then we write $f(\alpha)$ for $a\alpha^2 + b\alpha + c$. If $f(\alpha) = 0$, then α is called a ***zero*** of the quadratic expression $f(x)$.

Examples

(1) Let $f(x) \equiv x^2 - 5x - 6$. Then $f(x)$ is a quadratic expression and 6 and –1 are zeros of $f(x)$.

(2) Let $f(x) \equiv x^2 + 1$. Then $f(x)$ is a quadratic expression and i and $-i$ are zeros of $f(x)$.

(3) Let $f(x) \equiv 2x^2 - ix + 1$ be a quadratic expression. In this case i and $-i/2$ are zeros of $f(x)$.

(4) The expression $x^2 + x$ is a quadratic expression and 0 and –1 are zeros of $x^2 + x$.

DEFINITION 4.3 If $f(x)$ is a quadratic expression, then $f(x) = 0$ is called a ***quadratic equation***. If α is a zero of $f(x)$, then α is called a ***root*** or a ***solution*** of the quadratic equation $f(x) = 0$. In other words, if $f(x) \equiv ax^2 + bx + c$, $a \neq 0$, then a complex number α is said to be a root or a solution of $f(x) = 0$, if $a\alpha^2 + b\alpha + c = 0$. The zeros of the quadratic expression $f(x)$ are same as the roots or solutions of the quadratic equation $f(x) = 0$. Note that α is a zero of $f(x)$ if and only if $x - \alpha$ is a factor of $f(x)$.

Examples

(1) 0 and $-i$ are the roots of $x^2 + ix = 0$.

(2) 2 is the only root of $x^2 - 4x + 4 = 0$.

(3) i and $-i$ are the roots of $x^2 + 1 = 0$.

(4) i is the only root of $x^2 - 2ix - 1 = 0$.

THEOREM 4.1 Let $f(x) \equiv ax^2 + bx + c$ be a quadratic expression. Then the roots of the quadratic equation $f(x) = 0$ are

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

that is,

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

PROOF First note that for $f(x) \equiv ax^2 + bx + c$ to be a quadratic equation, it is necessary that $a \neq 0$. Let α be any complex number. Then

$$\begin{aligned}\alpha \text{ is a root of } f(x)=0 &\Leftrightarrow a\alpha^2+b\alpha+c=0\\ &\Leftrightarrow 4a(a\alpha^2+b\alpha+c)=0\\ &\Leftrightarrow (2a\alpha+b)^2-b^2+4ac=0\\ &\Leftrightarrow (2a\alpha+b)^2=b^2-4ac\\ &\Leftrightarrow 2a\alpha+b=\pm\sqrt{b^2-4ac}\\ &\Leftrightarrow \alpha=\frac{-b\pm\sqrt{b^2-4ac}}{2a}\end{aligned}$$

■

Note that, in the above, $\sqrt{b^2-4ac}$ denotes a square root of b^2-4ac; that is, it is a complex number β such that $\beta^2=b^2-4ac$. From the above theorem, it follows that any quadratic equation has two roots, which are not necessarily distinct. This is demonstrated in the examples described before. In the following some more examples are considered.

Examples

(1) Consider the quadratic equation $f(x)\equiv x^2+x+1=0$. Comparing with the standard form ax^2+bx+c, we have $a=1=b=c$. Therefore, the roots of the given equation are

$$\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-1\pm\sqrt{1^2-4\times1\times1}}{2\times1}=\frac{-1\pm i\sqrt{3}}{2}$$

Recall from the previous chapter that these are precisely the cube roots of unity other than the unity.

(2) The roots of the quadratic equation $x^2+4ix-4=0$ are

$$\frac{-4i\pm\sqrt{(4i)^2-(4(-4)\times1)}}{2\times1}=\frac{-4i\pm\sqrt{-16+16}}{2}=-2i$$

$-2i$ is a repeated root or a double root of the given equation.

(3) Consider the equation $3x^2+2x+1=0$. The roots of this equation are

$$\frac{-2\pm\sqrt{(2)^2-4\times3\times1}}{2\times3}=\frac{1}{3}(-1\pm i\sqrt{2})$$

(4) Consider the equation $\sqrt{3}(x^2+2)+10(x-\sqrt{3})=0$. To find its roots, we have to first transform this into the standard form $ax^2+bx+c=0$. We thus obtain

$$\sqrt{3}(x^2+2)+10(x-\sqrt{3})=\sqrt{3}x^2+10x-8\sqrt{3}$$

Therefore, the roots of the given equation are

$$\frac{-10\pm\sqrt{(10)^2-4\sqrt{3}(-8\sqrt{3})}}{2\sqrt{3}}=\frac{-10\pm14}{2\sqrt{3}}=\frac{2}{\sqrt{3}},-4\sqrt{3}$$

DEFINITION 4.4 Let $f(x)\equiv ax^2+bx+c$, $a\neq0$. Then the ***discriminant*** of the quadratic expression $f(x)$ or the quadratic equation $f(x)=0$ is defined as b^2-4ac and is denoted by $\Delta[f(x)]$ or simply Δ.

It is evident that the roots of a quadratic equation $f(x)=0$ are real or imaginary according as the discriminant of $f(x)$ is non-negative or negative, respectively. In the following we list the various natures of roots of a quadratic equation which mainly depend on the nature of the discriminant. The proof of the following theorem is a straight-forward verification.

THEOREM 4.2 Let α and β be the roots of the quadratic equation $f(x)\equiv ax^2+bx+c=0$, where a, b and c are real or complex numbers and $a\neq0$. Let Δ be the discriminant of $f(x)$, that is, $\Delta=b^2-4ac$. Then the following hold good:

1. $\alpha=\beta\Leftrightarrow\Delta=0$ (i.e., $b^2=4ac$), and in this case

$$\alpha=\frac{-b}{2a}=\beta$$

2. If a, b and c are real numbers, then

(i) $\Delta > 0 \Leftrightarrow \alpha$ and β are real numbers and $\alpha \neq \beta$.

(ii) $\Delta < 0 \Leftrightarrow \alpha$ and β are non-real complex numbers which are conjugate to each other.

PROOF The proof is left as an exercise for the readers. ■

Examples

(1) The equation $x^2 + 5x + 7 = 0$ has no real roots, since the discriminant $(5)^2 - 4 \times 7 \times 1 = -3 < 0$.

(2) Suppose that we wish to find the value of k such that the equation $x^2 + 2(k+2)x + 9k = 0$ has equal roots. The discriminant is given by

$$\Delta = b^2 - 4ac = [2(k+2)]^2 - 4 \times 1 \times 9k$$
$$= 4k^2 + 16k + 16 - 36k$$
$$= 4k^2 - 20k + 16$$

Since the roots are equal, therefore the discriminant should be zero, that is

$$\Delta = 0 \Leftrightarrow k^2 - 5k + 4 = 0$$
$$\Leftrightarrow k = \frac{-(-5) \pm \sqrt{(-5)^2 - 4 \times 1 \times 4}}{2}$$
$$\Leftrightarrow k = \frac{5 \pm 3}{2} = 4 \quad \text{or} \quad 1$$

(3) If a, b and c are rational numbers, then the roots of the equation

$$x^2 - 2ax + a^2 - b^2 + 2bc - c^2 = 0$$

are also rational, for the discriminant is given by

$$\Delta = (-2a)^2 - 4 \times 1 \times (a^2 - b^2 + 2bc - c^2)$$
$$= 4a^2 - 4a^2 + 4b^2 - 8bc + 4c^2$$
$$= 4b^2 - 8bc + 4c^2$$
$$= 4(b-c)^2$$

Since b and c are rational numbers, $(b-c)^2$ is a non-negative rational number and hence $\Delta \geq 0$, so that the given equation has real roots. Also, the roots are

$$\frac{-(-2a) \pm \sqrt{4(b-c)^2}}{2} = a \pm (b-c)$$

which are rational numbers, since a, b and c are so.

THEOREM 4.3 Let α and β be the roots of the quadratic equation $ax^2 + bx + c = 0$. Then

$$\alpha + \beta = \frac{-b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a}$$

PROOF The values of α and β are given by

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

and hence

$$\alpha + \beta = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{-b}{a}$$

and

$$\alpha\beta = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \times \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{b^2 - (b^2 - 4ac)}{4a^2} = \frac{c}{a}$$ ■

Also, we can write down a quadratic equation if the roots are known. In other words, if α and β are any given complex numbers, then $a(x-\alpha)(x-\beta) = a[x + (-\alpha - \beta)x + \alpha\beta] = 0$ is a quadratic equation whose roots are α and β, where a is an arbitrary non-zero real or complex number. This can also be verified by observing

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$
$$= a(x^2 - (\alpha + \beta)x + \alpha\beta)$$
$$= a(x-\alpha)(x-\beta)$$

QUICK LOOK 1

If the coefficient of x^2 in a quadratic equation is unity (i.e., 1), then

1. The sum of the roots is equal to the coefficient of x with its sign changed; that is, $\alpha + \beta + b = 0$, where b is the coefficient of x.
2. The product of the roots is equal to the constant term.
3. The equation can be written as $(x-\alpha)(x-\beta) = 0$, where α and β are the roots.

Example 4.1

Find the quadratic equation whose roots are 2 and $-i$.

Solution: The required quadratic expression is

$$(x-2)[x-(-i)] = (x-2)(x+i) = x^2 + (i-2)x - 2i$$

Hence the equation is $x^2 + (i-2)x - 2i = 0$.

Example 4.2

Find the quadratic equation whose roots are $1+i$ and $1-i$ and in which the coefficient of x^2 is 3.

Solution: The required quadratic expression is

$$\begin{aligned} 3[x-(1+i)](x-(1-i)) &= 3[(x-1)-i)][(x-1)+i] \\ &= 3[(x-1)^2+1] \\ &= 3x^2 - 6x + 6 \end{aligned}$$

Hence the equation is $3x^2 - 6x + 6 = 0$.

Example 4.3

If α and β are roots of the quadratic equation $ax^2 + bx + c = 0$ and z is any complex number, then find the quadratic equation whose roots are $z\alpha$ and $z\beta$.

Solution: We have

$$\alpha + \beta = \frac{-b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a}$$

The equation whose roots are $z\alpha$ and $z\beta$ is

$$\begin{aligned} 0 &= (x - z\alpha)(x - z\beta) \\ &= x^2 - (z\alpha + z\beta)x + z\alpha \times z\beta \\ &= x^2 + z[-(\alpha+\beta)]x + z^2\alpha\beta \\ &= x^2 + z\left(\frac{b}{a}\right)x + z^2\frac{c}{a} \end{aligned}$$

that is,

$$ax^2 + zbx + z^2c = 0$$

Example 4.4

If α and β are the roots of a quadratic equation $ax^2 + bx + c = 0$, then find the quadratic equation whose roots are $\alpha + z$ and $\beta + z$, where z is any given complex number.

Solution: We have

$$(\alpha+z) + (\beta+z) = (\alpha+\beta) + 2z = \frac{-b}{a} + 2z$$

and $$(\alpha+z)\cdot(\beta+z) = \alpha\beta + (\alpha+\beta)z + z^2 = \frac{c}{a} - \frac{b}{a}z + z^2$$

Therefore, the required equation is

$$\begin{aligned} 0 &= a[x-(\alpha+z)] \times [x-(\beta+z)] \\ &= ax^2 + a[-(\alpha+z)-(\beta+z)]x + a(\alpha+z)(\beta+z) \\ &= ax^2 + a\left(\frac{b}{a} - 2z\right)x + a\left(\frac{c}{a} - \frac{b}{a}z + z^2\right) \\ &= ax^2 + (b - 2az)x + (c - bz + az^2) \end{aligned}$$

Therefore, the quadratic equation whose roots are $\alpha + z$ and $\beta + z$ is

$$ax^2 + (b-2az)x + (c - bz + az^2) = 0$$

Example 4.5

Let α and β be the roots of a quadratic equation $ax^2 + bx + c = 0$ and p and q be any complex numbers. Then find the quadratic equation whose roots are $p\alpha + q$ and $p\beta + q$.

Solution: Consider,

$$(p\alpha + q) + (p\beta + q) = p(\alpha + \beta) + 2q = -\frac{pb}{a} + 2q$$

and $\quad (p\alpha + q)\cdot(p\beta + q) = p^2\alpha\beta + pq(\alpha + \beta) + q^2$

$$= \frac{p^2c}{a} - \frac{pqb}{a} + q^2$$

Therefore, the required equation is

$$x^2 - \left(\frac{-pb}{a} + 2q\right)x + \left(\frac{p^2c}{a} - \frac{pqb}{a} + q^2\right) = 0$$

$$ax^2 + (pb - 2aq)x + (p^2c - pqb + q^2a) = 0$$

Example 4.6

If α and β are roots of the quadratic equation

$$ax^2 + bx + c = 0 \quad \text{and} \quad c \neq 0$$

find the quadratic equation whose roots are $1/\alpha$ and $1/\beta$.

Solution: First, let us observe that $\alpha \neq 0$ and $\beta \neq 0$, as α and β are roots of $ax^2 + bx^2 + c = 0$ and $c \neq 0$. Now, consider

$$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta} = \frac{-b/a}{c/a} = -\frac{b}{c}$$

and $\quad \left(\frac{1}{\alpha}\right)\left(\frac{1}{\beta}\right) = \frac{1}{\alpha\beta} = \frac{1}{c/a} = \frac{a}{c}$

Therefore, the required equation is

$$x^2 - \left(\frac{-b}{c}\right)x + \frac{a}{c} = 0$$

That is,

$$cx^2 + bx + a = 0$$

The results obtained in the examples given above are summarized in the following and the reader can easily supplement formal proofs of these.

QUICK LOOK 2

Let $f(x) \equiv ax^2 + bx + c = 0$ be a quadratic equation and α and β be its roots. Then the following hold good.

1. $f(x - z) = 0$ is an equation whose roots are $\alpha + z$ and $\beta + z$, for any given complex number z.
2. $f(x/z) = 0$ is an equation whose roots are $z\alpha$ and $z\beta$ for any non-zero complex number z.
3. $f(-x) = 0$ is an equation whose roots are $-\alpha$ and $-\beta$.
4. If $\alpha\beta \neq 0$ and $c \neq 0$, $f(1/x) = 0$ is an equation whose roots are $1/\alpha$ and $1/\beta$.
5. For any complex numbers z_1 and z_2 with $z_1 \neq 0$, $f[(x - z_2)/z_1] = 0$ is an equation whose roots are $z_1\alpha + z_2$ and $z_1\beta + z_2$.

Note: If $ax^2 + bx + c = 0$ is a quadratic equation, then for any non-zero complex number d, the equation

$$dax^2 + dbx + dc = 0$$

has the same roots as $ax^2 + bx + c = 0$. Therefore, given α and β, the quadratic equation whose roots are α and β is not unique. However any two such equations are equivalent in the sense that their coefficients are proportional.

THEOREM 4.4 Two quadratic equations

$$ax^2 + bx + c = 0 \quad \text{and} \quad a'x^2 + b'x + c' = 0$$

have same roots if and only if the triples (a, b, c) and (a', b', c') are proportional and, in this case,

$$ax^2 + bx + c = \frac{a}{a'}(a'x^2 + b'x + c')$$

PROOF Suppose that α and β are the roots of $ax^2 + bx + c = 0$ and $a'x^2 + b'x + c' = 0$ simultaneously. Then by Theorem 4.3 we have

$$\frac{-b}{a} = \alpha + \beta = -\frac{b'}{a'} \quad \text{and} \quad \frac{c}{a} = \alpha\beta = \frac{c'}{a'}$$

Now

$$(a, b, c) = a\left(1, \frac{b}{a}, \frac{c}{a}\right) = a\left(1, \frac{b'}{a'}, \frac{c'}{a'}\right) = \frac{a}{a'}(a', b', c')$$

Therefore, (a, b, c) and (a', b', c') are proportional and

$$ax^2 + bx + c = \frac{a}{a'}(a'x^2 + b'x + c')$$

Conversely, suppose that (a, b, c) and (a', b', c') are proportional. Then, there is non-zero d such that

$$(a, b, c) = d(a', b', c')$$

and hence $ax^2 + bx + c = d(a'x^2 + b'x + c')$. Therefore, for any complex number α,

$$a\alpha^2 + b\alpha + c = 0 \Leftrightarrow a'\alpha^2 + b'\alpha + c' = 0$$

Example

The quadratic equations $2x^2 + 3x + 1 = 0$ and $6x^2 + 9x + 3 = 0$ have same roots since $3(2, 3, 1) = (6, 9, 3)$.

Example 4.7

Let α and β be the roots of the quadratic equation

$$ax^2 + bx + c = 0, \quad c \neq 0$$

Find the quadratic equation whose roots are

$$\frac{1-\alpha}{\alpha} \quad \text{and} \quad \frac{1-\beta}{\beta}$$

Solution: We have

$$\alpha + \beta = \frac{-b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a}$$

Now, consider

$$\frac{1-\alpha}{\alpha} + \frac{1-\beta}{\beta} = \frac{\beta(1-\alpha) + \alpha(1-\beta)}{\alpha\beta}$$

$$= \frac{(\alpha+\beta) - 2\alpha\beta}{\alpha\beta}$$

$$= \frac{(-b/a) - 2(c/a)}{c/a}$$

$$= \frac{-b-2c}{c} = -\left(\frac{b}{c} + 2\right)$$

Also

$$\left(\frac{1-\alpha}{\alpha}\right)\left(\frac{1-\beta}{\beta}\right) = \frac{1-(\alpha+\beta)+\alpha\beta}{\alpha\beta}$$

$$= \frac{1+(b/a)+(c/a)}{c/a}$$

$$= \frac{a+b+c}{c}$$

Therefore, the quadratic equation whose roots are $(1-\alpha)/\alpha$ and $(1-\beta)/\beta$ is

$$x^2 - \left[-\left(\frac{b}{c} + 2\right)\right]x + \frac{a+b+c}{c} = 0$$

$$cx^2 + (b+2c)x + (a+b+c) = 0$$

Example 4.8

If α and β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then evaluate the following:

(i) $\alpha^2 + \beta^2$ **(ii)** $\alpha^3 + \beta^3$ **(iii)** $\alpha^4 + \beta^4$

Solution: We know that

$$\alpha + \beta = \frac{-b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a}$$

(i) $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

$$= \left(-\frac{b}{a}\right)^2 - 2\left(\frac{c}{a}\right) = \frac{1}{a^2}(b^2 - 2ac)$$

(ii) $\alpha^3 + \beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)$

$$= \left(-\frac{b}{a}\right)^3 - 3\left(\frac{c}{a}\right)\left(-\frac{b}{a}\right) = \frac{1}{a^3}(3abc - b^3)$$

(iii) $\alpha^4 + \beta^4 = (\alpha^2 + \beta^2)^2 - 2\alpha^2\beta^2$

$$= \left[\frac{1}{a^2}(b^2 - 2ac)\right]^2 - 2\left(\frac{c}{a}\right)^2$$

$$= \frac{1}{a^4}[(b^2 - 2ac)^2 - 2c^2a^2]$$

$$= \frac{1}{a^4}[b^4 + 2c^2a^2 - 4ab^2c]$$

Example 4.9

Find the quadratic equation whose roots are α and β, where $\alpha + \beta = 1$ and $\alpha^2 + \beta^2 = 13$.

Solution: We have

$$\alpha\beta = \frac{1}{2}[(\alpha + \beta)^2 - (\alpha^2 + \beta^2)] = \frac{1}{2}(1 - 13) = -6$$

Therefore, the required equation is

$$0 = x^2 - (\alpha + \beta)x + \alpha\beta = x^2 - x - 6$$

Certain polynomial equations of degree greater than two can be reduced to quadratic equations by suitable substitutions. These are demonstrated in the following examples.

Example 4.10

Find the solutions of the equation

$$x^4 - 4x^2 - 5 = 0$$

Solution: Put $y = x^2$. Then the given equation is reduced to

$$y^2 - 4y - 5 = 0$$

$$(y - 5)(y + 1) = 0$$

This gives $y = 5, -1$. Therefore

$$x^2 = 5, -1$$

$$x = \pm\sqrt{5}, \pm i$$

Hence $\sqrt{5}, -\sqrt{5}$, i and $-i$ are the solutions of the given equation.

Example 4.11

Solve the equation $x^4 - 3x^3 + 2x^2 - 3x + 1 = 0$.

Solution: Since zero is not a solution of this equation, we can divide both sides of the equation by x^2 and get an equation whose roots are same as that of the given equation. That is

$$x^2 - 3x + 2 - \frac{3}{x} + \frac{1}{x^2} = 0$$

$$x^2 + \frac{1}{x^2} - 3\left(x + \frac{1}{x}\right) + 2 = 0$$

Putting $y = x + (1/x)$, we get

$$(y^2 - 2) - 3y + 2 = 0$$

$$y^2 - 3y = 0$$

This gives $y = 0, 3$. When $y = 0$, we have

$$x + \frac{1}{x} = 0 \Rightarrow x^2 + 1 = 0 \Rightarrow x = \pm i$$

When $y = 3$, we have

$$x + \frac{1}{x} = 3 \Rightarrow x^2 - 3x + 1 = 0$$

$$\Rightarrow x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4}}{2} = \frac{3 \pm \sqrt{5}}{2}$$

Thus $i, -i, (3 + \sqrt{5})/2$ and $(3 - \sqrt{5})/2$ are the solutions of the given equation.

Example 4.12

Solve $x^{2/5} + 3x^{1/5} - 4 = 0$.

Solution: By substituting $y = x^{1/5}$, the given equation reduces to a quadratic equation given by

$$y^2 + 3y - 4 = 0$$

$$(y + 4)(y - 1) = 0$$

$$y = 1 \quad \text{or} \quad -4$$

Now

$$y = 1 \Rightarrow x^{1/5} = 1 \Rightarrow x = 1$$

$$y = -4 \Rightarrow x^{1/5} = -4 \Rightarrow x = (-4)^5 = -1024$$

Therefore -1024 and 1 are the solutions of the given equation.

Example 4.13

Solve $4^x + 3 \cdot 4^{-x} - 4 = 0$.

Solution: Substituting $y = 4^x$, we get

$$y + \frac{3}{y} - 4 = 0$$

$$y^2 - 4y + 3 = 0$$

$$(y-3)(y-1) = 0$$

$$y = 3 \quad \text{or} \quad 1$$

Now

$$y = 1 \Rightarrow 4^x = 1 \Rightarrow x = 0$$

$$y = 3 \Rightarrow 4^x = 3 \Rightarrow x = \log_4 3$$

Therefore 0 and $\log_4 3$ are the solutions of the given equation.

Example 4.14

Solve the following:

$$\sqrt{\frac{x}{1-x}} + \sqrt{\frac{1-x}{x}} = \frac{13}{6}$$

Solution: Substituting y for $\sqrt{x/(1-x)}$, we get

$$y + \frac{1}{y} = \frac{13}{6}$$

$$6y^2 - 13y + 6 = 0$$

$$(2y-3)(3y-2) = 0$$

$$y = \frac{3}{2} \quad \text{or} \quad \frac{2}{3}$$

Now

$$y = \frac{3}{2} \Rightarrow \sqrt{\frac{x}{1-x}} = \frac{3}{2}$$

$$\Rightarrow 4x = 9(1-x)$$

$$\Rightarrow x = \frac{9}{13}$$

$$y = \frac{2}{3} \Rightarrow \sqrt{\frac{x}{1-x}} = \frac{2}{3}$$

$$\Rightarrow 9x = 4(1-x)$$

$$\Rightarrow x = \frac{4}{13}$$

Therefore 9/13 and 4/13 are the solutions of the given equation.

Example 4.15

Find all pairs of consecutive positive odd integers such that the sum of their squares is 290.

Solution: Let x be a positive odd integer. Then $x + 2$ will be the next odd integer. We have to find all the value of x for which

$$x^2 + (x+2)^2 = 290$$

$$2x^2 + 4x - 286 = 0$$

$$x^2 + 2x - 143 = 0$$

$$(x+13)(x-11) = 0$$

$$x = -13 \quad \text{or} \quad x = 11$$

But x is given to be odd positive integer. Therefore $x = 11$ and $x + 2 = 13$. Thus, (11, 13) is the unique pair of consecutive positive odd integers such that the sum of their squares is 290.

Example 4.16

Derive a necessary condition that one root of the quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$ and $c \neq 0$, is n times the other, where n is a positive integer.

Solution: Let α and $n\alpha$ be roots of the equation $ax^2 + bx + c = 0$. Then

$$\alpha + n\alpha = -\frac{b}{a} \quad \text{and} \quad \alpha \cdot n\alpha = \frac{c}{a}$$

Therefore

$$\alpha = \frac{-b}{a(n+1)} \quad \text{and} \quad \left(\frac{-b}{a(n+1)}\right)^2 n = \frac{c}{a}$$

Simplifying the second equation we get $nab^2 = (n+1)^2 a^2 c$. Now since $a \neq 0$, $nb^2 = (n+1)^2 ac$.

THEOREM 4.5 If a, b and c are real numbers and $a \neq 0$, then $(4ac - b^2)/4a$ is the maximum or minimum value of quadratic equation of $f(x) \equiv ax^2 + bx + c$ according as $a < 0$ or $a > 0$, respectively.

PROOF We have

$$f(x) \equiv ax^2 + bx + c \equiv a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

$$\equiv a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right] \equiv a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

If $a < 0$, then

$$f(x) \leq \frac{4ac - b^2}{4a} = f\left(\frac{-b}{2a}\right) \quad \text{for all } x \in \mathbb{R}$$

Hence $(4ac - b^2)/4a$ is the maximum value of $f(x)$.
If $a > 0$, then

$$f\left(\frac{-b}{2a}\right) = \frac{4ac - b^2}{4a} \leq f(x) \quad \text{for all } x \in \mathbb{R}$$

Hence $(4ac - b^2)/4a$ is the minimum value of $f(x)$. ■

QUICK LOOK 3

1. If a, b and c are real numbers and $a < 0$, then $f(-b/2a)$ is the maximum value of $f(x) \equiv ax^2 + bx + c$.
2. If a, b and c are real numbers and $a > 0$, then $f(-b/2a)$ is the minimum value of $f(x) \equiv ax^2 + bx + c$.

Examples

(1) The maximum value of $2x - x^2 + 3$ is

$$2\left(\frac{-2}{2(-1)}\right) - \left(\frac{-2}{2(-1)}\right)^2 + 3 = 2 - 1 + 3 = 4$$

(2) The minimum value of $x^2 + 3x + 2$ is

$$\left(\frac{-3}{2(1)}\right)^2 + 3\left(\frac{-3}{2(1)}\right) + 2 = \frac{9}{4} - \frac{9}{2} + 2 = -\frac{1}{4}$$

THEOREM 4.6 Let $f(x) = ax^2 + bx + c$, where a, b and c are real numbers and $a \neq 0$.

1. If α and β are real roots of $f(x) = 0$ and $\alpha < \beta$, then
   (i) $f(x)$ and a will have the same sign for all real $x < \alpha$ or $x > \beta$.
   (ii) $f(x)$ and a will have opposite sign for all real x such that $\alpha < x < \beta$.
2. If $f(x) = 0$ has imaginary roots, then $f(x)$ and a will have the same sign for all real x.

Proof **1.** It is given that $f(x) \equiv a(x-\alpha)(x-\beta)$. Therefore

$$\frac{f(x)}{a} = (x-\alpha)(x-\beta)$$

(i) If $x < \alpha$, then $x < \beta$ also and hence $x-\alpha$ and $x-\beta$ are both negative, so that $f(x)/a > 0$. Similarly, if $x > \beta$, then $x > \alpha$ also and hence both $x-\alpha$ and $x-\beta$ are positive, so that $f(x)/a > 0$. Therefore, in either case, $f(x)/a$ is positive, and hence $f(x)$ and a are either both positive or both negative.

(ii) If $\alpha < x < \beta$, then $x-\alpha > 0$ and $x-\beta < 0$ and hence $f(x)/a < 0$ which implies that one of $f(x)$ and a are positive and the other is negative.

2. Suppose that $f(x) = 0$ has imaginary roots. Then $b^2 - 4ac < 0$ and

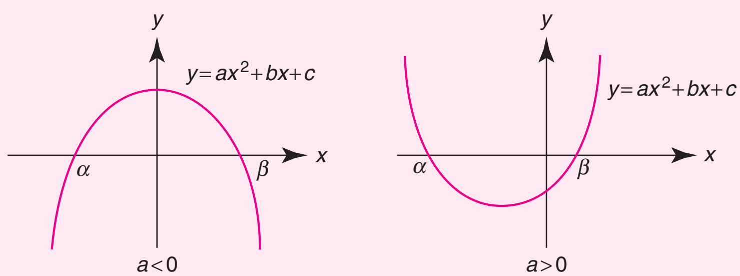
$$\frac{f(x)}{a} = \left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2} > 0$$

for all real x. Hence either both $f(x)$ and a are positive or both are negative. ∎

QUICK LOOK 4

Let $f(x) \equiv ax^2 + bx + c$, where a, b and c are real numbers and $a \neq 0$. Consider the graph of the curve $y = ax^2 + bx + c$. Different cases considered in Theorem 4.6 are described next by means of the graph of $y = ax^2 + bx + c$.

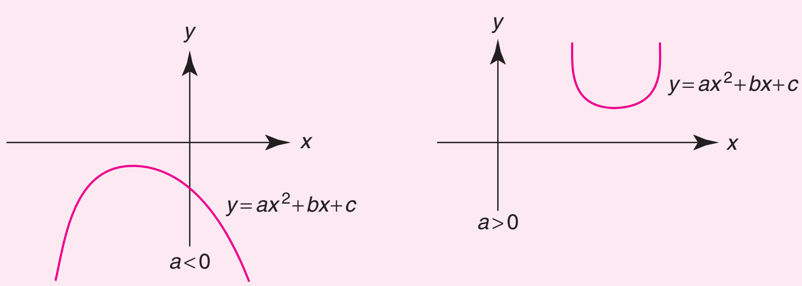
1.



$f(x) < 0$ for all $x \notin [\alpha, \beta]$
$f(x) > 0$ for all $x \in (\alpha, \beta)$

$f(x) > 0$ for all $x \notin [\alpha, \beta]$
$f(x) < 0$ for all $x \in (\alpha, \beta)$

2.



$f(x) = 0$ has no real roots

$f(x) = 0$ has no real roots

Examples

(1) $2x^2 - 11x + 15 > 0$ for all $x < 5/2$ or $x > 3$ [by Theorem 4.6 (1(i))] and $2x^2 - 11x + 15 < 0$ for all $5/2 < x < 3$ [by Theorem 4.6 (1(ii))] since 5/2 and 3 are roots of this quadratic expression in which the coefficient of x^2 is $2 > 0$.

(2) $-2x^2 + x + 15 < 0$ for all $x < -5/2$ or $x > 3$ and $-2x^2 + x + 15 > 0$ for all $-5/2 < x < 3$ since $-(5/2)$ and 3 are roots of this quadratic expression in which the coefficient of $x^2 = -2 < 0$.

WORKED-OUT PROBLEMS

Single Correct Choice Type Questions

1. If the equations

$$x^2 + ax + 1 = 0 \quad \text{and} \quad x^2 - x - a = 0$$

have a real common root, then the value of a is

(A) 0 (B) 1 (C) –1 (D) 2

Solution: Let α be a real common root. Then

$$\alpha^2 + a\alpha + 1 = 0$$

$$\alpha^2 - \alpha - a = 0$$

Therefore

$$\alpha(a+1) + (a+1) = 0$$

$$(a+1)(\alpha+1) = 0$$

If $a = -1$, then the equations are same and also cannot have a real root. Therefore $a + 1 \neq 0$ and hence $\alpha = -1$, so that $a = 2$.

Answer: (D)

2. If the roots of the equation $x^2 + px + q = 0$ are cubes of the roots of the equations $x^2 + mx + n = 0$, then

(A) $p = m^3 + 3mn$ (B) $p = m^3 - 3mn$

(C) $p + q = m^3$ (D) $\dfrac{p}{q} = \left(\dfrac{m}{n}\right)^3$

Solution: Let α and β be the roots of the equation $x^2 + mx + n = 0$. Therefore

$$\alpha + \beta = -m, \alpha\beta = n$$

Also since α and β are the roots of the equation $x^2 + mx + n = 0$, so that α^3 and β^3 are the roots of the equation $x^2 + px + q = 0$. Now,

$$\alpha^3 + \beta^3 = -p \quad \text{and} \quad \alpha^3\beta^3 = q$$

We have

$$\begin{aligned} -p &= \alpha^3 + \beta^3 \\ &= (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta) \\ &= -m^3 - 3n(-m) \end{aligned}$$

Therefore $p = m^3 - 3mn$.

Answer: (B)

3. If $(x^2 - 5x + 4)(y^2 + y + 1) < 2y$ for all real numbers y then x belongs to the interval

(A) (3, 4) (B) (3, 5) (C) (2, 3) (D) (–1, 2)

Solution: Let $m = x^2 - 5x + 4$. Then $my^2 + (m-2)y + m < 0$ for all real y. Therefore, $m < 0$ (by taking $y = 0$) and $(m-2)^2 - 4m^2 < 0$. Hence we have

$$m < 0 \quad \text{and} \quad 3m^2 + 4m - 4 > 0$$

$$\Rightarrow m < 0 \quad \text{and} \quad (3m-2)(m+2) > 0$$

This gives $m < -2$ and so

$$x^2 - 5x + 6 < 0 \Rightarrow (x-2)(x-3) < 0 \Rightarrow x \in (2, 3)$$

Answer: (C)

4. If p is prime number and both the roots of the equation $x^2 + px - (444)p = 0$ are integers, then p is equal to

(A) 2 (B) 3 (C) 31 (D) 37

Solution: Suppose the roots of $x^2 + px - (444)p = 0$ are integers. Then the discriminant

$$p^2 + 4(444)p = p\{p + 4 \times (444)\}$$

must be a perfect square. Therefore p divides $p + 4 \times (444)$. This implies

$$p \text{ divides } 4 \times (444) = 2^4 \times 3 \times 37$$

Therefore

$$p = 2 \quad \text{or} \quad 3 \quad \text{or} \quad 37$$

If $p = 2$ or 3 then $p^2 + 4(444)p$ is not a perfect square and when $p = 37$, it is a perfect square. Therefore, $p = 37$.

Answer: (D)

5. If a, b and c are distinct real numbers, then the number of real solutions of the equation

$$\frac{(x-a)(x-b)}{(c-a)(c-b)} + \frac{(x-b)(x-c)}{(a-b)(a-c)} + \frac{(x-c)(x-a)}{(b-c)(b-a)} + 1 = 0$$

is

(A) 0 (B) 1 (C) 2 (D) infinite

Solution: Let $p(x) = 0$ be the given equation. Then

$$p(a) = p(b) = p(c) = 2$$

Since $p(x)$ is a polynomial of degree 2 and a, b and c are distinct real numbers, it follows that $p(x) \equiv 2$, that is $p(x) = 2$ for all x.

Answer: (A)

6. The number of real solutions of the equation

$$\sqrt{x + 14 - 8\sqrt{x-2}} + \sqrt{x + 23 - 10\sqrt{x-2}} = 3$$

is

(A) 2 (B) 4 (C) 8 (D) infinite

Solution: The given equation is

$$\sqrt{(\sqrt{x-2} - 4)^2} + \sqrt{(\sqrt{x-2} - 5)^2} = 3$$

$$\left|\sqrt{x-2} - 4\right| + \left|\sqrt{x-2} - 5\right| = 3$$

Put $\sqrt{x-2} - 5 = y$. Then, the given equation becomes

$$|y+1| + |y| = 3$$

Case 1: Suppose $y \geq 0$. Then $y + 1 + y = 3$ or $y = 1$. Therefore

$$\sqrt{x-2} - 5 = 1 \Rightarrow x = 38$$

Case 2: Suppose $y \leq -1$. Then $y + 1 \leq 0$. This implies

$$-(y+1) - y = 3 \quad \text{or} \quad y = -2$$

Hence

$$\sqrt{x-2} - 5 = -2 \Rightarrow x = 11$$

Note that $-1 < y < 0$ is impossible (for, otherwise, $3 = |y+1| + |y| = y + 1 - y$). Thus, $x = 38$ or 11.

Answer: (A)

7. If a and b are roots of the equation $(x+c)(x+d) - k = 0$, then the roots of the equation $(x-a)(x-b) + k = 0$ are

(A) c, d (B) $-c, -d$ (C) $-c, d$ (D) $c, -d$

Solution: If a and b are roots of the equation $(x+c)(x+d) - k$, then

$$(x+c)(x+d) - k = (x-a)(x-b)$$

and hence

$$(x-a)(x-b) + k = (x+c)(x+d)$$

Therefore, $-c, -d$ are the roots of $(x-a)(x-b) + k = 0$.

Answer: (B)

8. The number of integer values of x satisfying

$$(x+1)^2 > 5x - 1 \quad \text{and} \quad (x+1)^2 < 7x - 3$$

simultaneously is

(A) 1 (B) 2 (C) 4 (D) 0

Solution: The first inequality,

$$\begin{aligned}(x+1)^2 > 5x - 1 &\Rightarrow x^2 - 3x + 2 > 0\\ &\Rightarrow (x-1)(x-2) > 0\\ &\Rightarrow x < 1 \quad \text{or} \quad x > 2 \qquad (4.1)\end{aligned}$$

The second inequality,

$$\begin{aligned}(x+1)^2 < 7x - 3 &\Rightarrow x^2 - 5x + 4 < 0\\ &\Rightarrow (x-1)(x-4) < 0\\ &\Rightarrow 1 < x < 4 \qquad (4.2)\end{aligned}$$

From Eqs. (4.1) and (4.2), we get $2 < x < 4$ and that x is an integer. Therefore $x = 3$.

Answer: (A)

9. The minimum value of "a" for which the real values if x such that

$$5^{1+x} + 5^{1-x}, \frac{a}{2}, 5^{2x} + 5^{-2x}$$

exist and are in arithmetic progression is

(***Note:*** p, q, r are said to be in arithmetic progression if $q - p = r - q$.)

(A) $-33/4$ (B) $33/4$ (C) -12 (D) 12

Solution: Put $y = 5^x + 5^{-x}$. Then $5y, a/2, y^2 - 2$ are in AP. Therefore

$$5y + y^2 - 2 = 2\left(\frac{a}{2}\right) = a$$

This implies that $y^2 + 5y - 2 - a = 0$ has real solutions. Hence

$$25 + 4(a+2) \geq 0$$

$$a \geq -\frac{33}{4} \qquad (4.3)$$

Also, since $(5^{x/2} - 5^{-x/2})^2 \geq 0$, we get that

$$y = 5^x + 5^{-x} \geq 2$$

Therefore

$$a = y^2 + 5y - 2 \geq 12 \qquad (4.4)$$

From Eqs. (4.3) and (4.4), we get $a \geq 12$. Therefore the minimum value of a is 12 and for this value of a, we have

$$\begin{aligned}y^2 + 5y - 14 &= 0\\ (y+7)(y-2) &= 0\\ y &= -7 \quad \text{or} \quad 2\end{aligned}$$

But $y = 5^x + 5^{-x} > 0$. Therefore $y = 2$. This implies

$$\begin{aligned}5^x + 5^{-x} &= 2\\ 5^{2x} - 2 \times 5^x + 1 &= 0\\ (5^x - 1)^2 &= 0\\ 5^x &= 1\\ x &= 0\end{aligned}$$

Therefore the following are in arithmetic progression:

$$5^{1+x} + 5^{1-x} = 10, \frac{a}{2} = 6 \quad \text{and} \quad 5^{2x} + 5^{-2x} = 2$$

Answer: (D)

10. Let a, b be positive real numbers. If the equations $x^2 + ax + 2b = 0$ and $x^2 + 2bx + a = 0$ have real roots, then minimum value of $a + b$ is

(A) 4 (B) 6 (C) 8 (D) 2

Solution: We have

$$x^2 + ax + 2b = 0 \text{ has real roots} \Rightarrow a^2 \geq 8b \qquad (4.5)$$

$$x^2 + 2bx + a = 0 \text{ has real roots} \Rightarrow b^2 \geq a \qquad (4.6)$$

Therefore

$$a \le b^2 \le \left(\frac{a^2}{8}\right)^2 = \frac{a^4}{64}$$

$$64 \le a^3 \quad \text{or} \quad a \ge 4 \tag{4.7}$$

Now,

$$b^2 \ge a \quad \text{and} \quad a \ge 4 \Rightarrow b \ge 2 \tag{4.8}$$

From Eqs. (4.7) and (4.8), we have $a + b \ge 6$ and, for values $a = 4$ and $b = 2$, the equations $x^2 + ax + 2b = (x + 2)^2$ and $x^2 + 2bx + a = (x + 2)^2$ have real roots.

Answer: (B)

11. Let a, b, c and d be non-zero real numbers. If c and d are roots of the equation $x^2 + ax + b = 0$ and a and b are roots of the equation $x^2 + cx + d = 0$, then the value of $-(a + b + c + d)$ is

(A) 1 (B) 2 (C) 3 (D) 4

Solution: Since c and d are roots of the equation $x^2 + ax + b = 0$, we have

$$c + d = -a \quad \text{and} \quad cd = b \tag{4.9}$$

Since a and b are roots of the equation $x^2 + cx + d = 0$, we have

$$a + b = -c \quad \text{and} \quad ab = d \tag{4.10}$$

From Eqs. (4.9) and (4.10) we have

$$a + b + c = 0 = a + c + d$$

and $\qquad cd = b, ab = d$

We thus have $b = d \ne 0$. Therefore $a = c = 1$ and $b = d = -2$. Hence

$$a + b + c + d = 0 + d = d = -2$$

$$-(a + b + c + d) = 2$$

Answer: (B)

12. If $(a - 1)(x^2 + x + 1)^2 - (a + 1)(x^4 + x^2 + 1) = 0$ has distinct real roots, then

(A) $|a| < 2$ (B) $|a| > 2$
(C) $|a| = 2$ (D) a is not a real number

Solution: Consider

$$x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4} > 0 \quad \text{(for all real } x\text{)}$$

Therefore, the given equation can be written as

$$(a - 1)(x^2 + x + 1) - (a + 1)(x^2 - x + 1) = 0$$

[Note that $(x^2 - x + 1)(x^2 + x + 1) = x^4 + x^2 + 1$]. Therefore,

$$(x^2 + 1)[a - 1 - (a + 1)] + x(a - 1 + a + 1) = 0$$

$$\Rightarrow -2(x^2 + 1) + 2ax = 0$$

$$\Rightarrow x^2 - ax + 1 = 0$$

This has distinct real roots if and only if $a^2 - 4 > 0$, that is, $|a| > 2$.

Answer: (B)

13. If a, b, c and d are distinct positive real numbers such that a and b are the roots of $x^2 - 10cx - 11d = 0$ and c and d are the roots of $x^2 - 10ax - 11b = 0$, then the value of $a + b + c + d$ is

(A) 1110 (B) 1010 (C) 1101 (D) 1210

Solution: Since a and b are the roots of $x^2 - 10cx - 11d = 0$ we have

$$\text{(i) } a + b = 10c \quad \text{and} \quad \text{(ii) } ab = -11d \tag{4.11}$$

Also since c and d are the roots of $x^2 - 10ax - 11b = 0$, we have

$$\text{(i) } c + d = 10a \quad \text{and} \quad \text{(ii) } cd = -11b \tag{4.12}$$

Adding part (i) of Eqs. (4.11) and (4.12), we get

$$a + b + c + d = 10(a + c) \Rightarrow b + d = 9(a + c) \tag{4.13}$$

Multiplying part (ii) of Eqs. (4.11) and (4.12), we get

$$abcd = 121\,bd \Rightarrow ac = 121 \tag{4.14}$$

Also,

$$a^2 - 10ca - 11d = 0 = c^2 - 10ca - 11b$$

$$\Rightarrow a^2 + c^2 - 20ca - 11(b + d) = 0$$

From Eqs. (4.13) and (4.14), we have

$$a^2 + c^2 - 20(121) - 99(a + c) = 0$$

$$(a + c)^2 - 2 \times 121 - 20 \times 121 - 99(a + c) = 0$$

$$(a + c - 121)(a + c + 22) = 0$$

$$a + c = 121$$

or $\qquad a + c = -22$

Since a, c are positive, $a + c \ne -22$. Therefore $a + c = 121$ and

$$a + b + c + d = (a + c) + 9(a + c) = 1210$$

Answer: (D)

14. The sum of all the real roots of the equation $|x - 2|^2 + |x - 2| - 2 = 0$ is

(A) 1 (B) 2 (C) 3 (D) 4

Try it out If we drop the condition that a, b, c and d are positive and assume that they are distinct non-zero real numbers, then also $a + b + c + d$ value may be 1210 (Try!)

Solution:
Case 1: Suppose $x \geq 2$. Then the equation becomes

$$(x-2)^2 + (x-2) - 2 = 0$$
$$x^2 - 3x = 0$$

Since $x \geq 2$, we get that $x = 3$.
Case 2: Suppose $x < 2$. Then the equation becomes

$$(2-x)^2 + (2-x) - 2 = 0$$
$$x^2 - 5x + 4 = 0$$
$$(x-1)(x-4) = 0$$

But $x < 2$. Therefore $x = 1$. Thus the real roots of the equation are 1 and 3 and their sum is 4.

Answer: (D)

15. If the product of the roots of the equation $x^2 - 3kx + 2e^{2\log k} - 1 = 0$ is 7, then the roots are real for $k =$

(A) 0 (B) 1 (C) 2 (D) 3

Solution: Observe that $\log k$ is defined when $k > 0$. The given equation is $x^2 - 3kx + 2k^2 - 1^2 = 0$. It is given that the product of the roots is 7. That is

$$2k^2 - 1 = 7$$
$$k^2 = 4$$
$$k = \pm 2$$

Since $k > 0$, we get that $k = 2$. Further, for $k = 2$, the given equation is $x^2 - 6x + 7 = 0$ whose roots are $3 \pm \sqrt{2}$, which are real.

Answer: (C)

16. Let a, b and c be the sides of a triangle, where a, b, c are distinct, and λ be a real number. If the roots of the equation $x^2 + 2(a+b+c)x + 3\lambda(ab+bc+ca) = 0$ are real, then

(A) $\lambda < \frac{4}{3}$ (B) $\lambda > \frac{5}{3}$

(C) $\frac{1}{3} < \lambda < \frac{5}{3}$ (D) $\frac{4}{3} < \lambda < \frac{5}{3}$

Solution: The given equation has real roots. Therefore,

$$4(a+b+c)^2 - 12\lambda(ab+bc+ca) \geq 0$$
$$a^2 + b^2 + c^2 + (2-3\lambda)(ab+bc+ca) \geq 0$$

$$\lambda \leq \frac{a^2+b^2+c^2}{3(ab+bc+ca)} + \frac{2}{3} \tag{4.15}$$

Since a, b and c are sides of a triangle, $(a-b)^2 < c^2$, $(b-c)^2 < a^2$ and $(c-a)^2 < b^2$, so that

$$\frac{a^2+b^2+c^2}{ab+bc+ca} < 2 \tag{4.16}$$

From Eqs. (4.15) and (4.16) we get

$$\lambda < \frac{2}{3} + \frac{2}{3} = \frac{4}{3}$$

Answer: (A)

17. Let α and β be roots of the equation $x^2 - px + r = 0$ and $\alpha/2$ and 2β be the roots of the equation $x^2 - qx + r = 0$. Then the value of r is

(A) $\frac{2}{9}(p-q)(2q-p)$ (B) $\frac{2}{9}(q-p)(2p-q)$

(C) $\frac{2}{9}(q-2p)(2q-p)$ (D) $\frac{2}{9}(2p-q)(2q-p)$

Solution: Since α and β are roots of the equation $x^2 - px + r = 0$, we have

$$\alpha + \beta = p \quad \text{and} \quad \alpha\beta = r$$

Since $\alpha/2$ and 2β are the roots of the equation $x^2 - qx + r = 0$, we have

$$\frac{\alpha}{2} + 2\beta = q \quad \text{and} \quad \frac{\alpha}{2} \times 2\beta = r$$

Therefore,

$$\alpha + \beta = p \quad \text{and} \quad \alpha + 4\beta = 2q$$

Solving the two equations we get

$$\beta = \frac{2q-p}{3} \quad \text{and} \quad \alpha = p - \beta = p - \frac{2q-p}{3} = \frac{2}{3}(2p-q)$$

Therefore

$$r = \alpha\beta = \frac{2}{9}(2p-q)(2q-p)$$

Answer: (D)

18. If $\sin^2\alpha\cos^2\alpha = \sin^2\beta$, then the roots of the equation $x^2 + 2x\cot\beta + 1 = 0$ are always

(A) equal (B) imaginary
(C) real and distinct (D) greater than 1

Solution: The discriminant of the given equation is

$$\begin{aligned}
4\cot^2\beta - 4 &= 4(\cot^2\beta - 1) \\
&= 4(\operatorname{cosec}^2\beta - 2) \\
&= \frac{4}{\sin^2\beta} - 8 = \frac{4}{\sin^2\alpha\cos^2\alpha} - 8 \\
&= 8\left(\frac{2}{\sin^2 2\alpha} - 1\right) \\
&= 8\frac{2 - \sin^2 2\alpha}{\sin^2 2\alpha} > 0
\end{aligned}$$

Answer: (C)

19. If λ is real and $(\lambda^2+\lambda-2)x^2+(\lambda+2)x<1$ for all real x, then λ belongs to the interval

(A) $(-2, 1)$ (B) $(-2, 2/5)$
(C) $(2/5, 1)$ (D) $(1, 2)$

Solution: Suppose that $(\lambda^2+\lambda-2)x^2+(\lambda+2)x-1<0$ for all real x. Then from Theorem 4.6,

$$\lambda^2+\lambda-2<0 \quad \text{and} \quad (\lambda+2)^2+4(\lambda^2+\lambda-2)<0$$
$$(\lambda+2)(\lambda-1)<0 \quad \text{and} \quad 5\lambda^2+8\lambda-4<0$$
$$-2<\lambda<1 \quad \text{and} \quad (\lambda+2)(5\lambda-2)<0$$
$$-2<\lambda<1 \quad \text{and} \quad -2<\lambda<\frac{2}{5}$$

These inequalities imply

$$\lambda \in \left(-2, \frac{2}{5}\right)$$

Answer: (B)

20. At least one of the equations $x^2+ax+b=0$ and $x^2+cx+d=0$ has real roots if

(A) $ac=2(b+d)$ (B) $ad=2(b+c)$
(C) $bc=2(a+d)$ (D) $ab=2(c+d)$

Solution: Suppose both the equations have imaginary roots and $ac=2(b+d)$. Then $a^2-4b<0$ and $c^2-4d<0$. Therefore

$$a^2+c^2-4(b+d)<0$$
$$a^2+c^2-2ac<0$$
$$(a-c)^2<0$$

which is impossible. Therefore $ac=2(b+d)$ implies that at least one of a^2-4b and c^2-4d is greater than or equal to 0.

Answer: (A)

21. For any real λ, the quadratic equation $(x-a)(x-c)+\lambda(x-b)(x-d)=0$ has always real roots if

(A) $a<b<c<d$ (B) $a<c<b<d$
(C) $a<c<d<b$ (D) $d<c<b<a$

Solution: The given equation is

$$(1+\lambda)x^2-(a+c+\lambda b+\lambda d)x+(ac+\lambda bd)=0$$

This equation has real roots if the discriminant

$$\Delta(\lambda)=(a+c+\lambda b+\lambda d)^2-4(1+\lambda)(ac+\lambda bd)\geq 0$$

for any λ. That is

$$\Delta(\lambda)=(b-d)^2\lambda^2+2(ab+ad+bc+dc-2bd-2ac)\lambda + (a-c)^2\geq 0 \quad \text{for any real } \lambda$$

$$\Delta(0)=(a-c)^2>0 \quad \text{for } a\neq c$$

It is enough if we show that $\Delta(\lambda)>0$ for any λ and hence to prove that the discriminant of $\Delta(\lambda)$ is negative. The discriminant of $\Delta(\lambda)$ is given by

$$\begin{aligned}&4(ab+ad+bc+cd-2bd-2ac)^2-4(a-c)^2(b-d)^2\\ &=4[ab+ad+bc+cd-2bd-2ac+(a-c)(b-d)]\\ &\quad\times[ab+ad+bc+cd-2bd-2ac-(a-c)(b-d)]\\ &=-16(b-a)(d-c)(c-b)(d-a)\end{aligned}$$

This is less than 0 if $a<b<c<d$. If $a<b<c<d$, then $\Delta(\lambda)>0$ for all real λ.

Answer: (A)

22. If α and β are the roots of $x^2+bx+c=0$ and are positive, then $\sqrt{\alpha}+\sqrt{\beta}$ is

(A) $b+2\sqrt{c}$ (B) $\sqrt{-b+2\sqrt{c}}$
(C) $\sqrt{b+2\sqrt{c}}$ (D) $\sqrt{2b-\sqrt{c}}$

Solution: Since α and β are the roots of $x^2+bx+c=0$, we have

$$\alpha+\beta=-b \quad \text{and} \quad \alpha\beta=c$$

Therefore

$$(\sqrt{\alpha}+\sqrt{\beta})^2=\alpha+\beta+2\sqrt{\alpha\beta}=-b+2\sqrt{c}$$
$$\sqrt{\alpha}+\sqrt{\beta}=\sqrt{-b+2\sqrt{c}}$$

Answer: (B)

23. Let α be a root of $ax^2+bx+c=0$ and β be a root of $-ax^2-bx+c=0$, where a, b and c are real numbers and $a\neq 0$. Then the equation

$$\frac{a}{2}x^2+bx+c=0$$

has a root γ such that

(A) $\gamma<\min\{\alpha, \beta\}$
(B) $\gamma>\max\{\alpha, \beta\}$
(C) γ lies between α and β
(D) $-\gamma$ lies between α and β

Solution: By hypothesis,

$$a\alpha^2+b\alpha+c=0 \quad \text{and} \quad a\beta^2-b\beta-c=0$$

Let

$$f(x)=\frac{a}{2}x^2+bx+c$$

Then

$$f(\alpha)=\frac{a}{2}\alpha^2+b\alpha+c$$

$$= \frac{1}{2}(a\alpha^2 + 2b\alpha + 2c)$$

$$= \frac{1}{2}(a\alpha^2 - 2a\alpha^2) = -\frac{a}{2}\alpha^2$$

and
$$f(\beta) = \frac{a}{2}\beta^2 + b\beta + c$$

$$= \frac{1}{2}(a\beta^2 + 2b\beta + 2c)$$

$$= \frac{1}{2}(a\beta^2 + a\beta^2) = a\beta^2$$

Therefore,

$$f(\alpha)f(\beta) = \frac{-a^2\alpha^2\beta^2}{2} < 0$$

Hence $f(x) = 0$ has a root in between α and β.

Answer: (C)

24. The number of equations of the form $ax^2 + bx + 1 = 0$, where $a, b \in \{1, 2, 3, 4\}$, having real roots is

(A) 15 (B) 9 (C) 7 (D) 8

Solution: The roots are real $\Leftrightarrow b^2 - 4a \geq 0 \Leftrightarrow b^2 \geq 4a$. In tabular form

a	$4a$	b	b^2	No. of required equations
1	4	2, 3, 4	4, 9, 16	3
2	8	3, 4	9, 16	2
3	12	4	16	1
4	16	4	16	1
			Total	7

Answer: (C)

25. If $x^2 + (a - b)x + 1 - a - b = 0$, where a and b are real numbers, has distinct real roots for all values of b, then

(A) $a < 1$ (B) $a > 1$
(C) $a < 0$ (D) $0 < a < 1$

Solution: We have

$$(a - b)^2 - 4(1 - a - b) > 0 \quad \text{for all real } b$$

$$\Rightarrow b^2 + 2(2 - a)b + a^2 - 4(1 - a) > 0 \quad \text{for all real } b$$

$$\Rightarrow 4(2 - a)^2 - 4(a^2 - 4 + 4a) < 0$$

$$\Rightarrow -16a + 16 + 16 - 16a < 0$$

$$\Rightarrow a > 1$$

Answer: (B)

26. The number of solutions of the equation $|x|^2 - 2|x| - 8 = 0$ which belong to the domain of the function $f(x) = \sqrt{5 - 2x}$ is

(A) 0 (B) 1 (C) 2 (D) 3

Solution: The domain of $f = \{x \mid x \leq 5/2\}$

$$|x|^2 - 2|x| - 8 = (|x| - 4)(|x| + 2) = 0$$

$$\Rightarrow |x| = 4 \quad (\text{since } |x| + 2 > 0)$$

$$\Rightarrow x = 4 \quad \text{or} \quad -4$$

Now $-4 < 5/2$ and -4 belongs to the domain of f.

Answer: (B)

27. The least integral value of k for which the quadratic expression $(k - 2)x^2 + 8x + k + 4$ is positive for all real x is

(A) 4 (B) -6 (C) 5 (D) 6

Solution: Let $f(x) = (k - 2)x^2 + 8x + k + 4$. From Theorem 4.6 we have

$f(x) > 0$ for all real $x \Rightarrow$ discriminant < 0 and coefficient of $x^2 > 0$

This implies

$$64 - 4(k - 2)(k + 4) < 0 \quad \text{and} \quad k > 2$$

$$k^2 + 2k - 24 > 0 \quad \text{and} \quad k > 2$$

$$(k + 6)(k - 4) > 0 \quad \text{and} \quad k > 2$$

$$k > 4 \quad \text{and} \quad k \text{ is an integer}$$

Therefore the least integral value of k is 5.

Answer: (C)

28. If the roots of the quadratic equation $(p - 3)x^2 - 2px + 5p = 0$ are real and positive, then

(A) $p > 0$ (B) $3 \leq p \leq 15/4$
(C) $3 < p \leq 15/4$ (D) $p > 15/4$

Solution: Let $f(x) = (p - 3)x^2 - 2px + 5p$. The roots of $f(x) = 0$ are real. This implies

$$4p^2 - 20p(p - 3) \geq 0$$

$$4p^2 - 15p \leq 0$$

$$p(4p - 15) \leq 0$$

$$0 \leq p \leq \frac{15}{4} \tag{4.17}$$

Now

1. $p = 0$ implies that the roots are 0, 0.
2. Roots are positive implies that $f(0)$ and the coefficient of x^2 must have the same sign. $f(0) = 5p$ and $p - 3$ have the same sign and $p \neq 3$.

By Eq. (4.17), we have $3 < p \leq 15/4$.

Answer: (C)

29. If α and β are the roots of the equation $(5+\sqrt{2})x^2 - (4+\sqrt{5})x + 8 + 2\sqrt{5} = 0$, then

$$\frac{2}{(1/\alpha)+(1/\beta)}$$

is equal to

(A) 2 (B) 4 (C) 1/2 (D) 1/4

Solution: Since α and β are the roots of the given equation we have

$$\alpha+\beta = \frac{4+\sqrt{5}}{5+\sqrt{2}} \quad \text{and} \quad \alpha\beta = \frac{8+2\sqrt{5}}{5+\sqrt{2}}$$

Therefore

$$\frac{\alpha+\beta}{\alpha\beta} = \frac{4+\sqrt{5}}{2(4+\sqrt{5})} = \frac{1}{2}$$

$$\frac{2}{(1/\alpha)+(1/\beta)} = \frac{2\alpha\beta}{\alpha+\beta} = 2\times 2 = 4$$

Answer: (B)

30. If $\alpha < \beta$ are the roots of the equation $x^2 + bx + c = 0$, where $c < 0 < b$, then

(A) $0 < \alpha < \beta$ (B) $\alpha < 0 < \beta < |\alpha|$
(C) $\alpha < \beta < 0$ (D) $\alpha < 0 < |\alpha| < \beta$

Solution: Since $\alpha < \beta$ are the roots of the equation $x^2 + bx + c = 0$,

$$\alpha + \beta = -b < 0 \quad (\text{since } 0 < b)$$

and $$\alpha\beta = c < 0$$

Since $\alpha < \beta$ and $\alpha\beta < 0$, we get that

$$\alpha < 0 < \beta \tag{4.18}$$

Also, $\beta = -b - \alpha < -\alpha$. Therefore $\beta < -\alpha$ and $\alpha < \beta$ and hence

$$|\alpha| > \beta \tag{4.19}$$

From Eqs. (4.18) and (4.19), we get

$$\alpha < 0 < \beta < |\alpha|$$

Answer: (B)

31. If $a < b$, then the equation $(x-a)(x-b) - 1 = 0$ has

(A) both the roots in $[a, b]$
(B) both the roots in $(-\infty, a)$
(C) both the roots in $(b, +\infty)$
(D) one root in $(-\infty, a)$ and another in $(b, +\infty)$

Solution: Let $f(x) = (x-a)(x-b) - 1$. Then

$$f(a) = -1 = f(b)$$

and coefficient of $x^2 = 1 > 0$. Therefore a and b must lie between the roots and $a < b$. Hence one root is less than a and another is greater than b.

Answer: (D)

32. The number of real solutions of the equation $(x^2 - 5x + 7)^2 - (x-2)(x-3) = 0$ is

(A) 1 (B) 2 (C) 3 (D) 0

Solution: The given equation is

$$(x^2 - 5x + 7)^2 - (x^2 - 5x + 6) = 0$$

Put $x^2 - 5x + 7 = t$. Then

$$t = \left(x - \frac{5}{2}\right)^2 + \frac{3}{4} > 0 \quad \text{for all } x \in \mathbb{R}$$

The given equation is equivalent to

$$t^2 - (t-1) = 0$$

$$t^2 - t + 1 = 0$$

$$t = \frac{1 \pm i\sqrt{3}}{2}$$

Therefore there is no real root of the given equation.

Answer: (D)

33. The number of real values of x satisfying the equation

$$x^3 + \frac{1}{x^3} + x^2 + \frac{1}{x^2} - 6\left(x + \frac{1}{x}\right) - 7 = 0$$

is

(A) 1 (B) 2 (C) 3 (D) 4

Solution: Put $x + (1/x) = t$. Then we have

$$x^2 + \frac{1}{x^2} = t^2 - 2 \quad \text{and} \quad x^3 + \frac{1}{x^3} = t^3 - 3t$$

Therefore the given equation transforms into

$$(t^3 - 3t) + (t^2 - 2) - 6t - 7 = 0$$

$$t^3 + t^2 - 9t - 9 = 0$$

$$(t+1)(t^2 - 9) = 0$$

Equating $t = -1$ and substituting the value to t back, we get

$$x + \frac{1}{x} = -1 \Rightarrow x^2 + x + 1 = 0 \Rightarrow x \text{ is not real}$$

Now for $t^2 = 9$ we get

$$t = \pm 3 \Rightarrow x^2 \mp 3x + 1 = 0$$

This gives

$$x = \frac{3 \pm \sqrt{5}}{2} \quad \text{and} \quad x = \frac{-3 \pm \sqrt{5}}{2}$$

Therefore four real values of x satisfy the given equation.

Answer: (D)

34. The number of quadratic equations, with coefficient of x^2 as 1, which are unaltered by squaring their roots is

(A) 2 (B) 4
(C) 6 (D) infinite

Solution: Let α and β be the roots of $x^2+bx+c=0$ for which α^2 and β^2 are also roots. But the equation whose roots are α^2 and β^2 is $x^2-(b^2-2c)x+c^2=0$. Therefore $b^2-2c=-b$ and $c^2=c$ (i.e., $c=0$ or 1).

(i) If $c=0$, then $b^2=-b$ and hence $b=0$ or -1.

(ii) If $c=1$, then $b^2+b-2=0$ and hence $b=1$ or -2.

Therefore the required equations are

$$x^2=0,\quad x^2-x=0,\quad x^2+x+1=0,\quad x^2-2x+1=0$$

Answer: (B)

35. The quadratic equations $x^2-6x+a=0$ and $x^2-cx+6=0$ have one root in common. The other roots of the first equation and the second equation are integers in the ratio 4:3. Then the common root is

(A) 4 (B) 3 (C) 2 (D) 1

Solution: Let α be the common root and let the other roots of the equations be 4β and 3β, respectively. Then

$$\alpha+4\beta=6,\quad 4\alpha\beta=a$$

and

$$\alpha+3\beta=c,\quad 3\alpha\beta=6$$

This implies

$$a=4\alpha\beta=4\cdot\frac{6}{3}=8$$

The first equation is $x^2-6x+8=0$ whose roots are 2 and 4.

If $\alpha=4$, then

$$\beta=\frac{6}{3\alpha}=\frac{1}{2}\quad\text{and}\quad 3\beta=\frac{3}{2}$$

which is not an integer, a contradiction to the hypothesis. Therefore, $\alpha=2$ is the common root, in which case the equations are $x^2-6x+8=0$ and $x^2-5x+6=0$, whose roots are 2, 4 and 2, 3, respectively.

Answer: (C)

36. The set of all real values of x for which $x^2-|x+2|+x>0$ is

(A) $(-\infty,2)\cup(2,\infty)$ (B) $(-\infty,-\sqrt{2})\cup(\sqrt{2},\infty)$
(C) $(-\infty,-1)\cup(1,\infty)$ (D) $(\sqrt{2},\infty)$

Solution:

Case 1: Suppose that $x<-2$. Then the given inequality is

$$x^2+x+2+x>0$$
$$(x+1)^2+1>0$$

which is always true. Therefore any $x\in(-\infty,-2)$ satisfies the given inequality.

Case 2: Suppose that $x\ge -2$. Then

$$x^2-(x+2)+x>0 \Leftrightarrow x^2-2>0$$
$$\Leftrightarrow x<-\sqrt{2}\quad\text{or}\quad x>\sqrt{2}$$

Therefore, the required set is $(-\infty,-\sqrt{2})\cup(\sqrt{2},\infty)$.

Answer: (B)

37. The smallest value of k for which both roots of the equation $x^2-8kx+16(k^2-k+1)=0$ are real, distinct and have values at least k is

(A) 0 (B) 1 (C) 3/2 (D) 2

Solution: Roots are real and distinct. Therefore discriminant > 0. That is

$$64k^2-64(k^2-k+1)>0$$
$$k>1 \tag{4.20}$$

For $k=1$, the given equation has roots 4, 4. Values of the roots at least 4 implies that the product of the roots ≥ 16. Therefore

$$16(k^2-k+1)\ge 16$$
$$k(k-1)\ge 0$$
$$k\le 0\quad\text{or}\quad k\ge 1 \tag{4.21}$$

$$\text{Roots}\ge 4\Rightarrow f(4)\ge 0$$
$$\Rightarrow 16-32k+16(k^2-k+1)\ge 0$$
$$\Rightarrow (k-1)(k-2)\ge 0$$
$$\Rightarrow k\le 1\quad\text{or}\quad k\ge 2 \tag{4.22}$$

From Eqs. (4.20), (4.21) and (4.22), we get $k\ge 2$.

Answer: (D)

38. If

$$-3<\frac{x^2+mx+1}{x^2+x+1}<3$$

for all real x, then

(A) $m<-1$ (B) $-1<m<6$
(C) $-1<m<5$ (D) $m>6$

Solution: We have

$$x^2+x+1=\left(x+\frac{1}{2}\right)^2+\frac{3}{4}>0$$

Therefore

$$-3(x^2+x+1)<x^2+mx+1<3(x^2+x+1)$$

$4x^2 + (m+3)x + 4 > 0$ and $2x^2 + (3-m)x + 2 < 0$ for all real x

$$(m+3)^2 - 64 < 0 \quad \text{and} \quad (3-m)^2 - 16 < 0$$
$$(m+3+8)(m+3-8) < 0 \text{ and } (3-m+4)(3-m-4) < 0$$
$$(m+11)(m-5) < 0 \quad \text{and} \quad (m-7)(m+1) < 0$$
$$-11 < m < 5 \quad \text{and} \quad -1 < m < 7$$

This gives $-1 < m < 5$.

Answer: (C)

39. The number of natural numbers n for which the equation $(x-8)x = n(n-10)$ has no real solutions is

(A) 2 (B) 3 (C) 4 (D) 5

Solution: $x^2 - 8x - n(n-10) = 0$ has no real solutions. This implies

$$64 + 4n(n-10) < 0$$
$$n^2 - 10n + 16 < 0$$
$$(n-8)(n-2) < 0$$
$$2 < n < 8$$

Therefore, $n = 3, 4, 5, 6, 7$. Also, when $n = 3, 4, 5, 6$ or 7, it can be seen that $(x-8)x = n(n-10)$ has no real solutions. Therefore the number of such n is 5.

Answer: (D)

Multiple Correct Choice Type Questions

1. Suppose a and b are integers and $b \neq -1$. If the quadratic equation $x^2 + ax + b + 1 = 0$ has a positive integer root, then

(A) the other root is also a positive integer
(B) the other root is an integer
(C) $a^2 + b^2$ is a prime number
(D) $a^2 + b^2$ has a factor other than 1 and itself

Solution: Let α and β be the roots and α be a positive integer. Then

$$\alpha + \beta = -a \quad \text{and} \quad \alpha\beta = b + 1$$

$\beta = -a - \alpha$ implies β is an integer and

$$a^2 + b^2 = (\alpha+\beta)^2 + (\alpha\beta - 1)^2$$
$$= \alpha^2 + \beta^2 + \alpha^2\beta^2 + 1$$
$$= (\alpha^2 + 1)(\beta^2 + 1)$$

Since $\alpha^2 + 1 > 1$ and $\beta^2 + 1 > 1$, it follows that $\alpha^2 + 1$ is a factor of $a^2 + b^2$ other than 1 and itself.

Answers: (B), (D)

Note: If $b = -1$, then $a^2 + b^2$ may be prime number; for example, the equation $x^2 - 2x = 0$ has a positive root 2 and the other root is 0. Here $a^2 + b^2 (= 5)$ is a prime number. In fact, $a^2 + b^2$ is prime implies $\alpha^2 + 1 = 1$ or $\beta^2 + 1 = 1$. This gives

$$\alpha = 0 \quad \text{or} \quad \beta = 0$$

or $$b = -1$$

2. If a, b and c are integers, then the discriminant of $ax^2 + bx + c$ is of the form (where k is an integer)

(A) $4k$ (B) $4k + 1$
(C) $4k + 2$ (D) $4k + 3$

Solution:

Case 1: Suppose b is even, that is, $b = 2m$. Then $b^2 - 4ac = 4(m^2 - ac) = 4k$.

Case 2: Suppose b is odd, that is, $b = 2m - 1$. Then

$$b^2 - 4ac = (2m-1)^2 - 4ac$$
$$= 4m^2 + 4m + 1 - 4ac$$
$$= 4(m^2 + m - ac) + 1$$
$$= 4k + 1$$

Answers: (A), (B)

3. If a and b are roots of the equation $x^2 + ax + b = 0$, then

(A) $a = 0, b = 1$ (B) $a = 0 = b$
(C) $a = 1, b = -1$ (D) $a = 1, b = -2$

Solution: If $a + b = -a$ and $ab = b$, then $a = 0 = b$ or $a = 1$, $b = -2$.

Answers: (B), (D)

4. Let a, b and c be real numbers and $a \neq 0$. Let α and β be the roots of $ax^2 + bx + c = 0$. If α' and β' are roots of the equation $a^3x^2 + (abc)x + c^3 = 0$, then

(A) $\alpha' = \alpha^3\beta^2$ (B) $\beta' = \beta^3\alpha^2$
(C) $\alpha' = \alpha^2\beta$ (D) $\beta' = \alpha\beta^2$

Solution: Since α and β are the roots of $ax^2 + bx + c = 0$, we have

$$\alpha + \beta = -\frac{b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a}$$

Also since α' and β' are roots of the equation $a^3x^2 + (abc)x + c^3 = 0$,

$$\alpha' + \beta' = \frac{-abc}{a^3} = \left(\frac{-b}{a}\right)\left(\frac{c}{a}\right) = (\alpha+\beta)\alpha\beta$$

$$\alpha'\beta' = \frac{c^3}{a^3} = (\alpha\beta)^3$$

Now

$$\begin{aligned}(\alpha' - \beta')^2 &= (\alpha' + \beta')^2 - 4\alpha'\beta' \\ &= (\alpha+\beta)^2\alpha^2\beta^2 - 4\alpha^3\beta^3 \\ &= (\alpha\beta)^2[(\alpha+\beta)^2 - 4\alpha\beta] \\ &= (\alpha\beta)^2(\alpha-\beta)^2\end{aligned}$$

Also

$$|\alpha' - \beta'| = |\alpha\beta(\alpha-\beta)|$$

Therefore

$$\alpha' - \beta' = \alpha\beta(\alpha-\beta) \Rightarrow \alpha' = \alpha^2\beta \quad \text{and} \quad \beta' = \alpha\beta^2$$

$$\alpha' - \beta' = -\alpha\beta(\alpha-\beta) \Rightarrow \alpha' = \alpha\beta^2 \quad \text{and} \quad \beta' = \alpha^2\beta$$

Answers: (C), (D)

5. If α and β are roots of the equation $x^2 - 2ax + b^2 = 0$ and γ and δ are the roots of the equation $x^2 - 2bx + a^2 = 0$, then

(A) $\alpha+\beta = 2\sqrt{\gamma\delta}$ (B) $\alpha+\beta = 2(\gamma+\delta)$
(C) $(\gamma+\delta)^2 = 4\alpha\beta$ (D) $(\alpha+\beta)(\gamma+\delta) = 4\gamma\delta$

Solution: Since α and β are roots of the equation $x^2 - 2ax + b^2 = 0$, we have

$$\alpha+\beta = 2a \quad \text{and} \quad \alpha\beta = b^2$$

Since γ and δ are roots of the equation $x^2 - 2bx + a^2 = 0$, we have

$$\gamma+\delta = 2b \quad \text{and} \quad \gamma\delta = a^2$$

Solving the two sets of equations we get

$$\alpha+\beta = 2\sqrt{\gamma\delta}$$

$$(\gamma+\delta)^2 = 4b^2 = 4\alpha\beta$$

Answers: (A), (C)

6. If α and β are the roots of $x^2 - p(x+1) - q = 0$, then

(A) $(\alpha+1)(\beta+1) = 1-q$

(B) $(\alpha+1)(\beta+1) = 1+q$

(C) $\dfrac{(\alpha+1)^2}{(\alpha+1)^2+q-1} + \dfrac{(\beta+1)^2}{(\beta+1)^2+q-1} = q$

(D) $\dfrac{\alpha^2+2\alpha+1}{\alpha^2+2\alpha+q} + \dfrac{\beta^2+2\beta+1}{\beta^2+2\beta+q} = 1$

Solution: Since α and β are the roots of $x^2 - p(x+1) - q = 0$, we have

$$\alpha+\beta = p \quad \text{and} \quad \alpha\beta = -(p+q)$$

Now

$$\begin{aligned}(\alpha+1)(\beta+1) &= \alpha\beta + \alpha + \beta + 1 \\ &= -(p+q) + p + 1 = 1 - q\end{aligned}$$

$$\begin{aligned}\frac{(\alpha+1)^2}{(\alpha+1)^2+q-1} + \frac{(\beta+1)^2}{(\beta+1)^2+q-1} &= \frac{(\alpha+1)^2}{(\alpha+1)^2-(\alpha+1)(\beta+1)} \\ &\quad + \frac{(\beta+1)^2}{(\beta+1)^2-(\alpha+1)(\beta+1)} \\ &= \frac{\alpha+1}{\alpha-\beta} + \frac{\beta+1}{\beta-\alpha} \\ &= \frac{(\alpha+1)-(\beta+1)}{\alpha-\beta} = 1\end{aligned}$$

Answers: (A), (D)

7. Let a, b and c be real numbers and $f(x) = ax^2 + bx + c$. Suppose that whenever x is an integer, $f(x)$ is also an integer. Then

(A) $2a$ is an integer (B) $a+b$ is an integer
(C) c is an integer (D) $a+b+c$ is an integer

Solution: By hypothesis, $f(-1)$, $f(0)$ and $f(1)$ are integers. Therefore $a(-1)^2 + b(-1) + c$, $a(0)^2 + b(0) + c$ and $a(1)^2 + b(1) + c$ are all integer. Hence

$$a - b + c, \quad c \quad \text{and} \quad a + b + c \quad \text{are integers}$$

Also

$$a - b \text{ and } a + b \text{ are integers} \Rightarrow 2a \text{ is an integer}$$

Answers: (A), (B), (C), (D)

8. If one root of the equation $3x^2 + px + 3 = 0$ is the square of the other, then p is equal to

(A) 1/3 (B) 1 (C) 3 (D) −6

Solution: Let α and α^2 be the roots of $3x^2 + px + 3 = 0$. Then

$$\alpha + \alpha^2 = \frac{-p}{3} \quad \text{and} \quad \alpha^3 = 1$$

Therefore

$$\alpha = 1, \quad \alpha = w = \frac{-1+i\sqrt{3}}{2} \quad \text{or} \quad \alpha = \frac{-1-i\sqrt{3}}{2}$$

(i) If $\alpha = 1$, then $p = -6$ so that the equation is $3x^2 - 6x + 3 = 0$ whose roots are 1, 1.

(ii) If $\alpha = w$ or w^2, then $p = -3(\alpha+\alpha^2) = -3(w+w^2) = -3(-1)$ and hence $p = 3$ so that the equation is $3x^2 + 3x + 3 = 0$, whose roots are w and w^2.

Answers: (C), (D)

9. Suppose that the three quadratic equations $ax^2 - 2bx + c = 0$, $bx^2 - 2cx + a = 0$ and $cx^2 - 2ax + b = 0$ all have only positive roots. Then

(A) $b^2 = ca$ (B) $c^2 = ab$
(C) $a^2 = bc$ (D) $a = b = c$

Solution: Let $\alpha > 0$ and $\beta > 0$ be the roots of $ax^2 - 2bx + c = 0$. Then

$$\frac{c}{a} = \alpha\beta > 0$$

and therefore a and c have the same sign. Similarly, b and c have the same sign and a and b have the same sign. Therefore, a, b and c have the same sign and hence $ab > 0$. Also $(-2b)^2 \geq 4ac$, that is, $b^2 \geq ac$. Similarly $c^2 \geq ab$ and $a^2 \geq bc$. Hence

$$b^2c^2 \geq a^2bc, \quad c^2a^2 \geq b^2ca \quad \text{and} \quad a^2b^2 \geq c^2ab$$

which gives

$$bc \geq a^2, \quad ca \geq b^2 \quad \text{and} \quad ab \geq c^2 \quad (\text{since } ab, bc \text{ and } ca \text{ are all positive})$$

But we have $a^2 \geq bc$, $b^2 \geq ca$ and $c^2 \geq ab$. Therefore

$$a^2 = bc, \quad b^2 = ca \quad \text{and} \quad c^2 = ab$$

$$a^3 = b^3 = c^3 = abc$$

$$a = b = c$$

Answers: (A), (B), (C), (D)

10. Let a and b be two real numbers. If the roots of the equation $x^2 - ax - b = 0$ have absolute values less than 1, then

(A) $|b| < 1$ (B) $a + b < 1$
(C) $b - a < 1$ (D) $a + b = 0$

Solution: Let α and β be the roots of $x^2 - ax - b = 0$. Then, $|\alpha| < 1$ and $|\beta| < 1$. Also

$$|b| = |-b| = |\alpha\beta| = |\alpha||\beta| < 1$$

Since the roots α and β lie between -1 and 1, we have $f(-1) > 0$ and $f(1) > 0$. Therefore

$$1 + a - b > 0 \quad \text{and} \quad 1 - a - b > 0$$

or

$$b - a < 1 \quad \text{and} \quad a + b < 1$$

Answers: (A), (B), (C)

Matrix-Match Type Questions

1. Match the items in Column I with those in Column II.

Column I	Column II
(A) If the difference of the roots of the equation $2x^2 - (a+1)x + (a-1) = 0$ is equal to their product, then the value(s) of a is (are)	(p) 1
(B) If the sum of the roots of the equation $x^2 - 2a(x-1) - 1 = 0$ is equal to the sum of their squares, then a is	(q) 0
(C) If one root of the equation $x^2 - x - 3m = 0$ $(m \neq 0)$ is twice one of the roots of $x^2 - x - m = 0$, then the value of m is	(r) 2
	(s) 1/2
(D) If the sum of the squares of the roots of the equation $x^2 - 4x + m = 0$ is equal to 16, then m is	(t) -1

Solution:

(A) Let α and β be the roots of $2x^2 - (a+1)x + (a-1) = 0$. Then

$$\alpha + \beta = \frac{a+1}{2} \quad \text{and} \quad \alpha\beta = \frac{a-1}{2}$$

Now

$$|\alpha - \beta|^2 = (\alpha\beta)^2 \Rightarrow (\alpha + \beta)^2 - 4\alpha\beta = (\alpha\beta)^2$$

$$\frac{(a+1)^2}{4} - \frac{4(a-1)}{2} = \frac{(a-1)^2}{4}$$

$$(a+1)^2 - 8(a-1) - (a-1)^2 = 0$$

$$a - 2(a-1) = 0$$

$$a = 2$$

Answer: (A) → (r)

(B) Let α and β be the roots of $x^2 - 2a(x-1) - 1 = 0$. Then $\alpha + \beta = 2a$ and $\alpha\beta = 2a - 1$. Now

$$\alpha^2 + \beta^2 = \alpha + \beta \Rightarrow (\alpha + \beta)^2 - 2\alpha\beta = \alpha + \beta$$

$$\Rightarrow (2a)^2 - 2(2a-1) = 2a$$

$$\Rightarrow 4a^2 - 6a + 2 = 0$$

$$\Rightarrow 2a^2 - 3a + 1 = 0$$

$$\Rightarrow (2a-1)(a-1) = 0$$

$$\Rightarrow a = \frac{1}{2}, 1$$

Answer: (B) → (p), (s)

(C) Let α be one root of $x^2 - x - m = 0$ and 2α be a root of $x^2 - x - 3m = 0$. Then

$$\alpha^2 - \alpha - m = 0 \quad \text{and} \quad (2\alpha)^2 - (2\alpha) - 3m = 0$$

Eliminating m, we have $\alpha = 0, -1$. Also $\alpha = 0 \Rightarrow m = 0$, a contradiction to hypothesis. Therefore, $\alpha = -1$ and $m = 2$.

Answer: (C) → (r)

(D) Let α and β be the roots of $x^2 - 4x + m = 0$. Then $\alpha + \beta = 4$ and $\alpha\beta = m$. Now

$$\alpha^2 + \beta^2 = 16 \Rightarrow (\alpha + \beta)^2 - 2\alpha\beta = 16$$
$$\Rightarrow 16 - 2m = 16$$
$$\Rightarrow m = 0$$

Answer: (D) → (q)

2. Match the items in Column I with those in Column II.

Column I	*Column II*
(A) If the roots of the equation $ax^2 + bx + c = 0$ are of the form $(k+1)/k$ and $(k+2)/(k+1)$, then $(a+b+c)^2 =$	(p) $\frac{1}{2}(a^2 + b^2)$
(B) If one root of the equation $ax^2 + bx + c = 0$ is the square of the other, then $b^3 + ac^2 + a^2c =$	(q) $2ac$
(C) If the sum of the roots of the equation $ax^2 + bx + c = 0$ is equal to the sum of their squares, then $b(a+b) =$	(r) $3abc$
(D) If the roots of the equation $\frac{1}{x+a} + \frac{1}{x+b} = \frac{1}{c}$ are equal in magnitude, but opposite in sign, then the product of the roots is	(s) $b^2 - 4ac$ (t) $-\frac{1}{2}(a^2 + b^2)$

Solution:

(A) Since $(k+1)/k$ and $(k+2)/(k+1)$ are the roots of the given equation we have

$$\frac{k+1}{k} + \frac{k+2}{k+1} = -\frac{b}{a} \tag{4.23}$$

and
$$\frac{k+2}{k} = \frac{c}{a} \tag{4.24}$$

From Eq. (4.24),

$$\frac{c}{a} - 1 = \frac{k+2}{k} - 1 = \frac{2}{k}$$

and hence

$$k = \frac{2a}{c-a}$$

Substituting this value for k in Eq. (4.23), we get

$$\frac{[2a/(c-a)]+1}{2a/(c-a)} + \frac{[2a/(c-a)]+2}{[2a/(c-a)]+1} = -\frac{b}{a}$$

$$\frac{a+c}{2a} + \frac{2c}{a+c} = -\frac{b}{a}$$

$$\frac{(a+c)^2 + 4ac}{2a(a+c)} = -\frac{b}{a}$$

$$a^2 + c^2 + 6ac = -2bc - 2ab$$

Adding b^2 to both the sides and splitting $6ac$, we get

$$a^2 + b^2 + c^2 + 2ab + 2bc + 2ca = b^2 - 4ca$$
$$(a+b+c)^2 = b^2 - 4ca$$

Answer: (A) → (s)

(B) Let α and α^2 be the roots of $ax^2 + bx + c = 0$. Then

$$\alpha + \alpha^2 = \frac{-b}{a} \quad \text{and} \quad \alpha^3 = \frac{c}{a}$$

Therefore

$$\left(\frac{c}{a}\right)^{2/3} + \left(\frac{c}{a}\right)^{1/3} = \frac{-b}{a}$$

$$\left(\frac{c}{a}\right)^2 + \frac{c}{a} + 3\left(\frac{c}{a}\right)^{2/3}\left(\frac{c}{a}\right)^{1/3}\left[\left(\frac{c}{a}\right)^{2/3} + \left(\frac{c}{a}\right)^{1/3}\right] = \frac{-b^3}{a^3}$$

$$\left(\frac{c}{a}\right)^2 + \frac{c}{a} + 3\frac{c}{a}\left(\frac{-b}{a}\right) = -\frac{b^3}{a^3}$$

$$\frac{c^2 + ca - 3bc}{a^2} = \frac{-b^3}{a^3}$$

$$b^3 + ac^2 + ca^2 = 3abc$$

Answer: (B) → (r)

(C) Let α and β be the roots of $ax^2 + bx + c = 0$, then

$$\alpha + \beta = -\frac{b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a}$$

Therefore

$$-\frac{b}{a} = \alpha + \beta = \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = \frac{b^2}{a^2} - \frac{2c}{a}$$

This gives

$$-ab = b^2 - 2ac$$
$$2ca = b^2 + ab$$

Answer: (C) → (q)

(D) The given equation is equivalent to

$$x^2 + [a + b - 2c]x + ab - bc - ca = 0$$

If α and $-\alpha$ are the roots of this, then

$$0 = \alpha + (-\alpha) = 2c - a - b \quad \text{and} \quad -\alpha^2 = ab - bc - ca$$

Therefore

$$a + b = 2c \quad \text{and} \quad -\alpha^2 = ab - c(a+b)$$

The product of the roots is

$$ab - c(a+b) = ab - \frac{(a+b)^2}{2} = -\left(\frac{a^2+b^2}{2}\right)$$

Answer: (D) → (t)

3. Match the items in Column I with those in Column II.

Column I	*Column II*
(A) The values of k for which both the roots of the equation $x^2 - 6kx + 2 - 2k + 9k^2 = 0$ are greater than 3 belong to	(p) $(2, 5)$ (q) $\left(-\infty, \frac{3}{2}\right)$
(B) If $\log_{0.1}(x^2+x) > \log_{0.5}(x^3-x) + \log_2(x-1)$, then x belongs to	(r) $(1, +\infty)$
(C) If $\lvert x-1-x^2\rvert \le \lvert x^2-3x+4\rvert$, then x belongs to	(s) $\left(\frac{11}{9}, +\infty\right)$
(D) If $\lvert x^2-2x-3\rvert < 3x-3$, then x lies in the interval	(t) $[2, 5]$

Solution:

(A) Let $f(x) = x^2 - 6kx + 2 - 2k + 9k^2$ and α and β be the roots of $f(x) = 0$. Then, since $3 < \alpha$ and $3 < \beta$, we have $6k > 6$ and therefore $k > 1$. Also

$$f(3) > 0$$
$$9k^2 - 20k + 11 > 0$$
$$(9k-11)(k-1) > 0$$
$$k < 1 \quad \text{or} \quad k > \frac{11}{9}$$

Since $k > 1$, it follows that $k > 11/9$.

Answer: (A) → (s)

(B) The inequality is defined for $x > 1$. Since

$$\log_2(x-1) = \log_{(0.5)^{-1}}(x-1) = -\log_{0.5}(x-1)$$

we have

$$\log_{0.1}(x^2+x) > \log_{0.5}(x^3-x) - \log_{0.5}(x-1)$$

Therefore

$$\log_{0.1}(x^2+x) > \log_{0.5}\left(\frac{x^3-x}{x-1}\right) = \log_{0.5}(x^2+x)$$

which further gives

$$x^2 + x > 1 \Rightarrow x^2 + x - 1 > 0$$

$$x < -\left(\frac{1+\sqrt{5}}{2}\right) \quad \text{or} \quad x > \frac{\sqrt{5}-1}{2} \quad \text{and} \quad x > 1$$

This implies $x \in (1, \infty)$

Answer: (B) → (r)

(C) We have

$$\lvert x-1-x^2\rvert \le \lvert x^2-3x+4\rvert$$
$$\Rightarrow \lvert x^2-x+1\rvert \le \lvert x^2-3x+4\rvert$$
$$\Rightarrow x^2 - x + 1 \le x^2 - 3x + 4 \quad \text{(since both are positive for all real } x\text{)}$$
$$\Rightarrow x \le \frac{3}{2}$$

Therefore

$$x \in \left(-\infty, \frac{3}{2}\right)$$

Answer: (C) → (q)

(D) We have

$$\lvert x^2-2x-3\rvert < 3x-3 \Rightarrow \lvert (x-3)(x+1)\rvert < 3x-3$$

Case 1: $x < -1$. Then

$$(x-3)(x+1) < 3x-3 \Leftrightarrow x^2 - 5x < 0 \Leftrightarrow 0 < x < 5$$

However, $x < -1$.

Case 2: $-1 < x < 3$. Then

$$(3-x)(x+1) < 3x-3 \Leftrightarrow -x^2 + 2x + 3 < 3x - 3$$
$$\Leftrightarrow 0 < x^2 + x - 6$$
$$\Leftrightarrow (x+3)(x-2) > 0$$

Therefore, $x > 2$. Hence

$$x \in (2, +\infty)$$

Case 3: $x \ge 3$. Then

$$(x-3)(x+1) < 3x-3 \Leftrightarrow x^2 - 2x - 3 < 3x - 3$$
$$\Leftrightarrow x^2 - 5x < 0$$
$$\Leftrightarrow 0 < x < 5$$
$$\Leftrightarrow 3 \le x < 5 \quad (\because x \ge 3)$$

From the above two cases $x \in (2, 5)$

Answer: (D) → (p)

4. Match the items in Column I with those in Column II.

Column I	*Column II*
(A) If $\frac{x}{x-3} \le \frac{1}{x}$, then x belongs to	(p) $(-\infty, +\infty)$
(B) $\frac{x^2+6x-7}{x^2+1} \le 2$ for all x belonging to	(q) $(0, 2)$
(C) $\frac{(x-1)^2(x+1)^3}{x^4(x-2)} \le 0$ for all x in	(r) $[1, 6]$ (s) $[-1, 0]$
(D) $1 < \frac{3x^2-7x+8}{x^2+1} \le 2$ for all x in	(t) $(0, 3)$

Solution:

(A) Let

$$f(x) = \frac{x}{x-3} - \frac{1}{x} = \frac{x^2 - x + 3}{x(x-3)}$$

Observe that

$$x^2 - x + 3 = \left(x - \frac{1}{2}\right)^2 + \frac{11}{4} > 0$$

and hence $f(x) \neq 0$ for all real $x \neq 0, 3$. Therefore

$$f(x) < 0 \Leftrightarrow x(x-3) < 0 \Leftrightarrow 0 < x < 3$$

Answer: (A) → (t)

(B) Let

$$f(x) = \frac{x^2 + 6x - 7}{x^2 + 1}$$

Note that $x^2 + 1 > 0$ for all x. Now,

$$f(x) \leq 2 \Leftrightarrow x^2 + 6x - 7 \leq 2(x^2 + 1)$$
$$\Leftrightarrow x^2 - 6x + 9 \geq 0$$
$$\Leftrightarrow (x-3)^2 \geq 0$$

which is true for all x. Also $f(3) = 2$.

Answer: (B) → (p)

(C) Let

$$f(x) = \frac{(x-1)^2 (x+1)^3}{x^4 (x-2)}$$

Now $f(x) = 0 \Leftrightarrow x = 1, -1$. To determine the change of sign of $f(x)$, we have to consider the points $-1, 0, 2$.

Case 1: $x < -1 \Rightarrow (x+1)^3 < 0$ and $x - 2 < 0$. Therefore $f(x) > 0$ for all $x < -1$.

Case 2: $-1 < x < 0 \Rightarrow x - 2 < 0$ and $f(x) < 0$.

Case 3: $0 < x < 2 \Rightarrow x - 2 < 0 \Rightarrow f(x) < 0$. Therefore $f(x) \leq 0$ for all $x \in (-1, 0) \cup (0, 2)$.

Answer: (C) → (q), (s)

(D) We have

$$1 < \frac{3x^2 - 7x + 8}{x^2 + 1} \leq 2$$

$$\Leftrightarrow x^2 + 1 < 3x^2 - 7x + 8 \leq 2(x^2 + 1)$$

$$\Leftrightarrow x^2 + 1 < 3x^2 - 7x + 8 \quad \text{and} \quad 3x^2 - 7x + 8 \leq 2x^2 + 2$$

$$\Leftrightarrow 2x^2 - 7x + 7 > 0 \quad \text{and} \quad x^2 - 7x + 6 \leq 0$$

Note that

$$2x^2 - 7x + 7 = 2\left(x - \frac{7}{4}\right)^2 + \frac{7}{8} > 0 \text{ for all } x$$

$$x^2 - 7x + 6 \leq 0 \Leftrightarrow (x-1)(x-6) \leq 0 \Leftrightarrow 1 \leq x \leq 6$$

Therefore, both the inequalities hold for $1 \leq x \leq 6$. Also note that

$$\frac{3x^2 - 7x + 8}{x^2 + 1} = 2$$

when $x = 1, 6$.

Answer: (D) → (r)

Comprehension-Type Questions

1. Passage: Let $f(x) = ax^2 + bx + c$, where a, b and c are real and $a \neq 0$. Let $\alpha < \beta$ be the roots of $f(x) = 0$. Then

(a) for all x such that $\alpha < x < \beta$, $f(x)$ and a have opposite signs.

(b) for $x < \alpha$ or $x > \beta$, $f(x)$ and a have the same sign.

Based on this, answer the following three questions.

(i) If both the roots of the equation $x^2 - mx + 1 = 0$ are less than unity, then

(A) $m \leq -2$ (B) $m > 2$
(C) $-1 \leq m \leq 3$ (D) $0 \leq m \leq 5/2$

(ii) If both the roots of the equation $x^2 - 6mx + 9m^2 - 2m + 2 = 0$ are greater than 3, then

(A) $m < 0$ (B) $m > 1$
(C) $0 < m < 1$ (D) $m > 11/9$

(iii) If both the roots of the equation $4x^2 - 2x + m = 0$ belong to the interval $(-1,1)$, then

(A) $-3 < m < -2$ (B) $0 < m < 2$
(C) $2 < m < 5/2$ (D) $-2 < m \leq 1/4$

Solution:

(i) $f(x) = x^2 - mx + 1$ and $\alpha \leq \beta$ are the roots of $f(x) = 0$. Now $\alpha < \beta < 1$ implies that $f(1)$ and the coefficients of x^2 have the same sign. This gives

$$1 - m + 1 = f(1) > 0$$
$$m < 2 \tag{4.25}$$

Also, discriminant is $m^2 - 4 \geq 0$. Therefore

$$m \leq -2 \quad \text{or} \quad m \geq 2 \tag{4.26}$$

From Eqs. (4.25) and (4.26), $m \leq -2$. Also, note that if $m = -2$, the roots are $-1, -1$.

Answer: (A)

(ii) Let $f(x) = x^2 - 6mx + 9m^2 - 2m + 2$. Let $\alpha > \beta > 3$ be the roots of $f(x) = 0$. Then $6 < \alpha + \beta = 6m$ and hence

$$m > 1 \tag{4.27}$$

Also $9m^2 - 2m + 2 = \alpha\beta > 9$. Therefore

$$9m^2 - 2m - 7 > 0$$

$$(9m+7)(m-1)>0$$

This gives

$$m<\frac{-7}{9} \quad \text{or} \quad m>1 \tag{4.28}$$

Also, $f(3)$ and the coefficient of x^2 have the same sign. Therefore, $f(3)>0$. This gives

$$9-18m+9m^2-2m+2>0$$
$$9m^2-20m+11>0$$
$$(9m-11)(m-1)>0$$
$$m<1 \quad \text{or} \quad m>\frac{11}{9} \tag{4.29}$$

From Eqs. (4.27)–(4.29), we get

$$\frac{11}{9}<m$$

Answer: (D)

(iii) Let α, β, where $\alpha \le \beta$, be the roots of $4x^2-2x+m=0$. Then $-1<\alpha, \beta<1$ and

$$\alpha+\beta=\frac{1}{2}, \quad \alpha\beta=\frac{m}{4}$$

Now $f(-1)$ and the coefficient of x^2 have the same sign. Therefore $f(-1)>0$ and hence $4+2+m>0$, that is

$$m>-6 \tag{4.30}$$

Also, $f(1)>0 \Rightarrow 4-2+m>0$. This implies

$$m>-2 \tag{4.31}$$

The discriminant is $4-16m \ge 0$. Therefore

$$m \le \frac{1}{4} \tag{4.32}$$

From Eqs. (4.30)–(4.32), we get

$$-2<m\le\frac{1}{4}$$

If $m=1/4$, then the given equation is

$$4x^2-2x+\frac{1}{4}=0$$
$$16x^2-8x+1=0$$

Therefore the roots are 1/4, 1/4. If the roots are distinct, then

$$-2<m<\frac{1}{4}$$

Answer: (D)

2. Passage: Let $f(x)=ax^2+bx+c$, where a, b and c are real numbers and $a \ne 0$. If $b^2-4ac<0$, then for all real x, $f(x)$ and a will have the same sign. If $\alpha<\beta$ are real roots of $f(x)=0$, then

(a) $f(x)$ and a are of opposite sign for all x, $\alpha<x<\beta$.

(b) $f(x)$ and a are of same sign for all x such that $x<\alpha$ or $x>\beta$.

Answer the following questions.

(i) If $(a-1)x^2-(a+1)x+(a+1)>0$ for all real x, then

(A) $a<-5/3$ (B) $-5/3<a<5/3$
(C) $a<5/3$ (D) $a>5/3$

(ii) If $(a+4)x^2-2ax+2a-6<0$ for all real x, then

(A) $a<-6$ (B) $-6<a<0$
(C) $-6<a<6$ (D) $a>6$

(iii) If the roots of the equation $(2-x)(x+1)=a$ are real and positive, then

(A) $a<-2$ (B) $-2<a<2$
(C) $2<a\le 9/4$ (D) $9/4<a<17/4$

Solution:

(i) Let $f(x)\equiv(a-1)x^2-(a+1)x+a+1$. Then the discriminant is given by

$$(a+1)^2-4(a^2-1)=-3a^2+2a+5$$

$f(x)>0$ for all real x, this implies

$$-3a^2+2a+5<0$$
$$3a^2-2a-5>0$$
$$(3a-5)(a+1)>0$$
$$a<-1 \quad \text{or} \quad a>\frac{5}{3} \tag{4.33}$$

$f(x)$ and $a-1$ are of same sign. This implies

$$a-1>0 \Rightarrow a>1 \tag{4.34}$$

From Eqs. (4.33) and (4.34), we have

$$a>\frac{5}{3}$$

Answer: (D)

(ii) Let $f(x)=(a+4)x^2-2ax+2a-6$. Now $f(x)<0$ for all real x, implies

$$4a^2-8(a+4)(a-3)<0 \quad \text{and} \quad a+4<0$$
$$a<-4 \quad \text{and} \quad a^2-2(a^2+a-12)<0$$
$$a<-4 \quad \text{and} \quad (a+6)(a-4)>0$$
$$a<-6$$

Answer: (A)

(iii) The given equation is equivalent to $x^2-x+a-2=0$. Let $f(x)\equiv x^2-x+a-2$. Roots of $f(x)=0$ are real and positive. Therefore discriminant ≥ 0. That is

$$1-4(a-2)\geq 0$$

$$a\leq \frac{9}{4}$$

Also, the roots of $f(x)=0$ are

$$\frac{1\pm\sqrt{1-4(a-2)}}{2}=\frac{1}{2}(1\pm\sqrt{9-4a})^2$$

These are given to be positive. Therefore,

$$1-\sqrt{9-4a}>0$$

$$9-4a<1$$

$$a>2$$

Therefore,

$$2<a\leq\frac{9}{4}$$

Note that, when $a=9/4$, $f(x)=0$ takes the form $4x^2-4x+1=0$, whose roots are $1/2, 1/2$.

Answer: (C)

Assertion–Reasoning Type Questions

In the following set of questions, a Statement I is given and a corresponding Statement II is given just below it. Mark the correct answer as:

(A) Both I and II are true and II is a correct reason for I
(B) Both I and II are true and II is not a correct reason for I
(C) I is true, but II is false
(D) I is false, but II is true

1. Statement I: Let a, b and c be real numbers and $a\neq 0$. If $4a+3b+2c$ and a have same sign, then not both the roots of the equation $ax^2+bx+c=0$ belong to the open interval $(1, 2)$.

Statement II: A quadratic equation $f(x)=0$ will have a root in the interval (a, b) if $f(a)\,f(b)<0$.

Solution: Let $f(x)=px^2+qx+r$. If $f(a)$ and $f(b)$ are of opposite sign, the curve (parabola) $y=f(x)$ must intersect x-axis at some point. This implies that $f(x)$ has a root in (a, b). Therefore, the Statement II is true.

Let α and β be roots of $ax^2+bx+c=0$. Then,

$$\alpha+\beta=\frac{-b}{a} \quad\text{and}\quad \alpha\beta=\frac{c}{a}$$

By hypothesis,

$$\frac{4a+3b+2c}{a}>0$$

$$4+3\left(\frac{b}{a}\right)+2\left(\frac{c}{a}\right)>0$$

$$4-3(\alpha+\beta)+2\alpha\beta>0$$

$$(\alpha\beta-2\alpha-\beta+2)+(\alpha\beta-\alpha-2\beta+2)>0$$

$$(\alpha-1)(\beta-2)+(\alpha-2)(\beta-1)>0 \quad (4.35)$$

If $1<\alpha, \beta<2$, then $(\alpha-1)(\beta-2)+(\alpha-2)(\beta-1)<0$ which is contradiction to Eq. (4.35). Therefore, at least one root lies outside $(1, 2)$.

Answer: (B)

2. Statement I: If $P(x)=ax^2+bx+c$ and $Q(x)=-ax^2+dx+c$, where $ac\neq 0$, then the equation $P(x)\,Q(x)=0$ has at least two real roots.

Statement II: A quadratic equation with real coefficients has real roots if and only if the discriminant is greater than or equal to zero.

Solution: Let $px^2+qx+r=0$ be a quadratic equation. The roots are

$$\frac{-q\pm\sqrt{q^2-4pr}}{2p}$$

These are real $\Leftrightarrow q^2-4pr\geq 0$. Therefore Statement II is true.

In Statement I, $ac\neq 0$. Therefore $ac>0$ or $ac<0$. If $ac<0$, then $b^2-4ac>0$, so that $P(x)=0$ has two real roots. If $ac>0$, then $d^2+4ac>0$ so that $Q(x)=0$ has two real roots. Further, the roots of $P(x)=0$ and $Q(x)=0$ are also the roots of $P(x)Q(x)=0$. Therefore, Statement I is true and Statement II is a correct reason for Statement I.

Answer: (A)

3. Statement I: If a, b and c are real, then the roots of the equation $(x-a)(x-b)+(x-b)(x-c)+(x-c)(x-a)=0$ are imaginary.

Statement II: If p, q and r are real and $p\neq 0$, then the roots of the equation $px^2+qx+r=0$ are real or imaginary according as $q^2-4pr\geq 0$ or $q^2-4pr<0$.

Solution: Statement II is obviously true. In Statement I, the given equation is $3x^2-2(a+b+c)x+ab+bc+ca=0$. The discriminant is

$$4(a+b+c)^2-12(ab+bc+ca)$$

$$=4[a^2+b^2+c^2-ab-bc-ca]$$

$$=2[(a-b)^2+(b-c)+(c-a)^2]\geq 0$$

Therefore, the equation has real roots. Statement I is false and Statement II is true.

Answer: (D)

4. **Statement I:** Let $f(x) = x^2 + ax + b$, where a and b are integers. Then, for each integer n, there corresponds an integer m such that $f(n)f(n+1) = f(m)$.

Statement II: If α and β are roots of $x^2 + px + q = 0$, then $x^2 + px + q = (x-\alpha)(x-\beta)$.

Solution: Let α and β be roots of $f(x) = 0$, where α and β may be imaginary. Then $f(x) \equiv (x-\alpha)(x-\beta)$, $\alpha + \beta = -a$ and $\alpha\beta = b$. Now,

$$\begin{aligned} f(n)f(n+1) &= (n-\alpha)(n-\beta)(n+1-\alpha)(n+1-\beta) \\ &= (n-\alpha)(n+1-\beta)(n-\beta)(n+1-\alpha) \\ &= [n(n+1) - n(\alpha+\beta) + \alpha\beta - \alpha] \\ &\quad \times [n(n+1) - n(\alpha+\beta) + \alpha\beta - \beta] \\ &= [n(n+1) + an + b - \alpha][n(n+1) + an + b - \beta] \end{aligned}$$

Put $m = n(n+1) + an + b$. Then m is an integer and $f(n)f(n+1) = (m-\alpha)(m-\beta) = f(m)$.

Answer: (A)

5. **Statement I:** If one root of $2x^2 - 2(2a+1)x + a(a+1) = 0$ is less than a and the other root is greater than a, then $a \in (-\infty, -1) \cup (0, +\infty)$.

Statement II: If $\alpha < \beta$ are the roots of the equation $f(x) \equiv ax^2 + bx + c = 0$, then for $\alpha < x < \beta$, $f(x)$ and a have opposite signs.

Solution: Roots are to be real and distinct. The discriminant is

$$4(2a+1)^2 - 8a(a+1) > 0$$
$$4a^2 + 4a + 1 - 2a^2 - 2a > 0$$
$$2a^2 + 2a + 1 = (a+1)^2 + a^2 > 0$$

Therefore a lies between the roots $\Rightarrow f(a)$ and coefficient of x^2 are of opposite sign. Hence $f(a) < 0$, which gives

$$2a^2 - 2(2a+1)a + a(a+1) < 0$$
$$a(a+1) > 0$$
$$a < -1 \quad \text{or} \quad a > 0$$
$$a \in (-\infty, -1) \cup (0, +\infty)$$

Therefore both Statements I and II are correct and Statement II is a correct reason for Statement I.

Answer: (A)

6. **Statement I:** If α and β are roots of the equation $ax^2 + bx + c = 0$ and $\alpha + \delta$ and $\beta + \delta$ are roots of the equation $px^2 + qx + r = 0$, then

$$\frac{b^2 - 4ac}{a^2} = \frac{q^2 - 4pr}{p^2}$$

Statement II: If α and β are roots of a quadratic equation $f(x) = 0$, then the equation whose roots are $\alpha + h$ and $\beta + h$ is $f(x-h) = 0$.

Solution: Let $\alpha' = \alpha + \delta$ and $\beta' = \beta + \delta$. Then $(\alpha' - \beta')^2 = (\alpha - \beta)^2$. That is,

$$(\alpha' - \beta')^2 - 4\alpha'\beta' = (\alpha - \beta)^2 - 4\alpha\beta$$

Therefore

$$\left(\frac{-q}{p}\right)^2 - 4\left(\frac{r}{p}\right) = \left(\frac{-b}{a}\right)^2 - 4\frac{c}{a}$$
$$\frac{q^2 - 4pr}{p^2} = \frac{b^2 - 4ac}{a^2}$$

So, Statement I is true.

For Statement II, put $y = x + h$. Then $x = y - h$. Therefore $\alpha + h$ and $\beta + h$ are the roots of $f(y-h) = 0$. By replacing y with x, Statement II is also true, but Statement II is not a correct reason for Statement I.

Answer: (B)

7. **Statement I:** If a, b, c, d and p are distinct real numbers such that

$$\begin{aligned} &(a^2 + b^2 + c^2)p^2 - 2(ab + bc + cd)p \\ &\quad + (b^2 + c^2 + d^2) \le 0 \end{aligned}$$

then a, b, c and d are in geometric progression, that is

$$\frac{a}{b} = \frac{b}{c} = \frac{c}{d}$$

Statement II: Sum of squares of real numbers is always non-negative and equal to zero if and only if each of the real numbers is zero.

Solution: Statement II is obviously true. In Statement I, the given inequality can be written as

$$\begin{aligned} &(a^2p^2 - 2abp + b^2) + (b^2p^2 - 2bcp + c^2) \\ &\quad + (c^2d^2 - 2cdp + d^2) \le 0 \\ &(ap - b)^2 + (bp - c)^2 + (cp - d)^2 \le 0 \end{aligned}$$

and hence

$$ap = b, \quad bp = c, \quad cp = d$$

or

$$\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \frac{1}{p}$$

Therefore, both Statements I and II are true and II is a correct reason for I.

Answer: (A)

SUMMARY

4.1 ***Quadratic expressions and equations:*** If a, b, c are real numbers and $a \neq 0$, the expression of the form $ax^2 + bx + c$ is called quadratic expression and $ax^2 + bx + c = 0$ is called quadratic equation.

4.2 Let $f(x) \equiv ax^2 + bx + c$ be a quadratic expression and α be a real (complex) number. Then we write $f(\alpha)$ for $a\alpha^2 + b\alpha + c$. If $f(\alpha) = 0$, the α is called a zero of $f(x)$ or a root of the equation $f(x) = 0$.

4.3 ***Roots:*** The roots of the quadratic equation $ax^2 + bx + c = 0$ are

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

4.4 ***Discriminant:*** $b^2 - 4ac$ is called the discriminant of the quadratic expression (equation) $ax^2 + bx + c = 0$.

4.5 ***Sum and product of the roots:*** If α and β are roots of the equation $ax^2 + bx + c = 0$, then

$$\alpha + \beta = \frac{-b}{2a} \quad \text{and} \quad \alpha\beta = \frac{c}{a}$$

4.6 Let $ax^2 + bx + c = 0$ be a quadratic equation and $\Delta = b^2 - 4ac$ be its discriminant. Then the following hold good.

(1) Roots are equal $\Leftrightarrow \Delta = 0$ (i.e., $b^2 = 4ac$).

(2) Roots are real and distinct $\Leftrightarrow \Delta > 0$.

(3) Roots are non-real complex (i.e., imaginary) $\Leftrightarrow \Delta > 0$.

4.7 ***Theorem:*** Two quadratic equations $ax^2 + bx + c = 0$ and $a'x^2 + b'x + c' = 0$ have same roots if and only if the triples (a, b, c) and (a', b', c') are proportional and in this case

$$ax^2 + bx + c = \frac{a}{a'}(a'x^2 + b'x + c')$$

4.8 ***Cube roots of unity:*** Roots of the equation $x^3 - 1 = 0$ are called cube roots of unity and they are

$$1, \quad \frac{-1}{2} \pm i\frac{\sqrt{3}}{2}$$

$-1/2 \pm i\sqrt{3}/2$ are called non-real cube roots of unity. Further each of them is the square of the other and the sum of the two non-real cube roots of unity is equal to -1. If $w \neq 1$ is a cube root of unity and n is any positive integer, then $1 + w^n + w^{2n}$ is equal to 3 or 0 according as n is a multiple of 3 or not.

4.9 ***Maximum and minimum values:*** If $f(x) \equiv ax^2 + bx + c$ and $a \neq 0$, then

$$f\left(\frac{-b}{2a}\right) = \frac{4ac - b^2}{4a}$$

is the maximum or minimum value of f according as $a < 0$ or $a > 0$.

4.10 ***Theorems (change of sign of $ax^2 + bx + c$):*** Let $f(x) \equiv ax^2 + bx + c$ where a, b, c are real and $a \neq 0$. If α and β are real roots of $f(x) = 0$ and $\alpha < \beta$, then

(1) **(i)** $f(x)$ and a (the coefficient of x^2) have the same sign for all $x < \alpha$ or $x > \beta$.

(ii) $f(x)$ and a will have opposite signs for all x such that $\alpha < x < \beta$.

(2) If $f(x) = 0$ has imaginary roots, then $f(x)$ and a will have the same sign for all real values of x.

4.11 If $f(x)$ is a quadratic expression and $f(p)f(q) < 0$ for some real numbers p and q, then the quadratic equation $f(x) = 0$ has a root in between p and q.

EXERCISES

Single Correct Choice Type Questions

1. The roots of the equation

$$(10)^{2/x} + (25)^{1/x} = \frac{17}{4}(50)^{1/x}$$

are

(A) 2, 1/2 (B) -2, 1/2 (C) 2, $-1/2$ (D) 1/2, $-1/2$

2. If $a \neq 0$ and $a(l + m)^2 + 2blm + c = 0$ and $a(l + n)^2 + 2bln + c = 0$, then

(A) $mn = l^2 + c/a$ (B) $lm = n^2 + c/a$

(C) $ln = m^2 + c/a$ (D) $mn = l^2 + bc/a$

3. If x is real, then the least value of

$$\frac{6x^2 - 22x + 21}{5x^2 - 18x + 17}$$

is

(A) 5/4 (B) 1 (C) 17/4 (D) $-5/4$

4. The roots of the equation $x^2-(m-3)x+m=0$ are such that exactly one of them lies in the interval $(1, 2)$. Then

(A) $5<m<7$ (B) $m<10$
(C) $2<m<5$ (D) $m>10$

5. If α and β are roots of the equation $2x^2+ax+b=0$, then one of the roots of the equation $2(\alpha x+\beta)^2+a(\alpha x+\beta)+b=0$ is

(A) 0 (B) $\dfrac{\alpha+2b}{\alpha^2}$
(C) $\dfrac{a\alpha+b}{2\alpha^2}$ (D) $\dfrac{a\alpha-2b}{2\alpha^2}$

6. If $a<b$ and $x^2+(a+b)x+ab<0$, then

(A) $a<x<b$ (B) $-b<x<-a$
(C) $x<a$ or $x>b$ (D) $x<-b$ or $x>-a$

7. If α and β are the roots of $x^2-2x+4=0$, then the value of $\alpha^6+\beta^6$ is

(A) 64 (B) 128 (C) 256 (D) 32

8. The greatest value of the expression

$$\frac{1}{4t^2+2t+1}$$

is

(A) 4/3 (B) 5/2 (C) 13/14 (D) 14/13

9. The roots of the equation $4^x-3\times 2^{x+2}+32=0$ are

(A) 1, 2 (B) 1, 3 (C) 2, 3 (D) 2, 1/2

10. If the equations $x^2-3x+a=0$ and $x^2+ax-3=0$ have a common root, then a possible value of a is

(A) 3 (B) 1 (C) –2 (D) 2

11. If $x^2-1\le 0$ and $x^2-x-2\ge 0$ hold simultaneously for a real x, then x belongs to the interval

(A) $(-1, 2)$ (B) $(-1, 1)$
(C) $[-1, 2)$ (D) $x=-1$

12. Let $\alpha\ne 1$ and $\alpha^{13}=1$. If $a=\alpha+\alpha^3+\alpha^4+\alpha^{-4}+\alpha^{-3}+\alpha^{-1}$ and $b=\alpha^2+\alpha^5+\alpha^6+\alpha^{-6}+\alpha^{-5}+\alpha^{-2}$ then the quadratic equation whose roots are a and b is

(A) $x^2+x+3=0$ (B) $x^2+x+4=0$
(C) $x^2+x-3=0$ (D) $x^2+x-4=0$

13. If $ax^2-2a^2x+1=0$ and $x^2-3ax+a^2=0$, $a\ne 0$, have a common root, then a^3 is a root of the equation

(A) $x^2-x-1=0$ (B) $x^2+x-1=0$
(C) $x^2+x+1=0$ (D) $x^2-x-2=0$

14. A sufficient condition for the equation $x^2+bx-4=0$ to have integer roots is that

(A) $b=0, \pm 3$ (B) $b=0, \pm 2$
(C) $b=0, \pm 1$ (D) $b=0, \pm 4$

15. The quadratic expression ax^2+bx+c assumes both positive and negative values if and only if

(A) $ab\ne 0$ (B) $b^2-4ac>0$
(C) $b^2-4ac\ge 0$ (D) $b^2-4ac<0$

16. If $a>0$ and one root of $ax^2+bx+c=0$ is less than –2 and the other is greater than 2, then

(A) $4a+2|b|+c<0$
(B) $4a+2|b|+c>0$
(C) $4a+2|b|+c=0$
(D) $a+b=c$

17. If b and c are real, then the equation $x^2+bx+c=0$ has both roots real and positive if and only if

(A) $b<0$ and $c>0$
(B) $bc<0$ and $b\ge 2\sqrt{c}$
(C) $bc<0$ and $b^2\ge 4c$
(D) $c>0$ and $b\le -2\sqrt{c}$

18. It is given that the quadratic expression ax^2+bx+c takes all negative values for all x less than 7. Then

(A) $ax^2+bx+c=0$ has equal roots
(B) a is negative
(C) a and b are both negative
(D) a and b are both positive

19. The value of a for which the equation $\cos^4 x-(a+2)\cos^2 x-(a+3)=0$ possesses solution, belongs to the interval

(A) $(-\infty, 3)$ (B) $(2, +\infty)$
(C) $[-3, -2]$ (D) $(0, +\infty)$

20. If the expression $ax+(1/x)-2\ge 0$ for all positive values of x, then the minimum value of a is

(A) 1 (B) 2 (C) 1/4 (D) 1/2

21. If a, b and c are real, $a\ne 0$, $b\ne c$ and the equations $x^2+abx+c=0$ and $x^2+cax+b=0$ have a common root, then

(A) $a^2(b+c)=-1$ (B) $b^2(c+a)=1$
(C) $c^2(a+b)=1$ (D) $a^2(b+c)=1$

22. If $(x^2+x+2)^2-(a-3)(x^2+x+2)(x^2+x+1)+(a-4)\times(x^2+x+1)^2=0$ has atleast one real root, then

(A) $0<a<5$ (B) $5<a\le 19/3$
(C) $5\le a<7$ (D) $a\ge 7$

Multiple Correct Choice Type Questions

1. The equation $x^{(3/4)(\log_2 x)^2+\log_2 x-(5/4)} = \sqrt{2}$ has

(A) atleast one real solution
(B) exactly three solutions
(C) exactly one irrational solution
(D) complex roots

2. If S is the set of all real values of x such that

$$\frac{2x-1}{2x^3+3x^2+x} > 0$$

then S is a superset of

(A) $(-\infty, -3/2)$ (B) $(-3/2, -1/4)$
(C) $(-1/4, 1/2)$ (D) $(1/2, 3)$

3. If $||x^2-5x+4|-|2x-3|| = |x^2-3x+1|$, then x belongs to the interval

(A) $(-\infty, 1]$ (B) $(1, 3/2)$
(C) $[3/2, 4]$ (D) $(4, +\infty)$

4. Let

$$y = \sqrt{\frac{(x+1)(x-3)}{x-2}}$$

Then the set of real values of x for which y is real is

(A) $[-1, 2)$ (B) $(2, 3)$
(C) $(-\infty, -1)$ (D) $[3, +\infty)$

5. Let a, b and c be distinct positive reals such that the quadratics ax^2+bx+c, bx^2+cx+a and cx^2+ax+b are all positive for all real x and

$$s = \frac{a^2+b^2+c^2}{ab+bc+ca}$$

Then

(A) $s \le 1$ (B) $1 < s < 4$
(C) $s \notin (-\infty, 1) \cup (4, +\infty)$ (D) $0 < s < 1$

6. If α and $1/\alpha$ $(\alpha > 0)$ are roots of $ax^2-bx+c=0$, then

(A) $c = a$ (B) $c \ge 2b$
(C) $b \ge 2a$ (D) $a \ge 2b$

7. If

$$\frac{k}{2x} = \frac{a}{x+c} + \frac{b}{x-c}$$

where $c \ne 0$, a and b are positive, has equal roots, then the value of k is

(A) $a+b$ (B) $a-b$
(C) $(\sqrt{a}+\sqrt{b})^2$ (D) $(\sqrt{a}-\sqrt{b})^2$

8. If the product of the roots of the equation

$$x^2 - 4mx + 3e^{2\log m} - 4 = 0$$

is 8, then the roots are

(A) real (B) non-real
(C) rational (D) irrational

9. If $3^{-\log_{1/9}[x^2-(10/3)x+1]} \le 1$, then x belongs to

(A) $[0, 1/3)$ (B) $(1/3, 1)$
(C) $(2, 3)$ (D) $(3, 10/3]$

10. If every pair of the equations $x^2+ax+bc=0$, $x^2+bx+ca=0$ and $x^2+cx+ab=0$ has a common root, then

(A) sum of these common roots is $-(1/2)(a+b+c)$
(B) sum of these common roots is $(1/2)(a+b+c)$
(C) product of the common roots is abc
(D) product of the common roots is $-(abc)$

11. If the equations $4x^2-11x+2k=0$ and $x^2-3x-k=0$ have a common root α, then

(A) $k = 0$ (B) $k = -17/36$
(C) $\alpha = 0$ (D) $\alpha = 17/6$

12. If a and b are real and $x^2+ax+b^2=0$ and $x^2+ax+a^2=0$ have a common root, then which of the following are true?

(A) $a = b$
(B) $a+b$ is the common root
(C) for real roots, $a = b = 0$
(D) no real values of a and b exist

13. For $a > 1$, the equation

$$(a+\sqrt{a^2-1})^{x^2-2x} + (a-\sqrt{a^2-1})^{x^2-2x} = 2a$$

has

(A) three real roots
(B) roots which are independent of a
(C) roots whose sum is 3
(D) roots whose product is -1

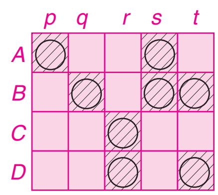
14. If a, b and c are positive real and $a = 2b+3c$, then the equation $ax^2+bx+c=0$ has real roots for

(A) $\left|\frac{b}{c}-4\right| \ge 2\sqrt{7}$ (B) $\left|\frac{c}{b}-4\right| \ge 2\sqrt{7}$
(C) $\left|\frac{a}{c}-11\right| \ge 4\sqrt{7}$ (D) $\left|\frac{a}{b}+4\right| \ge 2\sqrt{\frac{13}{3}}$

Matrix-Match Type Questions

In each of the following questions, statements are given in two columns, which have to be matched. The statements in Column I are labeled as **(A)**, **(B)**, **(C)** and **(D)**, while those in Column II are labeled as **(p)**, **(q)**, **(r)**, **(s)** and **(t)**. Any given statement in Column I can have correct matching with *one or more* statements in Column II. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following example.

Example: If the correct matches are (A) → (p), (s); (B) → (q), (s), (t); (C) → (r); (D) → (r), (t); that is if the matches are (A) → (p) and (s); (B) → (q), (s) and (t); (C) → (r); and (D) → (r), (t), then the correct darkening of bubbles will look as follows:



1. Match the items in Column I with those in Column II.

Column I	*Column II*
(A) If α and β are roots of $x^2 + x + 1 = 0$ and k is a positive integer and not a multiple of 3, then the equation whose roots are α^k and β^k is	(p) $x^2 + (a + 4)x + 2a = 0$
(B) If α and β are roots of $x^2 + x + 1 = 0$, then the equation whose roots are α^{2009} and β^{2009} is	(q) $x^2 - x + 1 = 0$
(C) If α and β are roots of $x^2 + ax + b = 0, b \neq 0$, then the equation whose roots are $1/\alpha, 1/\beta$ is	(r) $x^2 + x + 1 = 0$
(D) If α and β are roots of $x^2 + ax - 4 = 0$, then the equation whose roots are $\alpha - 2$ and $\beta - 2$ is	(s) $bx^2 - ax - 1 = 0$
	(t) $bx^2 + ax + 1 = 0$

2. Let α and β be roots of the equation $ax^2 + bx + c = 0$ and $\alpha\beta \neq 0$. Then match the items in Column I with those in Column II.

Column I	*Column II*
(A) The equation whose roots are $\alpha + \beta$ and $\alpha\beta$ is	(p) $cx^2 + bx + a = 0$
(B) The equation whose roots are α^2 and β^2 is	(q) $a^2x^2 + (2ac - b^2)x + c^2 = 0$
(C) The equation whose roots are $1/\alpha$ and $1/\beta$ is	(r) $a^2x^2 + a(b - c)x - bc = 0$
(D) The equation whose roots are $\alpha - c$ and $\beta - c$ is	(s) $ax^2 + (2ac + b)x + ac^2 + bc + c = 0$
	(t) $cx^2 - bx + a = 0$

3. Match the items in Column I with those in Column II.

Column I	*Column II*
(A) The maximum value of $$\frac{x^2 - 6x + 4}{x^2 + 2x + 4}$$ (x is real) is	(p) 0
(B) The correct value of a for which the equation $(a^2 + 4a + 3)x^2 + (a^2 - a - 2)x + a(a + 1) = 0$ has more than two roots is	(q) 1
(C) The number of real values of x satisfying $5^x + 5^{-x} = \log_{10}^{25}$ is	(r) –1
(D) If the ratio of the roots of the equation $ax^2 + bx + b = 0$ (a and b positive) is in the ratio $l:m$ (l and m positive), then $$\sqrt{\frac{l}{m}} + \sqrt{\frac{m}{l}} - \sqrt{\frac{b}{a}}$$ is equal to	(s) –1/3
	(t) 5

4. For the equation $(x^2 - 6x)^2 = 81 + 2(x - 3)^2$, match the items in Column I with those in Column II.

Column I	*Column II*
(A) The number of rational roots is	(p) 12
(B) The number of irrational roots is	(q) 6
(C) Sum of all the real roots is	(r) 2
	(s) 99
(D) Product of the real roots is	(t) –99

5. Let

$$f(x) = \frac{x^2 - 6x + 5}{x^2 - 5x + 6}$$

Then match the items in Column I with those in Column II.

Column I	Column II
(A) If $-1 < x < 1$, then $f(x)$ satisfies	(p) $0 < f(x) < 1$
(B) If $1 < x < 2$, then $f(x)$ satisfies	(q) $f(x) < 0$
(C) If $3 < x < 5$, then $f(x)$ satisfies	(r) $f(x) > 0$
(D) If $x > 5$, then $f(x)$ satisfies	(s) $f(x) < 1$
	(t) $f(x) = 0$

6. Match the items in Column I with those in Column II.

Column I	Column II
(A) If x is real, the expression $\frac{(x+3)^2 - 24}{2(x-2)} (x \neq 2)$ admits all values except those in the interval	(p) $[1, 7]$
(B) If the expression $\frac{px^2 + 3x - 4}{p + 3x - 4x^2}$ $(p + 3x - 4x^2 \neq 0)$ takes all real values, then p lies in the interval	(q) $[-1/11, 1]$
(C) If x is real, then $\frac{x}{x^2 - 5x + 9}$ must lie in the interval	(r) $(-3, -2)$
(D) If $x \neq -2$ and $x \neq -3$, then $\frac{x^2 - 4x + 5}{x^2 + 5x + 6} < 0$ for all x in the interval	(s) $(4, 6)$
	(t) $(3, 5)$

Comprehension-Type Questions

1. **Passage:** To solve equations of the form

$$(ax^2 + bx + c)(ax^2 + bx + d) = k$$

use the substitution $ax^2 + bx = y$, so that the given equation transforms into a quadratic equation in y which can be solved. Answer the following three questions.

(i) The number of real roots of the equation

$$\frac{1}{x(x+2)} - \frac{1}{(x+1)^2} = \frac{1}{12}$$

is

(A) 2 (B) 1 (C) 0 (D) 4

(ii) The equation

$$\frac{24}{x^2 + 2x - 8} - \frac{15}{x^2 + 2x - 3} = 2$$

has

(A) all positive solutions
(B) three positive and one negative solutions
(C) two non-negative and two negative solutions
(D) two real and two imaginary solutions

(iii) The real solution set of the equation

$$(x-2)(x+1)(x+4)(x+7) = 19$$

contains

(A) four elements (B) three elements
(C) two elements (D) no elements

2. **Passage:** Let $f(x) = ax^2 + bx + c$ and $a \neq 0$. If α and β are roots of $f(x) = 0$, then $\alpha + \beta = -b/a$ and $\alpha\beta = c/a$. Further, if α and β are real roots with $\alpha < \beta$, then $f(x)$ and a have the same sign for all $x < \alpha$ or $x > \beta$, and $f(x)$ and a have opposite sign for all $\alpha < x < \beta$. Consider the quadratic equation

$$(1 + m)x^2 - 2(1 + 3m)x + (1 + 8m) = 0$$

Now, answer the following three questions.

(i) The number of real values m such that the roots of the given quadratic equation are in the ratio 2:3 is

(A) 2 (B) 4
(C) 0 (D) infinite

(ii) The set of values of m such that both the roots of the equation are positive is

(A) $(-\infty, +\infty)$
(B) $-1 < m < 3$
(C) $3 \le m < \infty$
(D) $(-\infty, -1) \cup (3, +\infty) \cup \{0\}$

(iii) The values of m for which the equation has equal roots are

(A) 0, 3 (B) −1, 3 (C) −1, 0 (D) 1, 3

3. Passage: Let $f(x) \equiv ax^2 + bx + c, a \ne 0$. Let α and β be roots of $f(x) = 0$. Then the following hold good.

(a) $\alpha + h$ and $\beta + h$ are roots of $f(x - h) = 0$ for all h.
(b) $\lambda\alpha + h$ and $\lambda\beta + h$ are roots of $f[(x - h)/\lambda] = 0$ for all h and for all $\lambda \ne 0$.

Now, answer the following three questions.

(i) If α and β are the roots of $ax^2 + bx + c = 0$, then the equation whose roots are

$$\frac{\alpha+1}{\alpha-2} \quad \text{and} \quad \frac{\beta+1}{\beta-2}$$

is

(A) $a(x+1)^2 + b(x+1)(x-2) + c(x-2)^2 = 0$
(B) $a(x-2)^2 + b(x+1)(x-2) + c(x+1)^2 = 0$
(C) $a(2x+3)^2 + b(x+1)(x+2) + c(x+2)^2 = 0$
(D) $a(2x+1)^2 + b(2x+1)(x-1) + c(x-1)^2 = 0$

(ii) If α and β are roots of the equation $2x^2 + 4x - 5 = 0$, then the equation whose roots are $2\alpha - 3$ and $2\beta - 3$ is

(A) $x^2 + 10x - 11 = 0$
(B) $11x^2 + 10x - 1 = 0$
(C) $x^2 + 10x + 11 = 0$
(D) $11x^2 - 10x + 1 = 0$

(iii) If α and β are roots of $ax^2 + bx + c = 0$, then the equation whose roots are $\alpha + (c/a)$ and $\beta + (c/a)$ is

(A) $a^2x^2 - 2(ac + b)x + c(a + b) = 0$
(B) $a^2x^2 - (ca + b)x + c(a + b) = 0$
(C) $a^2x^2 + 2(ac + b)x - c(a + b + c) = 0$
(D) $a^2x^2 - a(b + 2c)x + c(a + b + c) = 0$

4. Passage: Let a, b and c be real numbers, $a \ne 0$ and $f(x) \equiv ax^2 + bx + c$. If $\alpha < \beta$ are roots of $f(x) = 0$, then it is known that

(A) $f(x) \cdot a < 0$ for all x in the open interval (α, β).
(B) $f(x) \cdot a > 0$ for all x such that either $x < \alpha$ or $x > \beta$.

Now, answer the following three questions.

(i) If the equation $(a^2 + 1)x^2 - (a + 1)x + (a^2 - a - 2) = 0$ has one positive and one negative root, then which one of the following is possible?

(A) $a \le -1$ (B) $-1 < a < 2$
(C) $2 \le a \le 5$ (D) $a > 5$

(ii) If $mx^2 - (m + 1)x + 3 = 0$ has roots belonging to $(1, 2)$, then

(A) $0 < m < 1$ (B) $1 \le m \le 2$
(C) $m < 0$ (D) no real value for m exists

(iii) If $x^2 - (m + 1)x + m^2 + m - 8 = 0$ has one root in the open interval $(-\infty, 1)$ and the other in $(1, +\infty)$, then

(A) $m < -2\sqrt{2}$ (B) $m > 2\sqrt{2}$
(C) $-2\sqrt{2} < m < 2\sqrt{2}$ (D) no real value for m exists

5. Passage: Let $f(x) \equiv ax^2 + bx + c$, where a, b and c are real and $a \ne 0$. Then $f(x) = 0$ has real roots or imaginary roots according as $b^2 - 4ac \ge 0$ or $b^2 - 4ac < 0$. Answer the following three questions.

(i) If the function

$$y = \frac{x^2 - x}{1 - mx}$$

takes all real values for real values of x, then

(A) $m < 0$ (B) $0 < m < 1$ (C) $m > 0$ (D) $m > 1$

(ii) If

$$y = \frac{x^2 + 2x + c}{x^2 + 4x + 3c}$$

takes all real values, then

(A) $0 < c < 1$ (B) $c < -1$
(C) $c > 1$ (D) $c > 0$

(iii) If

$$\frac{x^2 + ax + 1}{x^2 + x + 1} < 3$$

for all real x, then

(A) $a < 0$ (B) $a < -1$
(C) $-1 < a < 7$ (D) $a > 7$

Assertion–Reasoning Type Questions

In each of the following, two statements, I and II, are given and one of the following four alternatives has to be chosen.

(A) Both I and II are correct and II is a correct reasoning for I.
(B) Both I and II are correct but II is not a correct reasoning for I.
(C) I is true, but II is not true.
(D) I is not true, but II is true.

1. Statement I: If $f(x) \equiv ax^2 + bx + c$ is positive for all x greater than 5, then $a > 0$, but b may be negative or may not be negative.

Statement II: If $f(x) \equiv ax^2 + bx + c > 0$ for all $x > 5$, then the equation $f(x) = 0$ may not have real roots or will have real roots less than or equal to 5.

2. **Statement I:** If a, b and c are positive integers and $ax^2 - bx + c = 0$ has two distinct roots in the integer $(0, 1)$, then $\log_5(abc) \geq 2$.

   Statement II: If a quadratic equation $f(x) = 0$ has roots in an interval (h, k), then $f(h), f(k) > 0$

3. **Statement I:** There are only two values for $\sin x$ satisfying the equation $2^{\sin^2 x} + 5 \times 2^{\cos^2 x} = 7$.

   Statement II: Maximum value of $\sin^2 x$ is 1.

4. **Statement I:** If $x = 1$ is a root of the quadratic equation $ax^2 + bx + c = 0$, then the roots of the equation $4ax^2 + 3bx + 2c = 0$ are imaginary.

   Statement II: For any polynomial equation, 1 is a root if and only if the sum of all the coefficients of the polynomial is zero.

5. Let a, b, c, p and q be real numbers and α and β be roots of the equation $x^2 + 2px + q = 0$. Suppose α and $1/\beta$ are roots of the equation $ax^2 + 2bx + c = 0$ where $\beta^2 \notin \{-1, 0, 1\}$.

   Statement I: $(p^2 - q)(b^2 - ac) \geq 0$

   Statement II: $b \neq pa$ or $c \neq qa$

6. **Statement I:** Let α, β, a and b be real numbers. If $\alpha + i\beta\,(\alpha \neq 0, \beta \neq 0)$ is a root of the equation $x^3 + bx + c = 0$, then 2α is a root of one of the following equations.

   (i) $x^3 - bx + c = 0$ **(ii)** $x^3 - bx - c = 0$
   (iii) $x^3 + bx - c = 0$ **(iv)** $x^3 + bx - 2c = 0$

   Statement II: Complex roots occur in conjugate pairs for any polynomial equation with real coefficients.

7. **Statement I:** The maximum value of

$$\frac{3x^2 + 9x + 17}{3x^2 + 9x + 7} \quad (x \text{ is real})$$

   is 8.

   Statement II: If $a, b,$ and c are real numbers and $a > 0$, then the minimum value of $ax^2 + bx + c$ (x is real) is

$$\frac{4ac - b^2}{4a}$$

8. **Statement I:** Suppose a, b and c are real numbers and $a \neq 0$. If the equation $ax^2 + bx + c = 0$ has two roots of which one is less than -1 and the other is greater than 1, then

$$1 + \frac{c}{a} + \left|\frac{b}{a}\right| < 0$$

   Statement II: Let $f(x) \equiv ax^2 + bx + c$, where a, b and c are real numbers and $a \neq 0$. If $f(x) = 0$ has real roots, then $af(x) < 0$ for all real x lying between the roots of $f(x) = 0$.

9. **Statement I:** Let

$$y = \frac{x^2 - x + 1}{x^2 + x + 1}$$

   where x is real. Then y cannot lie between 1/3 and 3.

   Statement II: If a, b and c are real, then the quadratic equation $ax^2 + bx + c = 0$ has real roots if and only if $b^2 - 4ac \geq 0$.

10. **Statement I:** For all real values of x, the range of the function

$$y = \frac{(ax - b)(cx - d)}{(bx - a)(dx - c)}$$

    is $\mathbb{R}$ if a, b, c and d are real, $a \neq b, c \neq d$ and $(a^2 - b^2)(c^2 - d^2) > 0$.

    Statement II: A quadratic equation will have real roots if its discriminant is greater than or equal to zero.

11. **Statement I:** Suppose a, b and c are real, $c > 0$, $a + b + c > 0$ and $a - b + c > 0$. Then both the roots of the equation $ax^2 + bx + c = 0$ lie between -1 and 1.

    Statement II: For a quadratic expression $f(x)$, if $f(p)$ and $f(q)$ are of opposite sign, then $f(x) = 0$ has a root in between p and q.

12. **Statement I:** Let $f(x)$ and $g(x)$ be quadratic expressions with rational coefficients. Suppose they have a common root of the form $\alpha + \sqrt{\beta}$ where β is not a perfect square of a rational number. Then $g(x) = \gamma f(x)$ for some rational number γ.

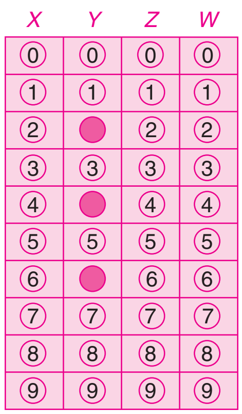
    Statement II: For a quadratic equation, with rational coefficients, if $a + \sqrt{b}$ (b is not a perfect square of a rational number) is a root, then $a - \sqrt{b}$ is also a root.

13. **Statement I:** If the equation $x^2 + px + q = 0$ has rational roots and p and q are integers, then the roots are integers.

    Statement II: A quadratic equation has rational roots if and only if its discriminant is a perfect square of a rational number.

Integer Answer Type Questions

The answer to each of the questions in this section is a non-negative integer. The appropriate bubbles below the respective question numbers have to be darkened. For example, as shown in the figure, if the correct answer to the question number Y is 246, then the bubbles under Y labeled as 2, 4, 6 are to be darkened.



1. The integer value of k for which
$$x^2 - 2(4k-1)x + 15k^2 - 2k - 7 > 0$$
for all real x is ______.

2. The number of negative integer solutions of $x^2 \times 2^{x+1} + 2^{|x-3|+2} = x^2 \times 2^{|x-3|+4} + 2^{x-1}$ is ______.

3. If $(\alpha + 5i)/2$ is a root of the equation $2x^2 - 6x + k = 0$, then the value of k is ______.

4. If the equation $x^2 - 4x + \log_{1/2} a = 0$ does not have distinct real roots, then the minimum value of $1/a$ is ______.

5. If a is the greatest negative integer satisfying
$$x^2 - 4x - 77 < 0 \quad \text{and} \quad x^2 > 4$$
simultaneously, then the value of $|a|$ is ______.

6. The number of values of k for which the quadratic equations $(2k-5)x^2 - 4x - 15 = 0$ and $(3k-8)x^2 - 5x - 21 = 0$ have a common root is ______.

7. The number of real roots of the equation $2x^2 - 6x - 5\sqrt{x^2 - 3x - 6} = 0$ is ______.

ANSWERS

Single Correct Choice Type Questions

1. (D)
2. (A)
3. (B)
4. (D)
5. (A)
6. (B)
7. (B)
8. (A)
9. (C)
10. (D)
11. (D)
12. (C)
13. (A)
14. (A)
15. (B)
16. (A)
17. (D)
18. (B)
19. (C)
20. (A)
21. (A)
22. (B)

Multiple Correct Choice Type Questions

1. (A), (B), (C)
2. (A), (D)
3. (A), (C)
4. (A), (D)
5. (B), (C)
6. (A), (C)
7. (C), (D)
8. (A), (D)
9. (A), (D)
10. (A), (C), (D)
11. (A), (B), (C), (D)
12. (A), (B), (C)
13. (A), (B), (C), (D)
14. (A), (C)

Matrix-Match Type Questions

1. (A) $\rightarrow$ (r), (B) $\rightarrow$ (r), (C) $\rightarrow$ (t), (D) $\rightarrow$ (p)
2. (A) $\rightarrow$ (r), (B) $\rightarrow$ (q), (C) $\rightarrow$ (p), (D) $\rightarrow$ (s)
3. (A) $\rightarrow$ (t), (B) $\rightarrow$ (r), (C) $\rightarrow$ (p), (D) $\rightarrow$ (p)
4. (A) $\rightarrow$ (r), (B) $\rightarrow$ (r), (C) $\rightarrow$ (p), (D) $\rightarrow$ (t)
5. (A) $\rightarrow$ (p), (r), (s), (B) $\rightarrow$ (q), (s), (C) $\rightarrow$ (q), (s), (D) $\rightarrow$ (p), (r), (s)
6. (A) $\rightarrow$ (s), (B) $\rightarrow$ (p), (C) $\rightarrow$ (q), (D) $\rightarrow$ (r)

Comprehension-Type Questions

1. (i) (A); (ii) (C); (iii) (A)
2. (i) (A); (ii) (D); (iii) (A)
3. (i) (D); (ii) (C); (iii) (D)
4. (i) (B); (ii) (D); (iii) (C)
5. (i) (D); (ii) (A); (iii) (C)

Assertion–Reasoning Type Questions

1. (A)
2. (A)
3. (A)
4. (D)
5. (B)
6. (A)
7. (D)
8. (A)
9. (D)
10. (A)
11. (A)
12. (A)
13. (A)

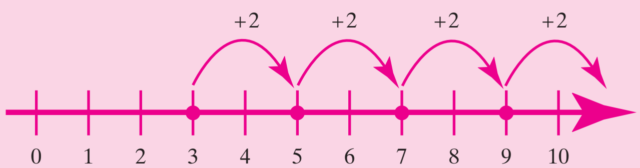
Integer Answer Type Questions

1. 3
2. 0
3. 17
4. 16
5. 3
6. 2
7. 4

Progressions, Sequences and Series

5

Progressions, Sequences and Series



Contents

Sequences: A sequence is an ordered list of objects (or events). It contains members (also called *elements* or *terms*), and the number of terms (possibly infinite) is called the *length* of the sequence. Order matters and the exactly same elements can appear multiple times at different positions in the sequence.

Series: The sum of terms of a sequence is a series.

We have defined the concept of a function and its domain, codomain and range in Chapter 1. A sequence is a function whose domain is the set of natural numbers and codomain is a given set. In this chapter, we discuss various aspects of sequences, in particular of sequences defined in certain recursive types.

5.1 | Sequences and Series

In this section, we will introduce the notion of a sequence and the corresponding series and their limits. Though the concept of limit is discussed in another volume of this series, we assume a certain intuitive idea about the limit or the approaching value. For example, the value of $1/n$ decreases as n increases and $1/n$ becomes nearer to zero (and it is never zero) as we take bigger values for n. A naive idea like this is enough to understand the concepts introduced in this chapter.

Sequence of Elements

DEFINITION 5.1 Let $\mathbb{Z}^+$ be the set of positive integers and X any set. Then a mapping $a:\mathbb{Z}^+ \to X$ is called a ***sequence of elements*** *in* X or, simply, a *sequence in* X. For any $n \in \mathbb{Z}^+$, we prefer to write a_n for the image $a(n)$. This a_n is called the ***nth term*** of the sequence.

Usually a sequence is denoted by its range $\{a_n | n \in \mathbb{Z}^+\}$ or simply $\{a_n\}$ or $\{a_1, a_2, a_3, \ldots\}$.

Examples

(1) $\{1/n\}$ is a sequence of real numbers. Here the sequence $a:\mathbb{Z}^+ \to \mathbb{R}$ is given by $a_n = 1/n$ for any $n \in \mathbb{Z}^+$.

(2) $\{n^2\}$ is a sequence of integers. Here $a_n = n^2$ for all $n \in \mathbb{Z}^+$.

(3) $\{\log_2 n\}$ is a sequence of real numbers. Here the sequence $a:\mathbb{Z}^+ \to \mathbb{R}$ is given by $a_n = \log_2 n$ for any $n \in \mathbb{Z}^+$.

(4) $\{i^n\}$ is a sequence of complex numbers. Here the sequence $a:\mathbb{Z}^+ \to \mathbb{C}$ is given by $a_n = i^n$ for any $n \in \mathbb{Z}^+$. Recall that $i^n = 1$ if n is a multiple of 4, $i^n = i$ if $n = 4m+1$, $i^n = -1$ if $n = 4m+2$ and $i^n = -i$ if $n = 4m+3$.

DEFINITION 5.2 A sequence $\{a_n\}$ is called ***finite*** if its range is a finite set. In other words, the set

$$\{a_n | n \in \mathbb{Z}^+\}$$

is a finite set. An ***infinite sequence*** is a sequence which is not finite.

Examples

(1) The sequence $\{i^n\}$ is finite, since $\{1, i, -1, -i\}$ is precisely the range.

(2) The sequence $\{(-1)^n\}$ is finite, since $\{1, -1\}$ is its range.

(3) For any $m > 1$, $\{m^n\}$ is an infinite sequence.

(4) The sequence $\{\log_2 n\}$ is infinite.

DEFINITION 5.3 A sequence $\{a_n\}$ is called ***constant*** if $a_1 = a_2 = \cdots$ (i.e., $a_n = a_m$ for all n and $m \in \mathbb{Z}^+$). $\{a_n\}$ is called ***ultimately constant*** if it is constant after a certain stage in the sense that, there is a positive integer m such that

$$a_m = a_{m+k} \quad \text{for all } k \in \mathbb{Z}^+$$

or

$$a_m = a_{m+1} = a_{m+2} = \cdots$$

Quite often, ultimately constant sequences are also called ***finite sequences*** for the simple reason that their ranges are finite. As per our terminology, any ultimately constant sequence is finite and not vice-versa; for, consider the following examples:

Examples

(1) The sequence $\{(-1)^n\}$ is finite but not ultimately constant, since $a_n \neq a_{n+1}$ for all $n \in \mathbb{Z}^+$, where $a_n = (-1)^n$, $a_n = 1$ if n is even and $a_n = -1$ if n is odd.

(2) The sequence $\{a_n\}$, where $a_n = [1/n]$ (the integral part of $1/n$), is ultimately constant, since $[1/n] = 0$ for all $n > 1$.

(3) Define the sequence $\{a_n\}$ by $a_n =$ the remainder obtained by dividing n with 2. Then a_n is 1 or 0 depending on whether n is odd or even, respectively. In this case $\{a_n\}$ is finite, but not ultimately constant. Here also, $a_n \neq a_{n+1}$ for all $n \in \mathbb{Z}^+$.

(4) The sequence $\{i^n\}$ is also finite, but not ultimately constant.

Quite often, sequences are defined recursively in the sense that a_n is defined in terms of $a_{n-1}, a_{n-2}, \ldots, a_2, a_1$. Of course, one has to define the first term a_1 or the first few terms.

Try it out

1. Let $a_1 = 2$ and $a_n = a_{n-1} + 2$ for any $n > 1$. Then show that

$$a_2 = a_1 + 2 = 2 + 2 = 4$$
$$a_3 = a_2 + 2 = 6$$
$$a_4 = a_3 + 2 = 8, \text{ etc.}$$

2. Let $a_1 = 1, a_2 = 4$ and $a_n = a_1 + a_2 + \cdots + a_{n-1}$ for $n \geq 3$. Then show that

$$a_3 = a_1 + a_2 = 5$$
$$a_4 = a_1 + a_2 + a_3 = 10$$
$$a_5 = a_1 + a_2 + a_3 + a_4 = 20, \text{ etc.}$$

Note that $a_n = 2a_{n-1}$ for $n > 3$.

3. Let $a_1 = 1, a_2 = 2$ and $a_n = a_{n-1} + a_{n-2}$ for any $n > 2$. Then show that

$$a_3 = a_2 + a_1 = 3$$
$$a_4 = a_3 + a_2 = 5$$
$$a_5 = a_4 + a_3 = 8$$
$$a_6 = a_5 + a_4 = 13$$
$$a_7 = a_6 + a_5 = 21, \text{ etc.}$$

4. Let

$$a_1 = \frac{1}{2} \quad \text{and} \quad a_n = \frac{a_{n-1}}{1 + 2a_{n-1}} \quad \text{for all } n > 1$$

Then show that

$$a_2 = \frac{1}{4}, a_3 = \frac{1}{6}, a_4 = \frac{1}{8}, a_5 = \frac{1}{10}, \text{ etc.}$$

Note that $a_n = 1/2n$ for all $n \in \mathbb{Z}^+$.

Series

DEFINITION 5.4 If $\{a_n\}$ is a sequence of real or complex numbers, then an expression of the form

$$a_1 + a_2 + a_3 + \cdots$$

is called a ***series***. If s_n is the sum of the first n-terms of the sequence $\{a_n\}$, that is,

$$s_n = a_1 + a_2 + \cdots + a_n$$

then $\{s_n\}$ is again a sequence and s_n is called the nth *partial sum* of the series.

Examples

(1) $1 + 2 + 3 + \cdots$ is a series and the partial sum is given by

$$s_n = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

(2) $1 + (-1) + 1 + (-1) + \cdots$ is a series and the partial sum is $s_n = 1$ or 0 depending on whether n is odd or even.

Note: An ultimately constant sequence $\{a_n\}$ is some times referred as a finite sequence and is expressed as $a_1, a_2, \ldots, a_n$ with the assumption that $a_n = a_{n+1} = \cdots$

Limit

DEFINITION 5.5 Let $\{a_n\}$ be a sequence of real numbers and a be a real number. Then a is said to be ***limit*** of the sequence $\{a_n\}$ if, for each $\in > 0$, there exists a positive integer n_0 such that

$$|a_n - a| < \in \quad \text{for all } n \geq n_0$$

That is, $a - \in < a_n < a + \in$ for all $n \geq n_0$.

THEOREM 5.1 Any sequence can have at most one limit.

PROOF Let $\{a_n\}$ be a sequence and a and b be limits of $\{a_n\}$. Suppose that $a \neq b$. Take

$$\in = \frac{1}{2}|a - b|$$

Since a is a limit of $\{a_n\}$, there exists $n_0 \in \mathbb{Z}^+$ such that

$$|a_n - a| < \in \quad \text{for all } n \geq n_0$$

Similarly, there exists $n_1 \in \mathbb{Z}^+$ such that

$$|a_n - b| < \in \quad \text{for all } n \geq n_1$$

Choose $n \in \mathbb{Z}^+$ such that $n > \max\{n_0, n_1\}$. Then

$$|a - b| \leq |a - a_n| + |a_n + b| < \in + \in = |a - b|$$

which is a contradiction. Thus $a = b$. ■

DEFINITION 5.6 If a is the limit of $\{a_n\}$, then we write $\lim\limits_{n\to\infty} a_n = a$ or, simply, $\lim a_n = a$ and denote it by $a_n \to a$.

Examples

(1) Consider the sequence $\{1/n\}$. Then $\lim(1/n) = 0$. For, if $\in > 0$ is given, choose a positive integer $n_0 > 1/\in$ so that, for any $n \geq n_0$

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} \leq \frac{1}{n_0} < \in$$

(2) The sequence $\{n\}$ has no limit.

(3) If $\lim a_n = a$ and $\lim b_n = b$, then $\lim(a_m \pm b_n) = a \pm b$ and $\lim a_n b_n = ab$.

(4) If a is the limit of a sequence $\{a_n\}$, then a is the limit of any sequence obtained by omitting a finite number of consecutive terms from $\{a_n\}$.

Convergent and Divergent Series

DEFINITION 5.7 Let $\{a_n\}$ be any sequence of real numbers and $s_n = a_1 + a_2 + \cdots + a_n$. If the sequence $\{s_n\}$ has limit s, then we write

$$\sum_{n=1}^{\infty} a_n = s$$

In this case, the infinite series $\sum_{n=1}^{\infty} a_n$ is said to be ***convergent*** to s. If $\{s_n\}$ has no limit, then the series $\sum_{n=1}^{\infty} a_n$ is said to be ***divergent***.

Examples

(1) $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

(2) $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

(3) If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim a_n = 0$.

(4) If $\lim a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ may not be convergent (see example given in point 2).

5.2 | Arithmetic Progressions

A sequence whose terms satisfy a specific condition is called a ***progression***. In this section we discuss sequences in which the difference between any two consecutive terms is a fixed constant. Such sequences are called *arithmetic progressions*. We begin with the formal definition in the following.

DEFINITION 5.8 A sequence $\{a_n\}$ is called an ***arithmetic progression*** if $a_{n+1} - a_n = a_n - a_{n-1}$ for all integers $n > 1$ and, in this case, $a_{n+1} - a_n$ is called the ***common difference***.

An arithmetic progression is also called an *arithmetic sequence*. Note that $\{a_n\}$ is an arithmetic progression if and only if

$$a_{n+1} + a_{n-1} = 2a_n$$

for all integers $n > 1$.

Before going for examples, let us have the following fundamental characterization of arithmetical progressions.

THEOREM 5.2 Let $\{t_n\}$ be a sequence of real numbers. Then $\{t_n\}$ is an arithmetic progression if and only if there exist unique real numbers a and d such that

$$t_n = a + (n-1)d$$

for all integers $n \geq 1$.

PROOF Suppose that $\{t_n\}$ is an arithmetic progression. Then

$$t_{n+1} - t_n = t_n - t_{n-1} \quad \text{for all } n > 1$$

Take $a = t_1$ and $d = t_2 - t_1$. Then

$$t_1 = a + (1-1)d$$
$$t_2 = t_1 + (t_2 - t_1) = a + (2-1)d$$
$$t_3 = t_2 + (t_3 - t_2) = t_2 + (t_2 - t_1) = a + d + d = a + 2d$$

and, in general, we can prove by induction that $t_n = t_{n-1} + (t_n - t_{n-1}) = a + (n-2)d + d = a + (n-1)d$ for any positive integer n.

Conversely, suppose that there are real numbers a and d such that $t_n = a + (n-1)d$ for all positive integers n. Then

$$t_{n+1} - t_n = a + (n+1-1)d - (a + (n-1)d) = d$$

for all $n \in \mathbb{Z}^+$ and therefore $\{t_n\}$ is an arithmetic progression. The uniqueness of a and d follows from the facts that $t_1 = a$ and $t_2 - t_1 = d$. ■

QUICK LOOK 1

1. In any arithmetic progression $\{t_n\}$ the first term t_1 and the common difference $t_{n+1} - t_n$ determine all the terms and, therefore, the first and second terms ($t_1 = a$ and $t_2 - t_1 = d$) of an arithmetic progression determine the whole sequence. Also, by examining the first three terms (in fact any three consecutive terms) we can get a clue that the given sequence is or is not an arithmetic progression.
2. A sequence $\{t_n\}$ is an arithmetic progression if and only if twice of any term is equal to the sum of its proceeding term and succeeding term.
3. Any arithmetic progression must be of the form $a, a+d, a+2d, a+3d, \ldots$ where a is the first term and d is the common difference. This is called the *general form of an arithmetic progression*.
4. The nth term of an arithmetic progression is

$$t_n = a + (n-1)d$$

where a is the first term t_1 and d is the common difference $t_{n+1} - t_n (= t_2 - t_1)$.

Examples

(1) The sequence $\{n\}$ is an arithmetic progression. Here the first term and the common difference are both equal to 1.

(2) Any constant sequence is an arithmetic progression the common difference being zero.

QUICK LOOK 2

1. If $\{a_n\}$ is an arithmetic progression, then for any real number k, $\{a_n + k\}$ is also an arithmetic progression with the same common difference as $\{a_n\}$, since $(a_{n+1} + k) - (a_n + k) = a_{n+1} - a_n$, for any $n \in \mathbb{Z}^+$.
2. If $\{a_n\}$ is an arithmetic progression with common difference d and k is any real number, then $\{ka_n\}$ is also an arithmetic progression whose common difference is kd.
3. If $\{a_n\}$ and $\{b_n\}$ are arithmetic progressions, then $\{a_n + b_n\}$ is also an arithmetic progression; however $\{a_n b_n\}$ is not so, in general. In this direction, we have the following.

THEOREM 5.3 Let $\{a_n\}$ and $\{b_n\}$ be arithmetic progressions. Then $\{a_n b_n\}$ is an arithmetic progression if and only if either $\{a_n\}$ or $\{b_n\}$ is a constant sequence.

PROOF If $\{a_n\}$ or $\{b_n\}$ is constant, then by point 2 of Quick look 2, $\{a_n b_n\}$ is an arithmetic progression. To prove the converse, let d and e be the common differences of $\{a_n\}$ and $\{b_n\}$, respectively. Then

$$\begin{aligned}
\{a_n b_n\} \text{ is an AP} &\Rightarrow a_{n+1}b_{n+1} - a_n b_n = a_n b_n - a_{n-1}b_{n-1} \\
&\Rightarrow (a_{n+1} - a_n)b_{n+1} + (b_{n+1} - b_n)a_n = (a_n - a_{n-1})b_n + (b_n - b_{n-1})a_{n-1} \\
&\Rightarrow db_{n+1} + e\,a_n = db_n + e\,a_{n-1} \\
&\Rightarrow d(b_{n+1} - b_n) = -e(a_n - a_{n-1}) \\
&\Rightarrow de = -de
\end{aligned}$$

$$\Rightarrow 2de = 0$$
$$\Rightarrow d = 0 \quad \text{or} \quad e = 0$$
$$\Rightarrow \{a_n\} \text{ is constant or } \{b_n\} \text{ is constant}$$

DEFINITION 5.9 Let $a_1, a_2, \ldots, a_n$ be given numbers. Then $a_1, a_2, \ldots, a_n$ are said to be in *arithmetic progression* if these are, in this order, consecutive terms of an arithmetic progression.

THEOREM 5.4 $a_1, a_2, \ldots, a_n$ are in arithmetic progression (where $n > 2$) if and only if $2a_r = a_{r+k} + a_{r-k}$ for all r and k such that $1 \le r - k < r < r + k \le n$.

PROOF If $2a_r = a_{r+k} + a_{r-k}$, then

$$a_{r+1} - a_r = a_r - a_{r-1} \quad \text{for all } 1 < r < n$$

and hence $a_r = a_1 + (r-1)(a_2 - a_1)$ for all $1 \le r \le n$. Therefore $a_1, a_2, \ldots, a_n$ are in arithmetic progression.

Try it out The converse is clear. It is left for the reader as an exercise.

QUICK LOOK 3

1. Any two real numbers a_1, a_2 are in arithmetic progression whose common difference is $a_2 - a_1$.
2. a_1, a_2, a_3 are in AP if and only if $2a_2 = a_1 + a_3$.
3. Three numbers in AP can be taken as $a - d, a, a + d$ for some a and d.
4. Four numbers in AP can be taken as $a - 3d, a - d, a + d, a + 3d$.
5. Five numbers in AP can be taken as $a - 2d, a - d, a, a + d, a + 2d$.
6. In general, $(2r + 1)$ numbers in AP can be taken as $a - rd, a - (r-1)d, \ldots, a, a + r, \ldots, a + rd$.
7. In general $2r$ numbers $(r \in \mathbb{Z}^+)$ in AP can be taken as $a - (2r-1)d, a - (2r-3)d, \ldots, a - d, a + d, a + 3d, \ldots, a + (2r-1)d$.
8. A sequence $\{a_n\}$ is an AP if and only if the nth term a_n is a linear expression in n.

THEOREM 5.5 The sum of the first n terms of an arithmetic progression is given by

$$s_n = n\left[a_1 + \frac{n-1}{2}d\right]$$

where a_1 is the first term and d is the common difference.

PROOF Let $\{a_n\}$ be an arithmetic progression and d the common difference. Then

$$a_n = a_1 + (n-1)d \quad \text{for all } n \in \mathbb{Z}^+$$

Let s_n be the sum of the first n terms in $\{a_n\}$. Then

$$\begin{aligned} s_n &= a_1 + a_2 + \cdots + a_n \\ &= \sum_{r=1}^{n} a_i = \sum_{r=1}^{n} (a_1 + (r-1)d) \\ &= n a_1 + \left(\sum_{r=1}^{n} (r-1)\right) d \end{aligned}$$

$$= na_1 + [0 + 1 + 2 + \cdots + (n-1)]d$$

$$= na_1 + \frac{n(n-1)}{2}d$$

$$= n\left(a_1 + \frac{n-1}{2}d\right)$$

The sum of the first n terms of an AP is given by

$$n\left[a_1 + \frac{n-1}{2}d\right] = \left(\frac{d}{2}\right)n^2 + \left(a_1 - \frac{d}{2}\right)n$$

which is a quadratic expression in n, with constant term zero. ■

Converse of Theorem 5.5 is proved in the following theorem.

THEOREM 5.6 A sequence is an arithmetic progression if and only if the sum of the first n terms is a quadratic expression in n with the constant term zero.

PROOF Let $\{a_n\}$ be a sequence and $s_n = a_1 + a_2 + \cdots + a_n$, for any $n \in \mathbb{Z}^+$. Then the nth term is given by

$$a_n = s_n - s_{n-1}$$

Now, suppose that s_n is a quadratic expression in n with constant term zero, that is,

$$s_n = an^2 + bn$$

where a and b are real numbers. Then

$$\begin{aligned} a_n &= s_n - s_{n-1} \\ &= an^2 + bn - [a(n-1)^2 + b(n-1)] \\ &= a[n^2 - (n-1)^2] + b[n - (n-1)] \\ &= (2n-1)a + b \end{aligned}$$

Therefore, the nth term is $a_n = (2n-1)a + b$ and so, for any $n > 1$,

$$\begin{aligned} a_n - a_{n-1} &= (2n-1)a + b - [(2(n-1)-1)a + b] \\ &= [2n - 1 - (2n-3)]a = 2a \end{aligned}$$

This shows that $a_n - a_{n-1}$ is a constant for all n and hence $\{a_n\}$ is an arithmetic progression with a common difference $2a$ and first term $a + b$. ■

QUICK LOOK 4

1. In the above, if $s_n = an^2 + bn + c$ with $c \neq 0$, then $\{a_n\}$ is not an arithmetic progression. In this case, it can be observed that

$$a_n - a_{n-1} = 2a \quad \text{for } n > 2$$

and

$$\begin{aligned} a_2 - a_1 &= (s_2 - s_1) - a_1 = s_2 - 2s_1 \\ &= (4a + 2b + c) - 2(a + b + c) \\ &= 2a - c \neq 2a \quad (\text{since } c \neq 0) \end{aligned}$$

Therefore $\{a_n\}$ is not an arithmetic progression. However, $a_2, a_3, a_4, a_5, \ldots$ are in AP, with common difference $2a$. That is, excluding a_1, $\{a_n\}$ is an AP.

2. If the sum of the first n terms of a sequence is $an^2 + bn$ for all $n \in \mathbb{Z}^+$, then the sequence is an AP whose first term is $a + b$ and the common difference is $2a$.

Example 5.1

If the first, second and nth term of an arithmetic progression are a, b and c respectively, then find the sum of the first n terms of the sequence.

Solution: The given sequence can be written as

$$a, b, b_3, b_4, \ldots, b_{n-1}, c, b_{n+1}, \ldots$$

The common difference must be $b - a$. Also, c which is the nth term equals

$$a + (n-1)d = a + (n-1)(b-a)$$

Therefore

$$n = \frac{c-a}{b-a} + 1 = \frac{c+b-2a}{b-a}$$

The sum of the first n terms is given by

$$n\left(a + \frac{n-1}{2}d\right) = n\left(a + \frac{c-a}{2(b-a)}(b-a)\right)$$

$$= \frac{c+b-2a}{b-a}\left(a + \frac{c-a}{2}\right)$$

$$= \frac{(c+b-2a)(a+c)}{2(b-a)}$$

Arithmetic Mean

DEFINITION 5.10 If three numbers a, b, c are in arithmetic progression, then b is called the *arithmetic mean* (AM) between a and c. In general, if $a, b_1, b_2, \ldots, b_n, c$ are in arithmetic progression, then $b_1, b_2, \ldots, b_n$ are called *n arithmetic means* (n AMs) between a and c.

THEOREM 5.7 If $A_1, A_2, \ldots, A_n$ are n arithmetic means between a and c, then

$$A_k = a + \frac{k(c-a)}{n+1} \quad \text{for } 1 \le k \le n$$

PROOF Let $a, A_1, A_2, \ldots, A_n, c$ be in arithmetic progression and d be the common difference. Then

$$A_1 = a + d, \quad A_2 = a + 2d, \quad \ldots, \quad A_k = a + kd, \quad A_n = a + nd$$

and

$$c = a + (n+1)d$$

Therefore,

$$d = \frac{c-a}{n+1} \quad \text{and} \quad A_k = a + \frac{c-a}{n+1}k$$

■

QUICK LOOK 5

1. If b is the arithmetic mean between a and c then

$$b = \frac{a+c}{2}$$

2. For any real numbers a and b,

$$a, \frac{a+b}{2}, b$$

are in arithmetic progression.

3. If $A_1, A_2, \ldots, A_n$ are n arithmetic means between a and b, then

$$A_1 + A_2 + \cdots + A_n = n\left(\frac{a+b}{2}\right)$$

That is, the sum of n arithmetic means between two given real numbers a and b is equal to n times of the AM of a and b.

4. If a is the first term and b is the nth term in an AP, then the sum of the first n terms is equal to

$$\frac{n}{2}(a+b)$$

THEOREM 5.8 If T_n and T_n' are the nth terms of two arithmetic progression and s_n and s_n' are their sums of the first n terms, respectively, then

$$\frac{T_n}{T_n'} = \frac{s_{2n-1}}{s_{2n-1}'}$$

PROOF Let the two AP's be

$$a, a+d, a+2d, \ldots \quad \text{and} \quad b, b+e, b+2e, \ldots$$

Then, by Theorem 5.4, we have

$$\frac{s_{2n-1}}{s_{2n-1}'} = \frac{(2n-1)[a+\{(2n-2)/2\}d]}{(2n-1)[b+\{(2n-2)/2\}e]}$$

$$= \frac{a+(n-1)d}{b+(n-1)e} = \frac{T_n}{T_n'}$$

Example 5.2

The nth terms of two AP's $\{a_n\}$ and $\{b_n\}$ are 10 and 15, respectively. If sum of the first n terms of $\{a_n\}$ is $30n$, then find the sum of the first 21 terms of $\{b_n\}$.

Solution: If s_n and t_n are the sums of the first n terms of $\{a_n\}$ and $\{b_n\}$, respectively, then by Theorem 5.7,

$$\frac{30(2n-1)}{t_{2n-1}} = \frac{s_{2n-1}}{t_{2n-1}} = \frac{a_n}{b_n} = \frac{10}{15}$$

and hence

$$t_{2n-1} = \frac{30(2n-1)\times 15}{10} = 45(2n-1)$$

Thus, the sum of the first 21 terms in $\{b_n\}$ is given by

$$t_{21} = t_{2\times 11-1} = 45(21) = 945$$

Example 5.3

The 22nd term and 46th term of an AP are 36 and 72, respectively. Find the general term of the AP.

Solution: The 22nd term is 36. This means

$$a + 21d = 36$$

The 46th term is 72 which implies

$$a + 45d = 72$$

Here, a is the first term and d is the common difference of the AP.

Then, solving the above two equations in two variables we get

$$24d = 36 \Rightarrow d = \frac{3}{2}$$

Substituting this value of d in any one of the above equations gives

$$a = \frac{9}{2}$$

Therefore, the nth term is given by

$$a_n = \frac{9}{2} + \frac{(n-1)3}{2}$$

Example 5.4

The sum of four integers in AP is 24 and their product is 945. Find these integers.

Solution: Let the four integers be $a-3d$, $a-d$, $a+d$, and $a+3d$. Then, it is given that

$$(a-3d)+(a-d)+(a+d)+(a+3d) = 24$$

and $$(a-3d)(a-d)(a+d)(a+3d) = 945$$

Therefore, from the first equation we get

$$4a = 24 \quad \text{or} \quad a = 6$$

Also from the second equation we have

$$(a^2 - d^2)(a^2 - 9d^2) = 945$$
$$(36 - d^2)(36 - 9d^2) = 945$$
$$9d^4 - 360d^2 + (36 \times 36) = 945$$
$$d^4 - 40d^2 + 144 = 105$$
$$d^4 - 40d^2 + 39 = 0$$
$$(d^2 - 1)(d^2 - 39) = 0$$

Since the terms of the given AP are integers, so is d. Therefore, $d^2 \neq 39$. This gives $d^2 = 1$ or $d = \pm 1$. Hence, the given integers are

$$3, 5, 7, 9 \quad \text{or} \quad 9, 7, 5, 3$$

5.3 | Geometric Progressions

A sequence in which the ratio of any term, and its immediate predecessor term is constant is called a ***geometric progression***. In this section we will discuss various properties of geometric progressions.

DEFINITION 5.11 A sequence $\{a_n\}$ of non-zero real numbers is called a ***geometric progression*** (GP) if

$$\frac{a_n}{a_{n-1}} = \frac{a_{n+1}}{a_n} \quad \text{for all } n > 1$$

DEFINITION 5.12 Let $\{a_n\}$ be a geometric progression and r be the constant a_{n+1}/a_n. Then r is called the ***common ratio***.

Examples

(1) The sequence $\{1, 2, 2^2, 2^3, \ldots\}$ is a geometric progression with common ratio 2.

(2) $\{3, -3/2, 3/4, -3/8, 3/16, -3/32, \ldots\}$ is a geometric progression with common ratio $-1/2$.

(3) The common ratio of a geometric progression is 1 if and only if it is a constant sequence.

(4) $\{3, -3, 3, -3, 3, -3, \ldots\}$ is a geometric progression with common ratio -1.

DEFINITION 5.13 Non-zero real numbers $t_1, t_2, \ldots, t_m$ are said to be in ***geometric progression*** (GP) if these are consecutive terms of a geometric progression.

QUICK LOOK 6

1. Any geometric progression with first term a and common ratio r can be expressed as
$$a, ar, ar^2, \ldots, ar^n, \ldots$$
This is known as the general form of a GP.
2. Three non-zero real numbers a, b and c are in GP if and only if $b^2 = ac$.
3. In general, non-zero real numbers $a_1, a_2, \ldots, a_n$ are in GP if and only if
$$a_i = \left(\frac{a_2}{a_1}\right)^{i-1} a_1 \quad \text{for all } 1 \le i \le n$$
4. The nth term of a GP with first term a and common ratio r is given by
$$t_n = r^{n-1}a, \quad \text{for any } n \in \mathbb{Z}^+$$

THEOREM 5.9 Let $a_1, a_2, \ldots, a_n$ be in GP with common ratio r.

1. For any non-zero constant λ, $\lambda a_1, \lambda a_2, \ldots, \lambda a_n$ are in GP with common ratio r.
2. For any real number $b > 1$, $\log_b a_1, \log_b a_2, \ldots, \log_b a_n$ are in AP with common difference $\log_b r$, provided $a_i > 0$ for $1 \le i \le n$.

PROOF

1. The first part is clear, since

$$\frac{\lambda a_n}{\lambda a_{n-1}} = \frac{a_n}{a_{n-1}} = r \quad \text{for all } n > 1$$

2. The second part follows from the fact that

$$\log_b a_n - \log_b a_{n-1} = \log_b \left(\frac{a_n}{a_{n-1}} \right) = \log_b r \quad \text{for all } n > 1$$

■

THEOREM 5.10 The sum of the first n terms of the GP with first term a and common ratio $r \neq 1$ is given by

$$s_n = \frac{a(1-r^n)}{1-r}$$

PROOF The GP with first term a and common ratio r can be expressed as

$$a, ar, ar^2, \ldots$$

If s_n is the sum of the first n terms, then

$$s_n = a + ar + \cdots + ar^{n-1}$$
$$s_n(1-r) = a - ar^n = a(1-r^n)$$

and therefore,

$$s_n = \frac{a(1-r^n)}{1-r} \quad \text{if } r \neq 1$$

■

QUICK LOOK 7

If the common ratio of a GP is 1, then the sum of the first n terms is na, where a is the first term.

DEFINITION 5.14 Let

$$s_n = \frac{a(1-r^n)}{1-r}$$

be the sum of the first n terms of a GP with first term a and common ratio r. If $|r| < 1$, then

$$\lim_{n\to\infty} s_n = \frac{a}{1-r}$$

is called the *sum to infinity* of the GP and this will be generally denoted by s_∞.

Example 5.5

Consider the sequence

$$\frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \frac{1}{256}, \ldots$$

Calculate the sum of first n terms and the sum to infinity.

Solution: The nth term of the sequence is $(1/4)^n$. The sequence is a GP with first term 1/4 and common ratio 1/4. Therefore,

$$s_n = \frac{1-(1/4)^n}{1-(1/4)} = \frac{4^n - 1}{3 \cdot 4^{n-1}} \quad \text{and} \quad s_\infty = \frac{1}{1-(1/4)} = \frac{4}{3}$$

Geometric Mean

DEFINITION 5.15 If three numbers a, b and c are in GP, then b is called the ***geometric mean*** (GM) of a and c or ***geometric mean*** between a and c.

QUICK LOOK 8

1. A number b is the GM between a and c if and only if
$$\frac{b}{a} = \frac{c}{b}$$
or, equivalently,
$$b^2 = ac$$
2. If x and y are any non-negative real numbers, then $x, \sqrt{xy}, y$ are in GP.

DEFINITION 5.16 If $a, g_1, g_2, \ldots, g_n, b$ are in GP, then $g_1, g_2, \ldots, g_n$ are called n ***geometric means*** or, simply, n GMs between a and b.

In the following we discuss the insertion of n GM's between two given non-zero real numbers, where n is a given positive integer.

THEOREM 5.11 Let a and b be two given non-zero real numbers and n a positive integer. If
$$g_k = a\left(\frac{b}{a}\right)^{k/(n+1)} \quad \text{for } 1 \le k \le n$$
then $g_1, g_2, \ldots, g_n$ are n geometric means between a and b.

PROOF Let
$$g_k = a\left(\frac{b}{a}\right)^{k/(n+1)} \quad \text{for } 1 \le k \le n$$
and consider $a, g_1, g_2, \ldots, g_n, b$. Then
$$\frac{g_1}{a} = \left(\frac{b}{a}\right)^{1/(n+1)}$$
$$\frac{g_2}{g_1} = \frac{a(b/a)^{2/(n+1)}}{a(b/a)^{1/(n+1)}} = \left(\frac{b}{a}\right)^{1/(n+1)}$$
$$\frac{g_k}{g_{k-1}} = \frac{a(b/a)^{k/(n+1)}}{a(b/a)^{(k-1)/(n+1)}} = \left(\frac{b}{a}\right)^{1/(n+1)} = \frac{b}{g_n}$$
Therefore $a, g_1, g_2, \ldots, g_n, b$ are in GP with common ratio $(b/a)^{1(n+1)}$ and hence $g_1, g_2, \ldots, g_n$ are the n GMs between a and b. ■

QUICK LOOK 9

If $g_1, g_2, \ldots, g_n$ are n geometric means between a and b, then their product is given by
$$g_1, g_2, \ldots, g_n = (\sqrt{ab})^n$$
since
$$\prod_{k=1}^{n} g_k = \prod_{k=1}^{n} a\left(\frac{b}{a}\right)^{k/(n+1)}$$
$$= a^n\left(\frac{b}{a}\right)^{(1+2+\cdots+n)/(n+1)}$$
$$= a^n\left(\frac{b}{a}\right)^{n/2} = (\sqrt{ab})^n$$

Example 5.6

Insert 8 geometric means between 1 and 16.

Solution: Let $1, g_1, g_2, \ldots, g_8, 16$ be in GP. Then

$$g_k = 1\left(\frac{16}{1}\right)^{k/(8+1)} \quad \text{for } 1 \le k \le 8$$

and hence $g_k = (16)^{k/9}$. Therefore,

$$(16)^{1/9},\ (16)^{2/9}, \ldots, (16)^{8/9}$$

are the 8 GMs between 1 and 16.

Arithmetic Geometric Progression

DEFINITION 5.17 A sequence of the form

$$a, (a+d)r, (a+2d)r^2, (a+3d)r^3, \ldots$$

is called ***arithmetic geometric progression*** (*AGP*) the nth term in AGP is $[a+(n-1)d]r^{n-1}$, where d and r are non-zero real numbers.

THEOREM 5.12 The sum of the first n terms of an AGP is given by

$$s_n = \frac{a}{1-r} + \frac{dr(1-r^{n-1})}{(1-r)^2} - \frac{(a+(n-1)d)r^n}{1-r}$$

If $|r| < 1$, the sum to infinity is

$$s_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$$

PROOF Let $a, (a+d)r, (a+2d)r^2, \ldots$ be an AGP and s_n be the sum of first n-term; that is,

$$s_n = a + (a+d)r + (a+2d)r^2 + \cdots + [a+(n-1)d]r^{n-1}$$

Then

$$rs_n = ar + (a+d)r^2 + (a+2d)r^3 + \cdots + [a+(n-1)d]r^n$$

Now using the two equation we get

$$(1-r)s_n = s_n - rs_n = a + \frac{dr(1-r^{n-1})}{1-r} - [a+(n-1)d]r^n$$

$$s_n = \frac{a}{1-r} + \frac{dr(1-r^{n-1})}{(1-r)^2} - \frac{[a+(n-1)d]r^n}{1-r}$$

If $|r| < 1$, then

$$s_\infty = \lim_{n\to\infty} s_n = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$$

■

Example 5.7

Find the sum to infinity of the series

$$1 + \frac{4}{5} + \frac{7}{5^2} + \frac{10}{5^3} + \cdots$$

Solution: The given series is of the form

$$a, (a+d)r, (a+2d)r^2, \ldots$$

which is an arithmetic geometric progression with $a = 1$, $d = 3$ and $r = 1/5$. Since $|r| < 1$, the sum to infinity of the AGP is given by

$$\frac{a}{1-r} + \frac{dr}{(1-r)^2} = \frac{1}{1-(1/5)} + \frac{3(1/5)}{[1-(1/5)]^2} = \frac{5}{4} + \frac{15}{16} = \frac{35}{16}$$

5.4 | Harmonic Progressions and Series

In this section we consider sequences in which the reciprocals of the terms form an arithmetic progression. Such sequences are called ***harmonic progressions***. Let us begin with the following.

DEFINITION 5.18 A sequence of non-zero real numbers is said to be a ***harmonic progression*** (*HP*) if their reciprocals form an arithmetic progression (AP). That is, $\{a_1, a_2, \ldots\}$ is said to be a HP if

(i) $0 \neq a_n \in \mathbb{R}$ for all n

(ii) $\{1/a_1, 1/a_2, \ldots\}$ is an AP

DEFINITION 5.19 Non-zero real numbers $a_1, a_2, \ldots, a_n$ are said to be in *HP* if they are consecutive terms of a HP, that is,

$$\frac{1}{a_i} - \frac{1}{a_{i-1}} = \frac{1}{a_{i+1}} - \frac{1}{a_i} \quad \text{for all } 1 < i < n$$

QUICK LOOK 10

The general form of an HP is

$$\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \ldots$$

where the nth term is

$$\frac{1}{a+(n-1)d}$$

Examples

(1) $\{1, 1/2, 1/3, 1/4, \ldots\}$ is an HP.

(2) For any $0 < a \in \mathbb{R}$ and $0 \neq d \in \mathbb{R}$,

$$\left\{\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \ldots\right\}$$

is an HP, if $a \neq -nd$ for all $n \in \mathbb{Z}^+$.

(3) The numbers 1/3, 1/7, 1/11 are in HP.

Harmonic Mean

DEFINITION 5.20 If a, b, c are in HP, then b is called the *harmonic mean* (*HM*) between a and c. Note that b is the HM between a and c if and only if

$$b = \frac{2ac}{a+c} \quad \left(\text{i.e., } \frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b}\right)$$

Examples

(1) Note that 1/5 is the HM of 1/3 and 1/7.

(2) If y is the AM of x and z, then $1/y$ is the HM of $1/x$ and $1/z$.

DEFINITION 5.21 $h_1, h_2, \ldots, h_n$ are said to be n *harmonic means* between two given real numbers a and b if $a, h_1, h_2, \ldots, h_n, b$ are in HP.

THEOREM 5.13 If $h_1, h_2, \ldots, h_n$ are n HMs between two non-zero real numbers a and b, then

$$h_K = \frac{ab(n+1)}{b(n+1) + K(a-b)} \quad \text{for any } 1 \leq K \leq n$$

Proof Suppose that $a, h_1, h_2, \ldots, h_n, b$ are in HP. Then

$$\frac{1}{a}, \frac{1}{h_1}, \frac{1}{h_2}, \ldots, \frac{1}{h_n}, \frac{1}{b}$$

are in AP. If d is the common difference of this AP, then

$$\frac{1}{h_1} = \frac{1}{a} + d, \frac{1}{h_2} = \frac{1}{a} + 2d, \ldots, h_n = \frac{1}{a} + nd$$

and

$$\frac{1}{b} = \frac{1}{a} + (n+1)d$$

and so

$$d = \frac{1}{n+1}\left[\frac{1}{b} - \frac{1}{a}\right] = \frac{a-b}{(n+1)ab}$$

Therefore

$$\frac{1}{h_K} = \frac{1}{a} + K\left(\frac{a-b}{(n+1)ab}\right) = \frac{b(n+1) + K(a-b)}{(n+1)ab}$$

$$h_K = \frac{(n+1)ab}{b(n+1) + K(a-b)}$$

■

Theorem 5.14 If A, G and H are the arithmetic, geometric and harmonic means, respectively, between two positive real numbers a and b, then

$$AH = G^2$$

that is, A, G, H are in GP or G is the GM of A and H.

Proof Since A, G and H are the arithmetic, geometric and harmonic means, respectively, we have

$$A = \frac{a+b}{2}, G = \sqrt{ab} \quad \text{and} \quad H = \frac{2ab}{a+b}$$

Therefore,

$$AH = \frac{a+b}{2} \cdot \frac{2ab}{a+b} = ab = G^2$$

■

Note: In Theorem 5.14, one has to take a and b to be non-zero, but in this theorem a and b must be positive. Also, note that $A \geq G \geq H$ and that the equality holds at the two places if and only if $a = b$. These are proved in the more general cases later (see Theorem 5.15).

Some inequality problems and maxima and minima problems can be solved by using the inequalities $A \geq G \geq H$, where A, G and H are arithmetic, geometric and harmonic means, respectively, of two positive real numbers. Let us begin with the following.

DEFINITION 5.22 Let $a_1, a_2, \ldots, a_n$ be positive real numbers $(n \geq 2)$. Then

(i) $(a_1 + a_2 + \cdots + a_n)/n$ is called the arithmetic mean (AM)
(ii) $(a_1 a_2 \cdots a_n)^{1/n}$ is called the geometric mean (GM)
(iii) $n/[(1/a_1) + (1/a_2) + \cdots + (1/a_n)]$ is called the harmonic mean (HM)

Theorem 5.15 If $a_1, a_2, \ldots, a_n$ $(n \geq 2)$ are positive real numbers and A and G be their AM and GM, respectively, then $A \geq G$ and the equality holds if and only if $a_i = a_j$ for all $1 \leq i, j \leq n$.

PROOF We will use mathematical induction on n. If $a_1, a_2, \ldots, a_n$ are all equal to each other, then clearly $A = G = a_1$. Suppose that, not all the $a_i s$ are equal. For $n = 2$

$$A = \frac{a_1 + a_2}{2} \quad \text{and} \quad G = \sqrt{a_1 a_2}$$

so that

$$A - G = \frac{(\sqrt{a_1} - \sqrt{a_2})^2}{2} > 0$$

and hence $A > G$. Now, consider the case $n = 3$. Let a_1, a_2, a_3 be positive real numbers and let

$$x = (a_1)^{1/3}, \quad y = (a_2)^{1/3} \quad \text{and} \quad z = (a_3)^{1/3}$$

Then x, y and z are positive and

$$\begin{aligned}
A - G &= \frac{a_1 + a_2 + a_3}{3} - (a_1 a_2 a_3)^{1/3} \\
&= \frac{1}{3}(x^3 + y^3 + z^3 - 3xyz) \\
&= \frac{1}{3}(x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx) \\
&= \frac{1}{6}(x + y + z)((x - y)^2 + (y - z)^2 + (z - x)^2) \geq 0
\end{aligned}$$

Hence $A \geq G$. Also, $A = G$ if and only if $x = y = z$ and hence $a_1 = a_2 = a_3$. Therefore the theorem is valid for $n = 2$ and $n = 3$.

Now, let $n > 3$ and assume that the theorem is valid for any $n - 1$ positive real numbers. Let $a_1, a_2, \ldots, a_n$ be any positive real numbers that are not all equal. We can suppose that $a_1 \geq a_2 \geq \cdots \geq a_n$ and $a_1 > a_n$. Let

$$A = \frac{a_1 + a_2 + \cdots + a_n}{n} \quad \text{and} \quad G = (a_1 a_2 \cdots a_n)^{1/n}$$

Consider $a_2, a_3, \ldots, a_{n-1}, a_1 a_n / G$ (which are $n - 1$ in number). By the induction hypothesis,

$$\frac{1}{n-1}\left(a_2 + a_3 + \cdots + a_{n-1} + \frac{a_1 a_n}{G}\right) > \left(a_2 a_3 \cdots a_{n-1} \frac{a_1 a_n}{G}\right)^{1/(n-1)}$$

$$a_2 + a_3 + \cdots + a_{n-1} + \frac{a_1 a_n}{G} > (n-1)\left(a_2 a_3 \cdots a_{n-1} \frac{a_1 a_n}{G}\right) = (n-1)G$$

Therefore

$$\begin{aligned}
nG &< G + a_2 + a_3 + \cdots + a_{n-1} + \frac{a_1 a_n}{G} \\
&= G + nA - (a_1 + a_n) + \frac{a_1 a_n}{G} \\
&= nA + \frac{G^2 - (a_1 + a_n)G + a_1 a_n}{G} \\
&= nA + \frac{(G - a_1)(G - a_n)}{G} \\
&< nA \quad (\text{since } a_1 > G > a_n)
\end{aligned}$$

Therefore $G < A$. ■

Note: If any two of the $a_i s$ are not equal, say $a_1 \neq a_2$, then we can write

$$\frac{a_1 + a_2 + \cdots + a_n}{n} = \frac{[(a_1 + a_2)/2] + [(a_1 + a_2)/2] + a_3 + a_4 + \cdots + a_n}{n}$$

$$\geq \left(\left(\frac{a_1 + a_2}{2}\right)^2 a_3 a_4 \cdots a_n\right)^{1/n}$$

$$> (a_1 a_2 a_3 a_4 \cdots a_n)^{1/n}$$

COROLLARY 5.1 If $a_1, a_2, \ldots, a_n$ are positive real numbers such that their sum is a fixed positive real numbers, then their product is greatest when each of

$$a_i = \frac{s}{n} \quad (i = 1, 2, \ldots, n)$$

PROOF By Theorem 5.15,

$$a_1 + a_2 + \cdots + a_n \geq n(a_1 a_2 \ldots a_n)^{1/n}$$

where equality holds if

$$a_1 = a_2 = a_3 = \cdots = a_n = \frac{s}{n}$$

Therefore greatest value of $a_1 a_2 a_3 \cdots a_n$ is $(s/n)^n$. ■

COROLLARY 5.2 If $a_1, a_2, \ldots, a_n$ are positive real numbers such that their product is a fixed positive real number P, then their sum is least when each of $a_1, a_2, \ldots, a_n$ is equal to $P^{1/n}$.

PROOF The following equality

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq (a_1 a_2 \cdots a_n)^{1/n} = P^{1/n}$$

holds when each $a_i = P^{1/n}$. Therefore, the least value of $a_1 + a_2 + \cdots + a_n$ is $nP^{1/n}$. ■

The following formulae and the methods of their derivation will help us in finding the sum to n terms of certain series.

Example 5.8

If a, b, c are in HP, then show that $a : a - b = a + c : a - c$.

Solution: If a, b, c are in HP then $1/a, 1/b, 1/c$ are in AP. Therefore

$$\frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b}$$

$$\frac{a-b}{ab} = \frac{b-c}{bc}$$

$$\frac{a-b}{a} = \frac{b-c}{c} = \frac{(a-b)+(b-c)}{a+c} = \frac{a-c}{a+c}$$

From the first and the last fractions we get

$$a : a - b = a + c : a - c$$

Example 5.9

Find the harmonic mean of the roots of the quadratic equation

$$(5+\sqrt{2})x^2 - (4+\sqrt{5})x + 2(4+\sqrt{5}) = 0$$

Solution: Let α and β be the roots of the given equation. Then

$$\alpha + \beta = \frac{4+\sqrt{5}}{5+\sqrt{2}} \quad \text{and} \quad \alpha\beta = \frac{2(4+\sqrt{5})}{5+\sqrt{2}}$$

Therefore, the harmonic mean of α and β is

$$\frac{2\alpha\beta}{\alpha+\beta} = 4$$

5.5 | Some Useful Formulae

I. Telescopic Series: Suppose that we have to find the sum to n terms of a series $u_1 + u_2 + u_3 + \cdots$. If $a_1 + a_2 + a_3 + \cdots$ is another series such that

$$u_K = a_K - a_{K+1} \quad \text{for all } K$$

then
$$u_1 + u_2 + \cdots + u_n = a_1 - a_{n+1}$$

For example, consider $\dfrac{1}{2\cdot 3} + \dfrac{1}{3\cdot 4} + \dfrac{1}{4\cdot 5} + \cdots$. Here, we have

$$\frac{1}{K(K+1)} = \frac{1}{K} - \frac{1}{K+1} \quad \text{for all } K \geq 2$$

II. Suppose that the nth term u_n of a given series is the product of r successive terms of an AP beginning with the nth term of the AP; that is, suppose that

$$u_n = [a + (n-1)d](a + nd)\cdots[a + (n + r - 2)d]$$

By choosing $a_n = u_n[a + (n + r - 1)d]$, we can write

$$u_n = \frac{1}{(r+1)d}[a_n - a_{n-1}]$$

so that the sum to n terms is equal to

$$\frac{1}{(r+1)d}(a_n - a_0)$$

For example, consider

(i) $1\cdot 2\cdot 3 + 2\cdot 3\cdot 4 + 3\cdot 4\cdot 5 + \cdots$

(ii) $1\cdot 3\cdot 5\cdot 7 + 3\cdot 5\cdot 7\cdot 9 + 5\cdot 7\cdot 9\cdot 11 + \cdots$

III. Suppose that the nth term of a series is the reciprocal of the nth term of the series given in II; that is,

$$u_n = \frac{1}{[a + (n-1)d][a + nd]\cdots[a + (n + r - 2)d]}$$

Then, we can choose

$$a_n = u_n[a + (n-1)d]$$

so that

$$u_n = \frac{1}{(r-1)d}(a_{n-1} - a_n)$$

and sum to n terms is given by

$$\frac{1}{(r-1)d}(a_0 - a_n)$$

For example, consider

(i) $\dfrac{1}{1\cdot 4\cdot 7} + \dfrac{1}{4\cdot 7\cdot 10} + \dfrac{1}{7\cdot 10\cdot 13} + \cdots$

(ii) $\dfrac{1}{1\cdot 3\cdot 5\cdot 7} + \dfrac{1}{3\cdot 5\cdot 7\cdot 9} + \dfrac{1}{5\cdot 7\cdot 9\cdot 11} + \cdots$

IV. Successive Differences Method: Suppose that in a given series, we cannot express the nth term by using induction. But when we take the successive differences of the series, ultimately we may arrive at an AP or a GP. Then we can find the nth term by the following method.

Let the given series be $u_1 + u_2 + u_3 + \cdots$. Let

$$\Delta u_n = \underbrace{(u_2 - u_1)}_{v_1} + \underbrace{(u_3 - u_2)}_{v_2} + \underbrace{(u_4 - u_3)}_{v_1} + \cdots$$

$$\Delta^2 u_n = (v_2 - v_1) + (v_3 - v_1) + (v_4 - v_3) + \cdots$$

(i) Suppose that $\Delta^K u_n$ becomes a series where all the terms are equal. Afterwards $\Delta^{K+1} u_n$ becomes a series in which each term is zero. We stop at $\Delta^K u_n$. Let the first terms of $\Delta u_n, \Delta^2 u_n, \Delta^3 u_n, \ldots, \Delta^K u_n$ be $d_1, d_2, d_3, \ldots, d_n$, respectively. Then

$$u_n = u_1 + \frac{d_1(n-1)}{1!} + \frac{d_2(n-1)(n-2)}{2!} + \frac{d_3(n-1)(n-2)(n-3)}{3!} + \cdots$$

$\Delta^K u_n$ is called the Kth-order differences of the given series.

Example

Let us find the sum to n terms of the series 2, 10, 30, 68, 130, 222, 350, …

$$\Delta u_n = 8, 20, 38, 62, 92, 128, \ldots$$

$$\Delta^2 u_n = 12, 18, 24, 30, 36, \ldots$$

$$\Delta^3 u_n = 6, 6, 6, 6, \ldots$$

$$\Delta^4 u_n = 0, 0, 0, \ldots$$

Now,

$$u_n = 2 + \frac{8(n-1)}{1!} + \frac{12(n-1)(n-2)}{2!} + \frac{6(n-1)(n-2)(n-3)}{3!}$$

$$= 2 + 8n - 8 + 6(n^2 - 3n + 2) + (n^2 - 3n + 2)(n-3)$$

$$= n^3 + n$$

One can check that $n^3 + n$ is the general term by giving values 1, 2, 3, … for n.

(ii) Suppose that $\Delta^{K+1} u_n$ forms a geometric progression. In this case, the nth term u_n is of the form

$$ar^{n-1} + a_0 + a_1(n-1) + a_2(n-1)(n-2) + \cdots$$

where r is the common ratio and the values of $a, a_0, a_1, a_2, \ldots$ can be evaluated by giving values 1, 2, 3, … to n and equating the terms to the corresponding terms of the given series.

Example 5.10

Consider the series $6 + 9 + 14 + 23 + 40 + \cdots$. Find the nth term and sum to n terms.

Solution: We have $\Delta u_n = 3, 5, 9, 17, \ldots$. Therefore

$$\Delta^2 u_n = 2, 4, 8, \ldots$$

which is a GP. Therefore

$$u_n = a2^{n-1} + a_0 + a_1(n-1)$$

Taking $n = 1$, we get

$$6 = u_1 = a + a_0 \tag{5.1}$$

Taking $n = 2$, we get

$$9 = u_1 = 2a + a_0 + a_1 \tag{5.2}$$

Taking $n = 3$, we get

$$14 = u_3 = 4a + a_0 + 2a_1 \tag{5.3}$$

Solving Eqs. (5.1)–(5.3), we get that $a = 2$, $a_0 = 4$ and $a_1 = 1$. Therefore

$$u_n = 2^n + 4 + (n-1) = 2^n + n + 3$$

Hence the sum to n terms is given by

$$\sum_{K=1}^{n} (2^K + K + 3) = (2 + 2^2 + \cdots + 2^n) + \frac{n(n+1)}{2} + 3n$$

$$= 2^{n+1} - 2 + \frac{n(n+1)}{2} + 3n$$

WORKED-OUT PROBLEMS

Single Correct Choice Type Questions

1. Sum of the first 20 terms of the AP

$$2, 3\frac{1}{4}, 4\frac{1}{2}, 5\frac{3}{4}, \ldots$$

is

(A) $274\frac{1}{2}$ (B) $277\frac{1}{2}$ (C) 277 (D) 274

Solution: First term a and the common difference d are given by

$$a = 2 \quad \text{and} \quad d = 1\frac{1}{4}$$

Therefore the sum of the first 20 terms is given by

$$\frac{20}{2}\left(4 + 19 \times \frac{5}{4}\right) = 277\frac{1}{2}$$

Answer: (B)

2. The third term of an AP is 18 and the seventh term is 30, then the sum of the first 17 terms is

(A) 812 (B) 512 (C) 612 (D) 712

Solution: Let "a" be the first term and "d" the common difference. Then

$$a + 2d = 18 \tag{5.4}$$

$$a + 6d = 30 \tag{5.5}$$

Solving Eqs. (5.4) and (5.5), we get $a = 12, d = 3$. Therefore sum of 17 terms is

$$\frac{17}{2}(24 + 16 \times 3) = 17 \times 36 = 612$$

Answer: (C)

3. The nth term of an AP is $4n + 1$. The sum of the first 15 terms is

(A) 495 (B) 555 (C) 395 (D) 695

Solution: The progression is: 5, 9, 13, 17, …. Here the first term is $a = 5$, the common difference is $d = 4$ and the number of terms is $n = 15$. Then the sum of the first 15 terms is

$$\frac{15}{2}(10 + 14 \times 4) = 15 \times 33 = 495$$

Answer: (A)

4. The sum of the first 15 terms of an AP is 600 and the common difference is 5. The first term is equal to

(A) 8 (B) 9 (C) 10 (D) 5

Solution: Using the formula for sum of an AP and substituting the values, we get

$$600 = \frac{15}{2}(2a + 14 \times 5)$$

$$a + 35 = 40$$

$$a = 5$$

Answer: (D)

5. If 17 arithmetic means are inserted between 7/2 and –83/2, then the 17th AM is

(A) –19 (B) –29 (C) –39 (D) –49

Solution: Let d be the common difference. Then

$$d = \frac{(-83/2) - (7/2)}{18} = -\frac{5}{2}$$

Therefore the 17th mean is given by (using Theorem 5.6)

$$\frac{7}{2} + 17\left(-\frac{5}{2}\right) = -39$$

Answer: (C)

6. The number of terms of the series 26, 21, 16, 11, … to be added so as to get the sum 74 is

(A) 5 (B) 4 (C) 3 (D) 6

Solution: In the given series $a = 26$, $d = -5$. Suppose sum of the first n terms is 74. This implies

$$74 = \frac{n}{2}[52 + (n-1)(-5)]$$

$$148 = n(57 - 5n)$$

$$5n^2 - 57n + 148 = 0$$

$$5n^2 - 20n - 37n + 148 = 0$$

$$5n(n-4) - 37(n-4) = 0$$

$$n = 4$$

Answer: (B)

7. The sum of the first n terms of two arithmetic series are in the ratio $(7n+1):(4n+27)$. The ratio of their 11th terms is

(A) 2:3 (B) 3:2 (C) 3:4 (D) 4:3

Solution: It is given that

$$\frac{S_n}{S'_n} = \frac{7n+1}{4n+27}$$

Therefore, using Theorem 5.8 we get

$$\frac{T_n}{T'_n} = \frac{7(2n-1)+1}{4(2n-1)+27} = \frac{14n-6}{8n+23}$$

Substituting $n = 11$ in this equation we get

$$\frac{T_{11}}{T'_{11}} = \frac{148}{111} = \frac{4}{3}$$

Answer: (D)

8. If $S_1, S_2, S_3, \ldots, S_p$ are the sums of first n terms of p arithmetic progressions whose first terms are 1, 2, 3, …, p and whose common differences are $1, 3, 5, 7, \ldots$, then the value of $S_1 + S_2 + S_3 + \cdots + S_p$ is

(A) $np(np+1)/2$ (B) $np(np-1)/2$

(C) $(np-1)(np+1)/2$ (D) $n(np+1)$

Solution: We have

$$S_1 = \frac{n}{2}[2+(n-1)1] = \frac{n(n+1)}{2}$$

$$S_2 = \frac{n}{2}[4+(n-1)3] = \frac{n(3n+1)}{2}$$

$$S_3 = \frac{n}{2}[6+(n-1)5] = \frac{n(5n+1)}{2}$$

$$S_k = \frac{n}{2}[2k+(n-1)(2k-1)] = \frac{n[(2k-1)n+1]}{2}$$

Therefore

$$\sum_{k=1}^{p} S_k = \frac{n}{2}[n\{1+3+5+\cdots+(2p-1)\}+p]$$

$$= \frac{n}{2}(np^2+p)$$

$$= \frac{np}{2}(np+1)$$

Answer: (A)

9. A total of n arithmetic means are inserted between x and $2y$ and further n arithmetic means are inserted between $2x$ and y. If the kth arithmetic means of both sets are equal, then a relation between x and y is

(A) $ky = (n-k)x$ (B) $ky = (n+1-k)x$

(C) $k(y+1) = (n-k)x$ (D) $k(y+1) = (n+1-k)x$

Solution: Let A_k and A'_k be the kth AMs between x and $2y$ and $2x$ and y, respectively. Then

$$A_k = x + k\frac{(2y-x)}{n+1}$$

and $$A'_k = 2x + \frac{k(y-2x)}{n+1} \quad \text{(Theorem 5.7)}$$

Therefore

$$A_k = A'_k \Rightarrow x + \frac{k(2y-x)}{n+1} = 2x + \frac{k(y-2x)}{n+1}$$

$$\Rightarrow k(2y-x) = x(n+1) + k(y-2x)$$

$$\Rightarrow ky = (n+1-k)x$$

Answer: (B)

10. Between two numbers whose sum is $2\frac{1}{6}$, an even number of arithmetic means is inserted; the sum of these means exceeds their number by unity. Then, the number of means is

(A) 8 (B) 10 (C) 12 (D) 16

Solution: Let x and y be the given numbers so that

$$x + y = \frac{13}{6} \quad (5.6)$$

Let $A_1, A_2, \ldots, A_{2n}$ be the means between x and y. Then

$$(2n)\left(\frac{x+y}{2}\right) = \text{Sum of the means} = 2n+1$$

Therefore by Eq. (5.6)

$$n\left(\frac{13}{6}\right) = 2n+1$$

This implies $n = 6$ and $2n = 12$ is the number of means.

Answer: (C)

11. The number of terms in an AP is even. The sum of the odd terms is 24 and the sum of the even terms is 30. If the last term exceeds the first term by $10\frac{1}{2}$, then the number of terms in the AP is

(A) 6 (B) 8 (C) 10 (D) 12

Solution: Let $a, a+d, \ldots, a+(2n-1)d$ be the $2n$ numbers. Therefore, by hypothesis

$$24 = \text{Sum of the odd terms} = \frac{n}{2}[2a+(n-1)2d]$$

This gives

$$n[a+(n-1)d] = 24 \quad (5.7)$$

Also, again by hypothesis,

$$30 = \text{Sum of the even terms} = \frac{n}{2}[2(a+d)+(n-1)2d]$$

This gives

$$n(a+nd) = 30 \quad (5.8)$$

Now it is given that the last term exceeds the first term by $10\frac{1}{2}$, so

$$a+(2n-1)d = a+\frac{21}{2}$$

$$(2n-1)d = \frac{21}{2} \quad (5.9)$$

From Eqs. (5.7) and (5.8) we get

$$\begin{aligned} 30 - nd &= 24 \\ nd &= 6 \end{aligned} \quad (5.10)$$

From Eqs. (5.9) and (5.10) we have

$$12 - d = \frac{21}{2}$$

$$d = \frac{3}{2} \quad \text{and} \quad 6 = nd = n\left(\frac{3}{2}\right) \Rightarrow n = 4$$

Therefore the number of terms is $2n = 8$.

Answer: (B)

12. The mth term of an AP is a and its nth term is b, and $m \neq n$. Then the sum of the first $(m+n)$ terms of the AP is

(A) $\frac{m+n}{2}\left[a+b+\frac{a-b}{m-n}\right]$

(B) $\frac{m+n}{2}\left[(a+b)-\frac{a-b}{m-n}\right]$

(C) $\frac{m+n}{2}\left[(a+b)+\frac{a+b}{m+n}\right]$

(D) $\frac{m+n}{2}\left[(a-b)-\frac{(a-b)}{m+n}\right]$

Solution: Let α be the first term and d the common difference. Then

$$\begin{aligned} \alpha+(m-1)d &= a \\ \alpha+(n-1)d &= b \end{aligned} \quad (5.11)$$

Therefore

$$d = \frac{a-b}{m-n}$$

Substituting the value of d in the first equation of Eq. (5.11), we get

$$\begin{aligned} \alpha &= a - \frac{(m-1)(a-b)}{m-n} \\ &= \frac{a(m-n)-(m-1)(a-b)}{m-n} \\ &= \frac{am-an-am+a+bm-b}{m-n} \\ &= \frac{b(m-1)-a(n-1)}{m-n} \end{aligned}$$

Therefore the sum of the first $(m+n)$ terms is given by

$$\begin{aligned} &\frac{m+n}{2}\left[2\left\{\frac{b(m-1)-a(n-1)}{m-n}\right\}+(m+n-1)\frac{(a-b)}{m-n}\right] \\ &= \frac{m+n}{2}\left[\frac{1}{m-n}[2b(m-1)-2a(n-1)+(a-b)(m+n-1)]\right] \\ &= \frac{m+n}{2}\left[\frac{1}{m-n}[bm-bn-b-an+am+a]\right] \\ &= \frac{m+n}{2}\left[\frac{1}{m-n}[m(a+b)-n(a+b)+(a-b)]\right] \\ &= \frac{m+n}{2}\left[\frac{1}{m-n}[(a+b)(m-n)+(a-b)]\right] \\ &= \frac{m+n}{2}\left[(a+b)+\frac{a-b}{m-n}\right] \end{aligned}$$

Answer: (A)

13. The sum of the first and fifth terms of an AP is 26 and the product of the second and fourth is 160. Then the sum of the first six terms of the progression is

(A) 59 or 69 (B) 69 or 87
(C) 87 or 109 (D) –69 or 87

Solution: Let a be the first term and d the common difference, then

$$a+(a+4d) = 26$$

$$\Rightarrow a+2d = 13 \quad (5.12)$$

$$(a+d)(a+3d) = 160 \quad (5.13)$$

$$(13-d)(13+d) = 160 \quad \text{[from Eq. (5.12)]}$$

$$169 - d^2 = 160$$

$$d = \pm 3 \quad \text{and} \quad a = 7, 19$$

Therefore the sum of the first six terms = 69, 87.

Answer: (B)

14. If the sum of first n terms of an AP is cn^2, then the sum of the squares of these terms is

(A) $[n(4n^2-1)c^2]/6$ (B) $[n(4n^2+1)c^2]/3$
(C) $[n(4n^2-1)c^2]/3$ (D) $[n(4n^2+1)c^2]/6$

Solution: Let a_n be the nth term. Therefore

$$\begin{aligned} a_n &= (\text{sum of first } n \text{ terms}) - [\text{sum of first } (n-1) \text{ terms}] \\ &= cn^2 - c(n-1)^2 \\ &= c(2n-1) \end{aligned}$$

This gives

$$\sum_{k=1}^{n} a_k^2 = c^2 \sum_{k=1}^{n} (2k-1)^2$$

$$= c^2 \sum_{k=1}^{n} (4k^2 - 4k + 1)$$

$$= c^2 \left[4\sum_{k=1}^{n} k^2 - 4\sum_{K=1}^{n} k + n \right]$$

$$= c^2 \left[\frac{4n(n+1)(2n+1)}{6} - \frac{4n(n+1)}{2} + n \right]$$

$$= c^2 \left[\frac{2n(n+1)(2n+1)}{3} - n(2n+1) \right]$$

$$= \frac{n(2n+1)}{3}[2(n+1) - 3]c^2$$

$$= \frac{n(4n^2-1)c^2}{3}$$

Answer: (C)

15. If the numbers 3^{2a-1}, 14, 3^{4-2a} $(0 < a < 1)$ are the first three terms of an AP, then its fifth term is equal to

(A) 33 (B) 43 (C) 53 (D) 63

Solution: By hypothesis $3^{2a-1} + 3^{4-2a} = 28$. Therefore

$$\frac{9^a}{3} + \frac{81}{9^a} = 28$$

Substituting $9^a = x$, we get

$$\frac{x}{3} + \frac{81}{x} = 28$$

$$x^2 - 84x + 243 = 0$$

$$(x-81)(x-3) = 0$$

$$x = 81 \quad \text{or} \quad x = 3$$

This gives

$$9^a = 81 \quad \text{or} \quad 9^a = 3$$

$$a = 2 \quad \text{or} \quad \frac{1}{2}$$

$$0 < a < 1 \Rightarrow a = \frac{1}{2}$$

Therefore, the numbers are 1, 14, 27, which are in AP with common difference 13. The fifth term is $1 + 4 \times 13 = 53$.

Answer: (C)

16. If the sum of the first $2n$ terms of the AP, 2, 5, 8, 11, … is equal to the sum of the first n terms of the AP. 57, 59, 61, 63, …, then n is equal to

(A) 10 (B) 12 (C) 11 (D) 13

Solution: The sum $s_{2n} = 2 + 5 + 8 + \cdots$ upto $2n$ terms is

$$\frac{2n}{2}[4 + (2n-1)3] = n(6n+1)$$

Now the sum $s'_n = 57 + 59 + 61 + \cdots$ upto n terms is

$$\frac{n}{2}[114 + (n-1)2] = n(n+56)$$

Therefore

$$s_{2n} = s'_n \Rightarrow n(6n+1) = n(n+56)$$

$$\Rightarrow 5n^2 - 55n = 0$$

$$\Rightarrow n = 11$$

Answer: (C)

17. In an AP, if s_n is the sum of the first n terms (n is odd) and s'_n is the sum of the first n odd terms, then $s_n / s'_n =$

(A) $2n/n+1$ (B) $n/n+1$

(C) $n+1/2n$ (D) $n+1/n$

Solution: Let a be the first term and d the common difference. Then

$$s_n = \frac{n}{2}[2a + (n-1)d]$$

$$s'_n = a + (a+2d) + (a+4d) + \cdots + \left[a + \left(\frac{n+1}{2} - 1 \right) 2d \right]$$

$$= \left(\frac{n+1}{4} \right) [2a + (n-1)d]$$

Therefore

$$\frac{s_n}{s'_n} = \frac{(n/2)[2a + (n-1)d]}{[(n+1)/4][2a + (n-1)d]} = \frac{n}{2} \cdot \frac{4}{n+1} = \frac{2n}{n+1}$$

Answer: (A)

18. The series of natural numbers is divided into groups (1), (2, 3, 4), (5, 6, 7, 8, 9)… and so on. The sum of the numbers in the nth group is

(A) $n^3 + (n+1)^3$ (B) $(n-1)^3 + n^3$

(C) $n^3 + 1 + (n-1)^3$ (D) $(n+1)^3 + (n-1)^3$

Solution: Clearly the nth group contains $2n - 1$ numbers. The last terms of each group are $1^2, 2^2, 3^2, \ldots$ and hence the last term of nth group is n^2. Also, the first term of each group is one more than the last term of its previous group. Therefore the first term of the nth group is

$$(n-1)^2 + 1$$

Hence the sum of the numbers in the nth group is

$$\frac{2n-1}{2}[(n-1)^2 + 1 + n^2] = (2n-1)(n^2 - n + 1)$$

$$= (n-1)^3 + n^3$$

Answer: (B)

19. If $\log_{10}2, \log_{10}(2^x-1), \log_{10}(2^x+3)$ are in AP, then

(A) $x=1$ (B) $x=2$
(C) $x=\log_2 5$ (D) $x=\log_{10}5$

Solution: By hypothesis $\log_{10}2, \log_{10}(2^x-1), \log_{10}(2^x+3)$ are in AP and so

$$\log_{10}2+\log_{10}(2^x+3)=2\log_{10}(2^x-1)$$
$$\log_{10}2(2^x+3)=\log_{10}(2^x-1)^2$$
$$2(2^x+3)=(2^x-1)^2$$
$$2^{2x}-4\cdot 2^x-5=0$$
$$a^2-4a-5=0 \quad (\text{where } a=2^x)$$
$$(a-5)(a+1)=0$$
$$2^x=-1 \quad \text{or} \quad 2^x=5$$

But 2^x cannot be negative. Therefore

$$2^x=5$$
$$x=\log_2 5$$

Answer: (C)

20. In a sequence $a_1, a_2, a_3, \ldots$ of real numbers it is observed that $a_p=\sqrt{2}$, $a_q=\sqrt{3}$ and $a_r=\sqrt{5}$, where p, q, r are positive integers such that $1\le p<q<r$. Then

(A) a_p, a_q, a_r can be terms of an AP
(B) $1/a_p, 1/a_q, 1/a_r$ can be terms of an AP
(C) a_p, a_q, a_r can be terms of an AP if and only if p, q, r are perfect-squares
(D) Neither a_p, a_q, a_r are in AP nor $1/a_p, 1/a_q, 1/a_r$ are in AP

Solution: Suppose

$$\sqrt{2}=a_p=a+(l-1)d$$
$$\sqrt{3}=a_q=a+(m-1)d$$
$$\sqrt{5}=a_r=a+(n-1)d$$

where l, m, n are positive integers in the increasing order. Therefore

$$(m-l)d=\sqrt{3}-\sqrt{2} \quad \text{and} \quad (n-m)d=\sqrt{5}-\sqrt{3}$$

and so

$$\frac{m-l}{n-m}=\frac{\sqrt{5}-\sqrt{3}}{\sqrt{3}-\sqrt{2}}$$

which is absurd because left-hand side (LHS) is rational.

Answer: (D)

21. The arithmetic mean of two numbers is $18\frac{3}{4}$ and the positive square root of their product is 15. The larger of the two numbers is

(A) 24 (B) 25 (C) 20 (D) 30

Solution: Let a and b be the positive numbers. Then

$$\frac{a+b}{2}=18\frac{3}{4}\Rightarrow a+b=\frac{75}{2}$$

and $$\sqrt{ab}=15\Rightarrow ab=225$$

Therefore

$$a-b=\pm\sqrt{\left(\frac{75}{2}\right)^2-4ab}$$
$$=\pm\sqrt{\frac{75^2}{4}-4\times 15^2}$$
$$=\pm\sqrt{\left(\frac{75}{2}+30\right)\left(\frac{75}{2}-30\right)}$$
$$=\pm\frac{\sqrt{135\times 15}}{2}=\pm\frac{45}{2}$$

Case 1: $a+b=75/2$ and $a-b=45/2$

Solving the two equations we get $a=30$ and $b=15/2$

Case 2: $a+b=75/2$ and $a-b=-45/2$. Solving the two equations we get

$$a=\frac{15}{2} \quad \text{and} \quad b=30$$

Therefore larger of the numbers is 30.

Answer: (D)

22. The sum of the integers from 1 to 100 which are divisible by exactly one of 2 and 5 is

(A) 2505 (B) 1055 (C) 2550 (D) 3050

Solution: Let A, B and C be the set of all integers from 1 to 100 that are divisible by 2, 5 and 10, respectively. Therefore

$$A=\{2, 4, 6, \ldots, 100\}, B=\{5, 10, 15, \ldots, 100\}$$

and $$C=\{10, 20, 30, \ldots, 100\}$$

Clearly, $C=A\cap B$. Therefore

(i) A contains 50 numbers which are in AP with first term 2 and common difference 2.

(ii) B contains 20 numbers that are in AP with first term 5 and common difference 5.

(iii) $C=A\cap B$ contains 10 numbers which are in AP with common difference 10 and first term 10.

Therefore the required sum is

$$\sum_{x\in A}x+\sum_{y\in B}y-\sum_{z\in C}z=\frac{50}{2}(2+100)+\frac{20}{2}(5+100)$$
$$-\frac{10}{2}(10+100)=3050$$

Answer: (D)

23. If the sum of the roots of the quadratic equation $ax^2 + bx + c = 0$ is equal to the sum of the squares of their reciprocals, then

(A) bc^2, ca^2, ab^2 are in AP
(B) bc^2, ab^2, ca^2 are in AP
(C) ca^2, bc^2, ab^2 are in AP
(D) ab, bc, ca are in AP

Solution: Let α and β be the roots of the equation. Then

$$\alpha + \beta = -\frac{b}{a}, \alpha\beta = \frac{c}{a}$$

Now

$$\alpha + \beta = \frac{1}{\alpha^2} + \frac{1}{\beta^2}$$

$$\Rightarrow -\frac{b}{a} = \frac{(b^2/a^2) - (2c/a)}{c^2/a^2}$$

$$-\frac{b}{a} = \frac{b^2 - 2ca}{c^2}$$

$$-bc^2 = ab^2 - 2ca^2$$

$$2ca^2 = ab^2 + bc^2$$

Therefore, bc^2, ca^2, ab^2 are in AP.

Answer: (A)

24. If the reciprocals of the roots of the equation $10x^3 - ax^2 - 54x - 27 = 0$ are in AP, then the value of a is

(A) 6 (B) 8 (C) –9 (D) 9

Solution: Let α, β, γ be the roots of the given equation. Then

$$\alpha + \beta + \gamma = \frac{a}{10}, \alpha\beta + \beta\gamma + \gamma\alpha = -\frac{54}{10}, \alpha\beta\gamma = \frac{27}{10}$$

Now the reciprocals of the roots of the equation are in AP, that is

$$\frac{1}{\alpha} + \frac{1}{\gamma} = \frac{2}{\beta}$$

Adding $1/\beta$ to both the sides we get

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{3}{\beta}$$

$$\frac{\beta\gamma + \gamma\alpha + \alpha\beta}{\alpha\beta\gamma} = \frac{3}{\beta}$$

$$-\frac{54}{27} = \frac{3}{\beta}$$

$$\beta = -\frac{3}{2}$$

Since β is a root of the given equation we get

$$10\beta^3 - a\beta^2 - 54\beta - 27 = 0$$

Substituting the value of β we get

$$10\left(-\frac{3}{2}\right)^3 - a\left(-\frac{3}{2}\right)^2 - 54\left(-\frac{3}{2}\right) - 27 = 0$$

$$10\left(-\frac{27}{8}\right) - \frac{9a}{4} + 81 - 27 = 0$$

$$\frac{9a}{4} = -\frac{10 \times 27}{8} + 54$$

$$a = 4\left(-\frac{30}{8} + 6\right) = \frac{18}{2} = 9$$

Answer: (D)

25. The fifth and 31st terms of an AP are, respectively, 1 and –77. If kth term of the given AP is –17, then k is

(A) 12 (B) 10 (C) 11 (D) 13

Solution: The fifth term is 1; therefore,

$$a + 4d = 1 \qquad (5.14)$$

The 31st term is –77; therefore,

$$a + 30d = -77 \qquad (5.15)$$

Solving Eqs. (5.14) and (5.15) we get

$$26d = 78 \Rightarrow d = -3$$

Substituting the value of d in either equation we get

$$a = 13$$

Now the kth term of the given AP is –17; therefore,

$$a + (k-1)d = -17$$

$$\Rightarrow 13 - 3(k-1) = -17$$

$$\Rightarrow 3k = 33$$

$$\Rightarrow k = 11$$

Answer: (C)

26. The sum of the four arithmetic means between 4 and 40 is

(A) 90 (B) 88 (C) 108 (D) 118

Solution: Sum of the four means is given by

$$4\frac{(4+40)}{2} = 88$$

Answer: (B)

27. In an increasing arithmetic progression, the sum of the first three terms is 27 and the sum of their squares is 275. The common difference of the AP is

(A) 6 (B) 8 (C) 2 (D) 4

Solution: Let $a-d$, a, $a+d$ be the first three terms. Since the terms are increasing, $d>0$. By hypothesis

$$3a = 27 \Rightarrow a = 9$$

$$(a-d)^2 + a^2 + (a+d)^2 = 275$$

and $$3a^2 + 2d^2 = 275$$

$$d^2 = 16 \Rightarrow d = \pm 4$$

Since $d>0$, we get $d=4$.

Answer: (D)

28. If $5^2 \cdot 5^4 \cdot 5^6 \cdots 5^{2n} = (0.04)^{-28}$, then n is equal to

(A) 7 (B) 5 (C) 6 (D) 3

Solution: The given equation can be written as

$$(5^2)\cdot(5^2)^2\cdot(5^2)^3\cdots(5^2)^n = (5^2)^{28}$$

$$25^{1+2+3+\cdots+n} = 25^{28}$$

$$25^{[n(n+1)/2]} = 25^{28}$$

Since the bases are the same, equating the powers we get

$$\frac{n(n+1)}{2} = 28$$

$$n(n+1) = 56$$

$$(n+8)(n-7) = 0$$

$$n = -8, 7$$

Now $n=-8$ is not possible. Hence $n=7$.

Answer: (A)

29. The interior angles of a polygon are in AP. The smallest angle is 120° and the common difference is 5°. The number of sides of the polygon is

(A) 11 (B) 9 (C) 12 (D) 13

Solution: Sum of the interior angles of a polygon of n sides equals $(2n-4)$ right angles. Therefore

$$\frac{n}{2}[240 + 5(n-1)] = (2n-4)90$$

$$5n^2 - 125n + 720 = 0$$

$$n^2 - 25n + 144 = 0$$

$$(n-9)(n-16) = 0$$

Now this gives two values of $n=9$ and 16. We take $n=9$, as $n=16$ is rejected because the last angle becomes

$$120° + (15\times 5)° = 195°$$

Therefore number of sides $= 9$.

Answer: (B)

30. The ratio of sum of m terms to the sum of n terms of an AP is $m^2:n^2$. If T_k is the kth term, then T_5/T_2 is

(A) 6 (B) 5 (C) 4 (D) 3

Solution: By hypothesis

$$\frac{s_m}{s_n} = \frac{m^2}{n^2}$$

Also

$$\frac{T_m}{T_n} = \frac{s_m - s_{m-1}}{s_n - s_{n-1}} = \frac{m^2-(m-1)^2}{n^2-(n-1)^2} = \frac{2m-1}{2n-1}$$

Substituting $m=5$ and $n=2$ in this equation we get

$$\frac{T_5}{T_2} = \frac{2(5)-1}{2(2)-1} = \frac{9}{3} = 3$$

Answer: (D)

31. The sum of the first eight terms of a GP whose nth term is $2\cdot 3^n (n=1,2,3,\ldots)$ is

(A) 19880 (B) 19860 (C) 19660 (D) 19680

Solution: The terms of the GP are

$$2\cdot 3, 2\cdot 3^2, 2\cdot 3^3, \ldots, 2\cdot 3^8$$

First term is 6 and the common ratio is 3. Therefore the sum of the first 8 terms is

$$\frac{6(1-3^8)}{1-3} = 19680$$

Answer: (D)

32. The difference between the fourth and the first term of a GP is 52 and the sum of the first three terms is 26. Then the sum of the first six terms is

(A) 720 (B) 725 (C) 728 (D) 780

Solution: Let the GP be $a, ar, ar^2, \ldots$. Then by hypothesis

$$ar^3 - a = 52$$

$$a + ar + ar^2 = 26$$

$$\frac{r^3-1}{1+r+r^2} = \frac{52}{26} = 2$$

Therefore

$$r - 1 = 2 \Rightarrow r = 3$$

Using this value we get $a=2$. Therefore the sum of the first six terms is

$$\frac{2(3^6-1)}{3-1} = 728$$

Answer: (C)

33. The sequence $\{a_n\}$ is a GP such that

$$\frac{a_4}{a_6}=\frac{1}{4} \quad \text{and} \quad a_2+a_5=216$$

If the common ratio is positive, then the first term is

(A) 12 (B) 11 (C) 103/7 (D) 13

Solution: Let $a_1=a, a_2=ar, a_3=ar^2, a_4=ar^3, a_5=ar^4$ and $a_6=ar^5$. Then

$$\frac{1}{4}=\frac{a_4}{a_6}=\frac{ar^3}{ar^5}=\frac{1}{r^2}$$

and hence $r=2$ (since $r>0$). Therefore

$$a_2+a_5=216 \Rightarrow a(r+r^4)=216 \Rightarrow a=12$$

Answer: (A)

34. a,b,c,d are in GP and are in ascending order such that $a+d=112$ and $b+c=48$. If the GP is continued with a as the first term, then the sum of the first six terms is

(A) 1156 (B) 1256 (C) 1356 (D) 1456

Solution: Let r be the common ratio so that $b=ar, c=ar^2$, and $d=ar^3$. Therefore

$$a+ar^3=112 \quad \text{and} \quad ar+ar^2=48$$

Dividing the first equation by the second and canceling a we get

$$\frac{1+r^3}{r+r^2}=\frac{112}{48}=\frac{7}{3}$$

$$\frac{(1+r)(1-r+r^2)}{r(1+r)}=\frac{7}{3}$$

$$3(1-r+r^2)=7r$$

$$3r^2-10r+3=0$$

$$(3r-1)(r-3)=0$$

$$r=3 \quad \text{or} \quad \frac{1}{3}$$

(i) $r=3 \Rightarrow a=4$

(ii) $r=1/3 \Rightarrow a=108$

But, it is given that $a<b<c<d$. Therefore, the GP is 4, 12, 36, 108, 324, 972, …. Hence the sum of the first 6 terms is

$$4+12+36+108+324+972=1456$$

Answer: (D)

35. The sum of the first n terms of the series

$$\frac{1}{2}+\frac{3}{4}+\frac{7}{8}+\frac{15}{16}+\cdots$$

is equal to

(A) 2^n-n-1 (B) $1-2^{-n}$

(C) $2^{-n}+n-1$ (D) 2^n+1

Solution: The given series is

$$\left(1-\frac{1}{2}\right)+\left(1-\frac{1}{4}\right)+\left(1-\frac{1}{16}\right)+\cdots$$

Therefore the sum of the first n terms is

$$(1+1+\cdots+n\text{ times})-\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}}\right)$$

$$=n-\frac{1}{2}\left(\frac{1-(1/2^n)}{1-(1/2)}\right)=2^{-n}+n-1$$

Answer: (C)

36. Let α and β be the roots of the quadratic equation $ax^2+bx+c=0$ and $\Delta=b^2-4ac$. If $\alpha+\beta, \alpha^2+\beta^2$ and $\alpha^3+\beta^3$ are in GP, then

(A) $\Delta \neq 0$ (B) $b\Delta=0$ (C) $c\Delta=0$ (D) $\Delta=0$

Solution: Since α and β be the roots of the given quadratic equation, we have

$$\alpha+\beta=\frac{-b}{a} \quad \text{and} \quad \alpha\beta=\frac{c}{a}$$

Now

$$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=\frac{b^2}{a^2}-\frac{2c}{a}$$

$$\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)=\frac{-b^3}{a^3}+\frac{3bc}{a^2}$$

Suppose that $\alpha+\beta, \alpha^2+\beta^2, \alpha^3+\beta^3$ are in GP. Then

$$(\alpha^2+\beta^2)^2=(\alpha+\beta)(\alpha^3+\beta^3)$$

$$\left(\frac{b^2}{a^2}-\frac{2c}{a}\right)^2=-\frac{b}{a}\left(-\frac{b^3}{a^3}+\frac{3bc}{a^2}\right)$$

$$(b^2-2ac)^2=-b(-b^3+3abc)$$

$$4c^2a^2+b^4-4b^2ac=b^4-3ab^2c$$

$$b^2ca-4c^2a^2=0$$

$$ca(b^2-4ac)=0$$

Since $a\neq 0$, $c\Delta=0$.

Answer: (C)

37. The first term of a GP $a_1, a_2, a_3, \ldots$ is unity. The value of $4a_2+5a_3$ is minimum when the common ratio is

(A) 1/3 (B) −1/3 (C) 2/5 (D) −2/5

Solution: Let r be the common ratio. Then

$$a_1=1, a_2=r, a_3=r^2, \ldots, a_n=r^{n-1}$$

Now $4a_2+5a_3=4r+5r^2$, which is minimum when

$$r=-\frac{4}{2\times 5}=-\frac{2}{5}$$

Answer: (D)

(Note that, if $a > 0$, then $ax^2 + bx + c$ assumes its minimum value at $x = -b/2a$.)

38. Three numbers are in AP. If 8 is added to the first number, we get a GP with sum of the terms is equal to 26. Then the common ratio of the GP when they are written in the ascending order, is

(A) 3 (B) 1/3 (C) 2 (D) 1/2

Solution: Let the three numbers which are in AP be $a-d$, a, $a+d$. Then $a-d+8$, a, $a+d$ are given to be in GP. Therefore

$$a^2 = (a-d+8)(a+d) = a^2 - d^2 + 8(a+d)$$

$$-d^2 + 8a + 8d = 0 \tag{5.16}$$

Also $(a-d+8) + a + (a+d) = 26$. Therefore

$$3a + 8 = 26$$

and hence $a = 6$.

From Eq. (5.16), we get

$$d^2 - 48 - 8d = 0$$

$$(d+4)(d-12) = 0$$

and hence $d = 12$ or -4.

(i) If $d = 12$, the GP $a-d+8, a, a+d$ is 2, 6, 18

(ii) If $d = -4$, the GP $a-d+8, a, a+d$ is 18, 6, 2

When we write the GP in the ascending order, the common ratio is 3.

Answer: (A)

39. Three distinct numbers a, b, c form a GP in that order and the numbers $a+b, b+c, c+a$ form an AP in that order. Then the common ratio of the GP is

(A) 1/2 (B) $-1/2$ (C) -2 (D) 2

Solution: Let $b = ar$ and $c = ar^2$. Then

$$2(ar + ar^2) = (a + ar) + (ar^2 + a)$$

$$2r + 2r^2 = 2 + r + r^2$$

$$r^2 + r - 2 = 0$$

$$(r+2)(r-1) = 0$$

$$r = -2 \quad \text{or} \quad 1$$

Since a, b and c are distinct, $r \neq 1$. Therefore $r = -2$.

Answer: (C)

40. The number of geometric progressions containing 27, 8 and 12 as three of their terms, is

(A) 1 (B) 2

(C) 5 (D) infinite

Solution: Let a be the first term and r the common ratio of a GP containing 27, 8 and 12 as lth, mth and nth terms, respectively. Then

$$ar^{l-1} = 27, \quad ar^{m-1} = 8 \quad \text{and} \quad ar^{n-1} = 12$$

$$\left(\frac{3}{2}\right)^3 = \frac{27}{8} = \frac{r^{l-1}}{r^{m-1}} = r^{l-m} \tag{5.17}$$

$$\left(\frac{3}{2}\right)^{-1} = \frac{8}{12} = \frac{r^{m-1}}{r^{n-1}} = r^{m-n} \tag{5.18}$$

From Eqs. (5.17) and (5.18), we have

$$r = \frac{3}{2}, l - m = 3 \quad \text{and} \quad m - n = -1$$

Therefore

$$l = m + 3 = n + 2 \quad \text{and} \quad m = n - 1$$

Each value of n determines the values of l and m. Therefore, there are infinitely many GP's satisfying the given conditions.

Answer: (D)

41. If $a < b < c$ are numbers lying between 2 and 18 such that

(i) $a + b + c = 25$

(ii) $2, a, b$ are three consecutive terms of an AP in that order

(iii) $b, c, 18$ are three consecutive terms of a GP in that order

then the product abc is equal to

(A) 480 (B) 680 (C) 440 (D) 640

Solution: Given $2 < a < b < c < 18$,

$$a + b + c = 25 \tag{5.19}$$

$$\frac{2+b}{2} = a \tag{5.20}$$

$$c^2 = 18b \tag{5.21}$$

From Eqs. (5.19) and (5.20), we get

$$3a + c = 27 \tag{5.22}$$

From Eqs. (5.20) and (5.21), we get

$$c^2 = 36(a-1) \tag{5.23}$$

From Eqs. (5.22) and (5.23), we get

$$(27 - 3a)^2 = 36(a-1)$$

$$(9 - a)^2 = 4(a-1)$$

$$a^2 - 22a + 85 = 0$$

$$(a-5)(a-17) = 0$$

$$a = 5 \quad \text{or} \quad 17$$

Case 1: Let $a = 5$. Then $b = 2a - 2 = 8$ and

$$c^2 = 18b = 18 \times 8 = 9 \times 16$$

implies $\quad c = 12$

Case 2: Let $a = 17$. Then $b = 2a - 2 = 32$ and

$$c^2 = 18b = 18 \times 32 = 9 \times 64$$

implies $\quad c = 3 \times 8 = 24$

But $2 < a < b < c < 18$. Therefore $a = 5$, $b = 8$, $c = 12$ and $abc = 480$.

Answer: (A)

42. An infinite GP has first term x and sum 5. Then

(A) $x < -10$ (B) $-10 < x < 0$

(C) $0 < x < 10$ (D) $x > 10$

Solution: Let r be the common ratio. Then $|r| < 1$. Therefore

$$\frac{x}{1-r} = 5 \quad \text{and} \quad r = 1 - \frac{x}{5}$$

Since

$$-1 < r < 1, -1 < 1 - \frac{x}{5} < 1$$

Therefore

$$-2 < -\frac{x}{5} < 0$$

$$-10 < -x < 0$$

$$0 < x < 10$$

Answer: (C)

43. If x, y, z are in GP, $x-2, y-6, z-58$ are in AP and $x-1, y-3, z-9$ are in GP, then the numbers x, y, z are

(A) 3, 9, 27 (B) 12, 36, 108

(C) 9, 36, 144 (D) 27, 81, 243

Solution: From the hypothesis, we have

$$y^2 = xz \tag{5.24}$$

$$2(y-6) = (x-2) + (z-58)$$

or

$$2y = x + z - 48 \tag{5.25}$$

and

$$(y-3)^2 = (x-1)(z-9) \tag{5.26}$$

Using Eq. (5.24) and (5.26), we get

$$6y = 9x + z \tag{5.27}$$

From Eqs. (5.25) and (5.27), we get

$$\begin{aligned} 6x - 2z + 144 &= 0 \\ 3x - z &= -72 \end{aligned} \tag{5.28}$$

$$\begin{aligned} 2y &= x + z - 48 \\ &= x + (3x + 72) - 48 \\ &= 4x + 24 \\ y &= 2x + 12 \end{aligned} \tag{5.29}$$

From Eqs. (5.24) and (5.29), we get

$$(2x+12)^2 = x(3x+72)$$

$$x^2 - 24x + 144 = 0$$

$$(x-12)^2 = 0$$

Therefore $x = 12$, $y = 36$, $z = 108$.

Answer: (B)

44. Let a, b, c be in GP. If p is the AM between a and b and q is the AM between b and c, then b is equal to

(A) $p + q/2$ (B) $p + q/pq$

(C) $p + q/2pq$ (D) $2pq/p + q$

Solution: Given that

$$b^2 = ac, \; p = \frac{a+b}{2} \quad \text{and} \quad q = \frac{b+c}{2}$$

Therefore $a = 2p - b$ and $c = 2q - b$. Hence

$$b^2 = ac = (2p-b)(2q-b) = 4pq - 2b(p+q) + b^2$$

This gives

$$b = \frac{2pq}{p+q}$$

Answer: (D)

45. If $a_1, a_2, a_3, \ldots$ is a GP satisfying the relation $a_k + a_{k+2} = 3a_{k+1}$ for all $k \geq 1$, then common ratio is

(A) $(\sqrt{3} \pm 1)/2$ (B) $(\sqrt{5} \pm 2)/2$

(C) $(3 \pm \sqrt{5})/2$ (D) $(4 \pm \sqrt{5})/2$

Solution: Let a be the first term and r the common ratio. Then

$$ar^{k-1} + ar^{k+1} = 3ar^k$$

Therefore

$$r^2 - 3r + 1 = 0$$

$$r = \frac{3 \pm \sqrt{5}}{2}$$

Answer: (C)

46. The value of

$$\frac{2n+1}{2n-1} + 3\left(\frac{2n+1}{2n-1}\right)^2 + 5\left(\frac{2n+1}{2n-1}\right)^3 + \cdots$$

upto n terms is

(A) $n(2n+1)$ (B) $(n+1)(2n+1)$

(C) $n(2n+3)$ (D) $(n+1)(2n+3)$

Solution: Let

$$x = \frac{2n+1}{2n-1}$$

and $$s_n = x + 3x^2 + 5x^3 + \cdots + (2n-1)x^n$$

Then

$$xs_n = x^2 + 3x^3 + \cdots + (2n-3)x^n + (2n-1)x^{n+1}$$

Therefore

$$(1-x)s_n = x + 2x^2 + 2x^3 + \cdots + 2x^n - (2n-1)x^{n+1}$$
$$= 2(x + x^2 + x^3 + \cdots + x^n) - x - (2n-1)x^{n+1}$$
$$\left(\frac{1-x}{x}\right)s_n = 2(1 + x + x^2 + \cdots + x^{n-1}) - 1 - (2n-1)x^n$$
$$= \frac{2(1-x^n)}{1-x} - 1 - (2n-1)x^n$$

Now,

$$\frac{1-x}{x} = \frac{1-[(2n+1)/(2n-1)]}{(2n+1)/(2n-1)} = \frac{-2}{2n+1}$$

Therefore

$$\left(\frac{-2}{2n+1}\right)s_n = \frac{2[1-\{(2n+1)/(2n-1)\}^n]}{1-[(2n+1)/(2n-1)]} - 1$$
$$-(2n-1)\left(\frac{2n+1}{2n-1}\right)^n$$
$$= -(2n-1)\left[1-\left(\frac{2n+1}{2n-1}\right)^n\right] - 1$$
$$-(2n-1)\left(\frac{2n+1}{2n-1}\right)^n$$
$$= -(2n-1) - 1 = -2n$$

Hence $s_n = n(2n+1)$.

Answer: (A)

47. Sum to infinity of a GP is 15 and the sum to infinity of their squares is 45. If a is the first term and r is the common ratio, then the sum of the first 5 terms of the AP with first term a and common difference $3r$ is

(A) 25 (B) 35 (C) 45 (D) 55

Solution: By the hypothesis, we have

$$\frac{a}{1-r} = 15 \tag{5.30}$$

$$\frac{a^2}{1-r^2} = 45$$

$$\frac{a^2}{(1-r)(1+r)} = 45$$

$$\frac{a}{1+r} = 3 \quad \text{[using Eq. (5.30)]} \tag{5.31}$$

Dividing Eq. (5.30) by Eq. (5.31) we get

$$\frac{1+r}{1-r} = \frac{15}{3} = 5$$

$$r = \frac{2}{3}$$

Using this value of r in Eq. (5.30) we get $a = 5$. Also $3r = 2$. Therefore

$$\sum_{k=1}^{5}[a+(k-1)3r] = \frac{5}{2}[2a+(k-1)(3r)] = 45$$

Hence, the required sum is 45.

Answer: (C)

48. The value of $0.423232323\cdots (= 0.4\overline{23})$ is

(A) 419/423 (B) 419/990

(C) 423/990 (D) 419/999

Solution: We have

$$0.4\overline{23} = \frac{4}{10} + \frac{23}{10^3} + \frac{23}{10^5} + \cdots + \infty$$
$$= \frac{4}{10} + \frac{23}{10^3}\left(1 + \frac{1}{10^2} + \frac{1}{10^3} + \cdots\right)$$
$$= \frac{4}{10} + \frac{23}{10^3}\left(\frac{1}{1-(1/10^2)}\right)$$
$$= \frac{4}{10} + \frac{23}{10^3}\left(\frac{10^2}{99}\right) = \frac{419}{990}$$

Answer: (B)

49. If x, y and z are, respectively, the fourth, seventh and 10th terms of a GP, then

(A) $x^2 = y^2 + z^2$ (B) $y^2 = zx$

(C) $x^2 = yz$ (D) $z^2 = xy$

Solution: Let the first term be a and common ratio r. Then

$$x = ar^3,\ y = ar^6,\ z = ar^9$$

Therefore,

$$y^2 = a^2 r^{12} = (ar^3)(ar^9) = xz$$

Answer: (B)

50. In a certain GP, if the first, second and eighth terms are x^{-4}, x^K and x^{52}, respectively, then the value of K is

(A) 2 (B) 3 (C) 4 (D) 0

Solution: Let $a, ar, ar^2, \ldots$ be the terms of the GP. Then

$$a = x^{-4},\ ar = x^K \quad \text{and} \quad ar^7 = x^{52}$$

Therefore

$$x^{-4} r = x^K \quad \text{and} \quad x^{-4} r^7 = x^{52}$$

$$r = x^{K+4} \quad \text{and} \quad r^7 = x^{56}$$

$$x^{7(K+4)} = x^{56}$$

Equating the powers we get

$$7(K+4) = 56$$

$$K + 4 = 8$$

$$K = 4$$

Answer: (C)

51. Three numbers a, b, c are in GP. If $a, b, c-64$ are in AP and $a, b-8, c-64$ are in GP, then the sum of the numbers may be

(A) 124 (B) 241 (C) 142 (D) 214

Solution: Let $b = ar$ and $c = ar^2$. Given that $a, ar, ar^2 - 4$ are in AP. Therefore

$$\begin{aligned} a + (ar^2 - 64) &= 2(ar) \\ a(r^2 - 2r + 1) &= 64 \end{aligned} \tag{5.32}$$

Again $a, ar-8, ar^2-64$ are in GP. Therefore

$$\begin{aligned} a(ar^2 - 64) &= (ar-8)^2 \\ a(16r - 64) &= 64 \end{aligned} \tag{5.33}$$

From Eqs. (5.32) and (5.33), we get

$$r^2 - 2r + 1 = 16r - 64$$

$$r^2 - 18r + 65 = 0$$

$$(r-5)(r-13) = 0$$

$$r = 5 \quad \text{or} \quad 13$$

If $r = 5$, then $a(80-64) = 64$ and hence $a = 4$. In this case the numbers are 4, 20, 100 and their sum is 124.

Answer: (A)

52. The product of nine GMs inserted between the numbers 2/9 and 9/2 is

(A) 9 (B) 1 (C) 3 (D) $3\sqrt{3}$

Solution: The product of n GMs inserted between two positive real numbers a and b is $(\sqrt{ab})^n$. Here $a = 2/9$, $b = 9/2$ and $n = 9$. Substituting these values we get the required product as

$$\left(\sqrt{\frac{2}{9} \cdot \frac{9}{2}}\right)^9 = 1$$

Answer: (B)

53. Let A be the arithmetic mean of x and y. If p and q are two GM's between x and y and $p^3 + q^3 = K(pq)A$, then the value of K is

(A) 1 (B) 2 (C) 3 (D) 4

Solution: It is given that A is the arithmetic mean of x and y; that is

$$A = \frac{x+y}{2}$$

Now x, p, q, y are in GP. Therefore

$$p = xr, q = xr^2 \quad \text{and} \quad y = xr^3$$

where $r = p/x$. Then

$$r = \left(\frac{y}{x}\right)^{1/3}$$

Hence

$$p = xr = x\left(\frac{y}{x}\right)^{1/3} = x^{2/3} \cdot y^{1/3}$$

$$q = x\left(\frac{y}{x}\right)^{2/3} = x^{1/3} \cdot y^{2/3} \quad \text{and} \quad pq = xy$$

Now

$$\begin{aligned} p^3 + q^3 &= x^2 y + xy^2 \\ &= xy(x+y) \\ &= (xy)(2A) = (pq)(2A) \end{aligned}$$

Therefore $K = 2$.

Answer: (B)

54. Let $f: \mathbb{R} \to \mathbb{R}$ be a function such that $f(1) = 2$ and it satisfies the relation $f(x+y) = f(x)f(y)$ for all natural numbers x and y. Then the value of the natural number a such that

$$\sum_{K=1}^{n} f(a+K) = 16(2^n - 1)$$

is

(A) 3 (B) 4 (C) 5 (D) 6

Solution: $f(1) = 2$

$$f(2) = f(1+1) = f(1)f(1) = 2^2$$

$$f(3) = f(2+1) = f(2)f(1) = 2^3$$

This implies

$$f(K) = 2^K \quad \text{for any natural number } K$$

Now

$$16(2^n - 1) = \sum_{K=1}^{n} f(a+K) = \sum_{K=1}^{n} f(a)f(K)$$

$$= 2^a(2 + 2^2 + \cdots + 2^n)$$
$$= 2^{a+1}(2^n - 1)$$

Therefore

$$2^{a+1} = 16 = (2)^4$$

Equating the powers we get $a = 3$.

Answer: (A)

55. In an HP, if the mth term is n and the nth term is m, then $(m+n)$th term is

(A) $m - n/m + n$ (B) $mn/m + n$
(C) $m + n/mn$ (D) $2mn/m + n$

Solution: By hypothesis

$$n = m\text{th term} = \frac{1}{a + (m-1)d}$$

$$m = n\text{th term} = \frac{1}{a + (n-1)d}$$

Therefore

$$[a + (m-1)d] - [a + (n-1)d] = \frac{1}{n} - \frac{1}{m} = \frac{m-n}{mn}$$
$$(m-n)d = \frac{m-n}{mn}$$
$$d = \frac{1}{mn}$$

Now

$$a + (m-1)d = \frac{1}{n}$$
$$a + \frac{m-1}{mn} = \frac{1}{n}$$
$$a = \frac{1}{n} - \frac{m-1}{mn} = \frac{1}{mn}$$

Therefore the $(m+n)$th term is

$$\left(\frac{1}{mn} + (m+n-1)\frac{1}{mn}\right)^{-1} = \frac{mn}{m+n}$$

Answer: (B)

56. If a, b, c are in HP (in this order), then

(A) $\dfrac{1}{b-a} + \dfrac{1}{b-c} = \dfrac{c+a}{ca}$ (B) $\dfrac{1}{a+b} + \dfrac{1}{b+c} = \dfrac{c+a}{ca}$

(C) $\dfrac{1}{c-a} + \dfrac{1}{b-a} = \dfrac{1}{a} + \dfrac{1}{c}$ (D) $\dfrac{1}{a} + \dfrac{1}{b} = \dfrac{1}{b} + \dfrac{1}{c}$

Solution: We have

$$\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$$

are in AP. Therefore

$$b = \frac{2ac}{a+c}$$

Now

$$\frac{1}{b-a} + \frac{1}{b-c} = \frac{1}{[2ac/(a+c)] - a} + \frac{1}{[2ac/(a+c)] - c}$$
$$= \frac{a+c}{ac - a^2} + \frac{a+c}{ac - c^2}$$
$$= (a+c)\left[\frac{1}{a(c-a)} + \frac{1}{c(a-c)}\right]$$
$$= \frac{a+c}{ac}$$

Answer: (A)

57. If $(m+1)$th term, $(n+1)$th term and $(r+1)$th terms of an AP are in GP and m, n and r are in HP, then the ratio of the common difference to the first term of AP is

(A) $-1/n$ (B) $1/n$ (C) $2/n$ (D) $-2/n$

Solution: By hypothesis,

$$(a + md)(a + rd) = (a + nd)^2 \quad \text{and} \quad n = \frac{2mr}{m+r}$$

Therefore

$$a^2 + ad(m+r) + mrd^2 = a^2 + 2and + n^2d^2$$
$$a(m+r) + mrd = 2an + n^2 d$$
$$d(n^2 - mr) = a(m + r - 2n)$$
$$\frac{d}{a} = \frac{(m+r) - 2n}{n^2 - mr} = \frac{(m+r) - 2n}{n^2 - [n(m+r)/2]}$$
$$\left(\because \quad mr = \frac{n(m+r)}{2}\right)$$
$$= \frac{2[m + r - 2n]}{n(2n - m - r)} = \frac{-2}{n}$$

Answer: (D)

58. Three numbers l, m and n are in GP. The lth, mth and nth terms of an AP are in HP. Then the ratio of the first term of the AP to its common difference is

(A) $m : 1$ (B) $1 : m$
(C) $1 : m + 1$ (D) $m + 1 : 1$

Solution: Since l, m and n are in GP, we have $m^2 = ln$. Let the AP be $a, a + d, a + 2d, \ldots$. Therefore

$$a + (l-1)d, a + (m-1)d \quad \text{and} \quad a + (n-1)d$$

are in HP. Hence

$$a+(m-1)d=\frac{2[a+(l-1)d][a+(n-1)d]}{[a+(l-1)d]+[a+(n-1)d]}$$

$$2[a+(l-1)d][a+(n-1)d]$$
$$=[a+(m-1)d][2a+d(l+m-2)]$$

$$2ad(l+n-2)+2(l-1)(n-1)d^2$$
$$=ad(l+n-2+2m-2)+(m-1)(l+n-2)d^2$$

$$2a(l+n-2)+2(l-1)(n-1)d$$
$$=a(l+n+2m-4)+(m-1)(l+n-2)d^2$$

$$a[2l+2n-4-l-n-2m+4]$$
$$=d[(m-1)(l+n-2)-2(l-1)(n-1)]$$

$$a(l+n-2m)$$
$$=d(m+1)(l+n-2m)\quad (\because m^2=ln)$$

$$\frac{a}{d}=m+1$$

Answer: (D)

59. If $a_1, a_2, a_3, \ldots, a_n$ are in HP and

$$f(K)=\left(\sum_{r=1}^{n} a_r\right)-a_K$$

then

$$\frac{a_1}{f(1)}, \frac{a_2}{f(2)}, \ldots, \frac{a_n}{f(n)}$$

are in

(A) AP (B) GP (C) HP (D) AGP

Solution: It is given that

$$f(K)+a_K=\sum_{r=1}^{n} a_r$$

and $\frac{1}{a_1}, \frac{1}{a_2}, \ldots, \frac{1}{a_n}$ are in AP

Therefore

$$\frac{\sum_{r=1}^{n} a_r}{a_1}, \frac{\sum_{r=1}^{n} a_r}{a_2}, \ldots, \frac{\sum_{r=1}^{n} a_r}{a_n} \text{ are in AP}$$

$$\frac{a_1+f(1)}{a_1}, \frac{a_2+f(2)}{a_2}, \ldots, \frac{a_n+f(n)}{a_n} \text{ are in AP}$$

$$\frac{f(1)}{a_1}, \frac{f(2)}{a_2}, \ldots, \frac{f(n)}{a_n} \text{ are in AP}$$

Finally,

$$\frac{a_1}{f(1)}, \frac{a_2}{f(2)}, \ldots, \frac{a_n}{f(n)} \text{ are in HP}$$

Answer: (C)

60. Two AMs A_1 and A_2, two GMs G_1 and G_2 and two HMs H_1 and H_2 are inserted between two given non-zero real numbers x and y. Then

$$\frac{1}{H_1}+\frac{1}{H_2}=$$

(A) $\frac{1}{A_1}+\frac{1}{A_2}$ (B) $\frac{1}{G_1}+\frac{1}{G_2}$

(C) $\frac{G_1G_2}{A_1+A_2}$ (D) $\frac{A_1+A_2}{G_1G_2}$

Solution: Given that

$$A_1=x+\frac{y-x}{3}=\frac{2x+y}{3} \quad \text{and} \quad A_2=\frac{x+2y}{3}$$

$$G_1=x\left(\frac{y}{x}\right)^{1/3}=x^{2/3}y^{1/3} \quad \text{and} \quad G_2=x^{1/3}y^{2/3}$$

$$H_1=\frac{3xy}{3y+(x-y)}=\frac{3xy}{x+2y} \quad \text{and} \quad H_2=\frac{3xy}{2x+y}$$

Therefore

$$\frac{1}{H_1}+\frac{1}{H_2}=\frac{(x+2y)+(2x+y)}{3xy}$$
$$=\frac{3(x+y)}{3xy}=\frac{x+y}{xy}=\frac{A_1+A_2}{G_1G_2}$$

Answer: (D)

61. Let $a_1, a_2, \ldots, a_{10}$ be in AP and $h_1, h_2, \ldots, h_{10}$ be in HP. If $a_1=h_1=2$ and $a_{10}=h_{10}=3$, then $a_4h_7=$

(A) 2 (B) 3 (C) 5 (D) 6

Solution: Given $a_1, a_2, \ldots, a_9$ are 8 AMs between 2 and 3 and $h_1, h_2, \ldots, h_9$ be 8 HMs between 2 and 3. Therefore

$$a_4=2+3\left(\frac{3-2}{9}\right)=\frac{7}{3}$$

and
$$h_7=\frac{6(9)}{3(9)+6(2-3)}=\frac{54}{21}$$

Therefore

$$a_4h_7=\frac{7}{3}\cdot\frac{54}{21}=6$$

Answer: (D)

62. If a, b, c are distinct real numbers and are in AP and a^2, b^2, c^2 are in HP, then

(A) $a, b, c/2$ are in GP (B) $a, b, -c/2$ are in GP

(C) $a/2, b, c$ are in GP (D) $a, b/2, c/2$ are in GP

Solution: By hypothesis,

$$2b=a+c \quad \text{and} \quad \frac{1}{b^2}-\frac{1}{a^2}=\frac{1}{c^2}-\frac{1}{b^2}$$

Therefore

$$\frac{(a-b)(a+b)}{a^2b^2} = \frac{(b-c)(b+c)}{b^2c^2}$$

$$\frac{a+b}{a^2} = \frac{b+c}{c^2} \quad (\text{since } b-a=c-b)$$

$$c^2a + c^2b = a^2b + a^2c$$

$$ac(c-a) + b(c^2-a^2) = 0$$

$$ac + b(c+a) = 0 \quad (\text{since } c \neq a)$$

$$ac + 2b^2 = 0 \quad (\text{since } c+a=2b)$$

Hence $a, b, -c/2$ are in GP.

Answer: (B)

63. If the roots of the equation $10x^3 - Kx^2 - 54x - 27 = 0$ are in HP, then K is equal to

(A) 3 (B) 6 (C) 9 (D) 12

Solution: Let α, β, γ be the roots of the given equation. Then

$$\alpha+\beta+\gamma = K, \alpha\beta+\beta\gamma+\gamma\alpha = -54, \alpha\beta\gamma = 27$$

Now α, β, γ are in HP, and hence $1/\alpha$, $1/\beta$, $1/\gamma$ are in AP. This gives

$$\frac{-54}{27} = \frac{\alpha\beta+\beta\gamma+\gamma\alpha}{\alpha\beta\gamma} = \frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} = \frac{3}{\beta}$$

$$\beta = \frac{-3}{2}$$

Since β is a root of the given equation, substituting the value of β in it we get

$$10\left(\frac{-3}{2}\right)^3 - K\left(\frac{-3}{2}\right)^2 - 54\left(\frac{-3}{2}\right) - 27 = 0$$

$$\frac{-270}{8} - \frac{9K}{4} + 81 - 27 = 0$$

$$\frac{9K}{4} = \frac{162}{8}$$

$$K = 9$$

Answer: (C)

64. Let G and H be, respectively, the GM and HM between two numbers. If $H:G = 4:5$, then the ratio of the numbers can be

(A) 1 : 4 (B) 5 : 9 (C) 9 : 2 (D) 3 : 4

Solution: Let a and b be the numbers. Then

$$G = \sqrt{ab} \quad \text{and} \quad H = \frac{2ab}{a+b}$$

Given that

$$\frac{H}{G} = \frac{4}{5}$$

Therefore

$$\frac{2ab}{\sqrt{ab}(a+b)} = \frac{4}{5}$$

$$10\sqrt{ab} = 4(a+b)$$

$$5\sqrt{\frac{a}{b}} = 2\left(\frac{a}{b}+1\right)$$

Let $x = \sqrt{a/b}$. Substituting this we get

$$5x = 2(x^2+1)$$

$$(x-2)(2x-1) = 0$$

$$x = 2 \quad \text{or} \quad \frac{1}{2}$$

(i) When $x = 2$, $a/b = 4$.

(ii) When $x = 1/2$, $a/b = 1/4$.

Answer: (A)

65. If a, b, c are in GP and $a-b, c-a, b-c$ are in HP, then the value of $a+4b+c$ is equal to

(A) 1 (B) 0 (C) $2abc$ (D) b^2+ac

Solution: It is given that $b = ar$ and $c = ar^2$. Also

$$\frac{2}{c-a} = \frac{1}{a-b} + \frac{1}{b-c}$$

$$\frac{2}{ar^2-a} = \frac{1}{a-ar} + \frac{1}{ar-ar^2}$$

$$\frac{2}{r^2-1} = \frac{1}{1-r} + \frac{1}{r(1-r)} = \frac{r+1}{r(1-r)}$$

$$\frac{-2}{r+1} = \frac{r+1}{r}$$

$$-2r = (r+1)^2$$

$$r^2 + 4r + 1 = 0$$

$$ar^2 + 4ar + a = 0$$

$$c + 4b + a = 0$$

Answer: (B)

66. If a, b, c are in AP, b, c, d are in GP and c, d, e are in HP, then a, c, e are in

(A) AP (B) GP (C) HP (D) AGP

Solution: It is given that

$$b = \frac{a+c}{2}, c^2 = bd \quad \text{and} \quad d = \frac{2ce}{c+e}$$

Therefore

$$c^2 = bd = \frac{a+c}{2}\cdot\frac{2ce}{c+e}$$

$$c^2(c+e) = ce(a+c)$$

$$c^2 = ea$$

This implies a, c, e are in GP.

Answer: (B)

67. If $a^x = b^y = c^z$ and a, b, c are in GP, then x, y, z, are in

(A) AP (B) GP (C) HP (D) AGP

Solution: From the hypothesis we have

$$a = b^{y/x}, c = b^{y/z} \quad \text{and} \quad b^2 = ac$$

Therefore

$$b^2 = b^{y/x}, b^{y/z} = b^{(y/x)+(y/z)}$$

$$2 = \frac{y}{x} + \frac{y}{z}$$

$$2xz = y(x+z)$$

$$y = \frac{2xz}{x+z}$$

Therefore x, y, z are in HP.

Answer: (C)

68. Sum to n terms of the series $1 + 5 + 19 + 65 + 211 + \cdots$ is equal to

(A) $1/2(3^{n+1} - 2^{n+2} + 1)$ (B) $1/2(3^n + 2^n - 1)$

(C) $3^{n+1} - 2^{n+1} - 1$ (D) $3^n + 2^n + 1$

Solution: We have

$$1 = 3-2,\quad 5 = 3^2 - 2^2,\quad 19 = 3^3 - 2^3,$$
$$65 = 3^4 - 2^4,\quad 211 = 3^5 - 2^5, \ldots$$

Therefore sum to n terms is

$$\sum_{K=1}^{n}(3^K - 2^K) = (3 + 3^2 + \cdots + 3^n) - (2 + 2^2 + \cdots + 2^n)$$

$$= \frac{3(3^n-1)}{3-1} - \frac{2(2^n-1)}{2-1}$$

$$= \frac{3}{2}(3^n - 1) - 2(2^n - 1)$$

$$= \frac{1}{2}(3^{n+1} - 2^{n+2} + 1)$$

Answer: (A)

69. For any positive integer n, let

$$f(n) = \frac{4n + \sqrt{4n^2-1}}{\sqrt{2n+1} + \sqrt{2n-1}}$$

Then $\sum_{K=1}^{40} f(K) =$

(A) 365 (B) 366 (C) 364 (D) 363

Solution: Let $x = \sqrt{2n+1}$ and $y = \sqrt{2n-1}$. Then

$$x^2 + y^2 = 4n$$

$$x^2 - y^2 = 2$$

$$xy = \sqrt{4n^2 - 1}$$

Therefore

$$f(n) = \frac{x^2 + y^2 + xy}{x+y} = \frac{x^3 - y^3}{x^2 - y^2} = \frac{1}{2}[(2n+1)^{3/2} - (2n-1)^{3/2}]$$

Substituting $n = 1$ to 40 we get

$$f(1) = \frac{1}{2}(3^{3/2} - 1^{3/2})$$

$$f(2) = \frac{1}{2}(5^{3/2} - 3^{3/2})$$

$$f(3) = \frac{1}{2}(7^{3/2} - 5^{3/2})$$

$$\vdots$$

$$f(40) = \frac{1}{2}(81^{3/2} - 79^{3/2})$$

Therefore

$$\sum_{n=1}^{40} f(n) = \frac{1}{2}(81^{3/2} - 1^{3/2}) = 364$$

Answer: (C)

70. Let

$$x = \sqrt{1 + \frac{1}{1^2} + \frac{1}{2^2}} + \sqrt{1 + \frac{1}{2^2} + \frac{1}{3^2}} + \sqrt{1 + \frac{1}{3^2} + \frac{1}{4^2}} + \cdots + \sqrt{1 + \frac{1}{2009^2} + \frac{1}{2010^2}}$$

Then

$$\frac{(2010)x - 2009}{2010} =$$

(A) 2010 (B) 2009 (C) 1999 (D) 2000

Solution: Let

$$T_n = \sqrt{1 + \frac{1}{n^2} + \frac{1}{(n+1)^2}} = \sqrt{\frac{n^2(n+1)^2 + (n+1)^2 + n^2}{n^2(n+1)^2}}$$

Then

$$T_n = \frac{n^2+n+1}{n(n+1)} = 1 + \frac{1}{n(n+1)} = 1 + \frac{1}{n} - \frac{1}{n+1}$$

Therefore,

$$x = \sum_{n=1}^{2009} T_n = \sum_{n=1}^{2009}\left(1 + \frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 2009 + \left(1 - \frac{1}{2009+1}\right)$$

$$= \frac{(2010)^2 - 1}{2010}$$

$$= \frac{2011 \cdot 2009}{2010}$$

Therefore

$$\frac{2010x - 2009}{2010} = 2009$$

Answer: (B)

71. Sum to infinity of the series

$$\frac{1}{2^2-1} + \frac{1}{4^2-1} + \frac{1}{6^2-1} + \cdots$$

is

(A) 1/3 (B) 1 (C) 2/3 (D) 1/2

Solution: The nth term

$$u_n = \frac{1}{4n^2-1} = \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$$

Therefore

$$s_n = u_1 + u_2 + \cdots + u_n = \frac{1}{2}\left(1 - \frac{1}{2n+1}\right)$$

Now sum to infinity is given by

$$s_\infty = \lim_{n\to\infty} s_n = \frac{1}{2}$$

Answer: (D)

72. Odd natural numbers are arranged in groups as (1), (3, 5), (7, 9, 11), (13, 15, 17, 19), (21, 23, 25, 27, 29), …. Then the sum of the natural numbers in the nth group is

(A) $n^3 + n$ (B) n^3 (C) $(n+1)^3$ (D) $n^3 - n$

Solution: The nth group consists of n natural odd numbers. Let u_n be the first number in the nth group. We thus have

$$u_1 = 1, u_2 = 3, u_3 = 7, u_4 = 13, u_5 = 21, u_6 = 31, \ldots$$

Now

$$\Delta u_n = 3-1, 7-3, 13-7, 21-13, 31-21$$

$$= 2, 4, 6, 8, 10$$

$$\Delta^2 u_n = 2, 2, 2, \ldots$$

which is an AP. Then, using Sec. 5.5, IV(i), we get

$$u_n = 1 + \frac{2(n-1)}{1!} + \frac{2(n-1)(n-2)}{2!}$$

$$= 1 + 2n - 2 + n^2 - 3n + 2$$

$$= n^2 - n + 1$$

The sum of the numbers in the nth group is

$$(n^2 - n + 1) + (n^2 - n + 3) + (n^2 - n + 5) + \cdots + (n^2 - n + 2n - 1)$$

$$= n(n^2 - n) + [1 + 3 + 5 + \cdots + (2n-1)]$$

$$= n^3 - n^2 + n^2 = n^3$$

Answer: (B)

73. Sum to n terms of the series 1, 2, 3, 6, 17, 54, 171, … is

(A) $1/8(3^n - 1) - n/12(2n^2 - 9n - 2)$

(B) $1/8(3^n + 1) + n/12(2n^2 - 9n + 2)$

(C) $1/8(3^n - 1) + n/12(2n^2 - 9n + 2)$

(D) $1/8(3^n + 1) - n/12(2n^2 + 9n - 2)$

Solution: Given series is 1, 2, 3, 6, 17, 54, 171, …

$$\Delta u_n = 1, 1, 3, 11, 37, 117, \ldots$$

$$\Delta^2 u_n = 0, 2, 8, 26, 80, \ldots$$

$$\Delta^3 u_n = 2, 6, 18, 54, \ldots,$$

which is a GP with common ratio 3. By Sec. 5.5, IV(ii), we get

$$u_n = a3^{n-1} + a_0 + a_1(n-1) + a_2(n-1)(n-2)$$

Therefore

$$1 = u_1 = a + a_0$$

$$2 = u_2 = 3a + a_0 + a_1$$

$$3 = u_3 = 9a + a_0 + 2a_1 + 2a_2$$

$$6 = u_4 = 27a + a_0 + 3a_1 + 6a_2$$

Solving the above set of equations for a, a_0, a_1, a_2, we get that

$$a = \frac{1}{4}, a_0 = \frac{3}{4}, a_1 = \frac{1}{2}, a_2 = -\frac{1}{2}$$

Hence

$$u_n = \frac{3^{n-1}}{4} + \frac{3}{4} + \frac{n-1}{2} - \frac{(n-1)(n-2)}{2}$$

$$s_n = \frac{1}{4}(1 + 3 + 3^2 + \cdots + 3^{n-1}) + \frac{3n}{4} + \frac{n(n-1)}{4} - \frac{1}{2}\frac{(n-2)(n-1)n}{3}$$

$$= \frac{1}{8}(3^n - 1) - \frac{n}{12}(2n^2 - 9n - 2)$$

Answer: (A)

74. Sum to n terms of the series

$$\frac{1^4}{1\cdot 3}+\frac{2^4}{3\cdot 5}+\frac{3^4}{5\cdot 7}+\cdots$$

is

(A) $n(n+1)(2n^2+1)/8(2n+1)$
(B) $n(n+1)(n^2+n+1)/6(2n+1)$
(C) $(n+1)[(2n+1)^2+1]/8(2n+1)$
(D) $n(n+1)[(2n+1)^2+1]/16(2n+1)$

Solution: The Kth term of the given series is

$$u_K=\frac{K^4}{(2K-1)(2K+1)}=\frac{K^2}{4}+\frac{1}{16}+\frac{1}{32}\left(\frac{1}{2K-1}-\frac{1}{2K+1}\right)$$

Therefore the sum of n terms is

$$\begin{aligned}s_n&=u_1+u_2+\cdots+u_n\\&=\frac{1}{4}\sum_{K=1}^{n}K^2+\frac{n}{16}+\frac{1}{32}\left(1-\frac{1}{2n+1}\right)\\&=\frac{n(n+1)(2n+1)}{4\cdot 6}+\frac{n}{16}+\frac{n}{16(2n+1)}\\&=\frac{n(n+1)(n^2+n+1)}{6(2n+1)}\end{aligned}$$

Answer: (B)

Multiple Correct Choice Type Questions

1. Consider the AP

$$20, 19\frac{1}{3}, 18\frac{2}{3}, 18, \ldots$$

Then

(A) sum of the first 25 terms is 300
(B) sum of the first 36 terms is 300
(C) sum of the terms from 26th term to 36th term is zero
(D) the sum of all the non-negative terms is 310

Solution: The given series is an AP with first term $a=20$ and the common difference $d=-2/3$. Let

$$s_n=\frac{n}{2}[2a+(n-1)d]$$

be the sum of the first n terms. The sum of the first 25 terms is given by

$$\begin{aligned}s_{25}&=\frac{25}{2}\left[40+24\left(-\frac{2}{3}\right)\right]\\&=25\left[20+12\left(-\frac{2}{3}\right)\right]=25\times 12=300\end{aligned}$$

Again sum of first 36 terms is

$$s_{36}=\frac{36}{2}\left[40+35\left(-\frac{2}{3}\right)\right]=18\left(\frac{50}{3}\right)=300$$

Now $s_{36}=s_{25}$ implies that the sum of the terms from 26th to 36th is zero.

If T_n is the last non-negative term, then

$$\begin{aligned}&T_n=20+(n-1)\left(-\frac{2}{3}\right)\ge 0\\ \Rightarrow\quad &60-2n+2\ge 0\\ \Rightarrow\quad &n\le 31\end{aligned}$$

When $n=31$, $T_n=0$. Therefore

$$s_{31}=\frac{31}{2}(20+0)=310$$

Answers: (A), (B), (C), (D)

2. The sum of the first 8 and 19 terms of an AP is 64 and 361, respectively. Then

(A) common difference is 2
(B) first term is 1
(C) sum of the first n terms is n^2
(D) nth term is $2n$

Solution: The sum of first 8 terms is 64; therefore,

$$2a+7d=16$$

The sum of first 19 terms is 361; therefore,

$$19a+171d=361$$

Solving these two equations in two variables, we get $d=2, a=1$.

Answers: (A), (B), (C)

3. Let an^2+bn be the sum of the first n terms of an AP. Then

(A) first term is $a+b$
(B) first term is $a-b$
(C) common difference is $2a$
(D) common differences is $b-a$

Solution: By hypothesis

$$s_n=an^2+bn$$

The nth term is given by

$$s_n-s_{n-1}=a[n^2-(n-1)^2]+b[n-(n-1)]=(2n-1)a+b$$

Therefore, the series is $a+b, 3a+b, 5a+b, \ldots$. In this case the first term is $a+b$ and the common difference is $2a$.

Answers: (A), (C)

4. The numbers a, b, c and A, B, C are in AP. The common difference of the second set is one more than the common difference of the first. If

$$a + b + c = A + B + C = 15$$

and $$\frac{abc}{ABC} = \frac{7}{8}$$

then

(A) $a = 7,\ A = 6$ (B) $B = b = 5$

(C) $a = 5,\ A = 6$ (D) $a = 6,\ A = 5$

(It is given that the two sets of numbers are in the descending order.)

Solution: Let $a = \alpha - d, b = \alpha, c = \alpha + d$. By hypothesis

$$A = \beta - (d+1), B = \beta, C = \beta + (d+1)$$

which gives

$$3\alpha = 15 = 3\beta \Rightarrow \alpha = 5 = \beta$$

Now

$$\frac{7}{8} = \frac{abc}{ABC} = \frac{\alpha(\alpha^2 - d^2)}{\beta[\beta^2 - (d+1)^2]}$$

$$7[25 - (d+1)^2] = 8(25 - d^2)$$

$$8d^2 - (d+1)^2\, 7 = 25$$

$$d^2 - 14d - 32 = 0$$

$$d = -2, 16$$

Therefore if $d = -2$, then

$$a = \alpha - d = 7, A = 5 - (-2+1) = 6 \quad \text{and} \quad b = 5 = B$$

Answers: (A), (B)

5. The second, 31st and last terms of an AP are, respectively, 31/4, 1/2 and –13/2. Then

(A) first term is 8 (B) number of terms is 58

(C) number of terms is 59 (D) first term is 6

Solution: Let a be the first term and d the common difference. Also let the nth term be the last term. Then

$$a + d = \frac{31}{4} \qquad (5.34)$$

$$a + 30d = \frac{1}{2} \qquad (5.35)$$

$$a + (n-1)d = -\frac{13}{2} \qquad (5.36)$$

From Eqs. (5.34) and (5.35) we get

$$d = -\frac{1}{4},\ a = 8$$

Substituting these values in Eq. (5.36), we get $n = 59$.

Answers: (A), (C)

6. If a^2, b^2, c^2 are in AP, then

(A) $1/(b+c), 1/(c+a), 1/(a+b)$ are in AP

(B) $a/(b+c), b/(c+a), c/(a+b)$ are in AP

(C) $(b+c-a)/a, (c+a-b)/b, (a+b-c)/c$ are not in AP

(D) $1/a, 1/b, 1/c$ are in AP

Solution: By hypothesis $b^2 - a^2 = c^2 - b^2$. Therefore

$$(b-a)(b+a) = (c-b)(c+b) \qquad (5.37)$$

(A) We have

$$\frac{1}{c+a} - \frac{1}{b+c} = \frac{b-a}{(c+a)(c+b)}$$

$$= \frac{b^2 - a^2}{(a+b)(b+c)(c+a)} \qquad (5.38)$$

$$\frac{1}{a+b} - \frac{1}{c+a} = \frac{c-b}{(a+b)(c+a)}$$

$$= \frac{c^2 - b^2}{(a+b)(b+c)(c+a)} \qquad (5.39)$$

Equations (5.38), (5.39) and $b^2 - a^2 = c^2 - b^2$ give

$$\frac{1}{c+a} - \frac{1}{b+c} = \frac{1}{a+b} - \frac{1}{c+a}$$

Therefore

$$\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$$

are in AP. Therefore (A) is true.

(B) We have

$$\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b} \quad \text{are in AP}$$

$$\Leftrightarrow \frac{a}{b+c}+1, \frac{b}{c+a}+1, \frac{c}{a+b}+1 \quad \text{are in AP}$$

$$\Leftrightarrow \frac{a+b+c}{b+c}, \frac{a+b+c}{c+a}, \frac{a+b+c}{a+b} \quad \text{are in AP}$$

$$\Leftrightarrow \frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b} \quad \text{are in AP and this is true}$$

Therefore

$$\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$$

are in AP. Therefore (B) is true.

(C) We have

$$\frac{b+c-a}{a}, \frac{c+a-b}{b} \quad \text{and} \quad \frac{a+b-c}{c}$$

are in AP. This implies

$$\frac{b+c}{a}, \frac{c+a}{b}, \frac{a+b}{c}$$

are in AP and this is not true according to B. Therefore (C) is not true.

(D) We have

$$\frac{1}{a}, \frac{1}{b}, \frac{1}{c} \text{ are in AP} \Leftrightarrow \frac{1}{b}-\frac{1}{a}=\frac{1}{c}-\frac{1}{b}$$

$$\Leftrightarrow \frac{a-b}{ab}=\frac{b-c}{bc}$$

$$\Leftrightarrow \frac{a-b}{b-c}=\frac{a}{c}$$

This is not true because from Eq. (5.37). Then

$$\frac{a-b}{b-c}=\frac{b+c}{b+a}$$

Therefore (D) is not true.

Answers: (A), (B)

7. Consider an infinite geometric series with first term a and common ratio r. If its sum is 4 and the second term is 3/4, then

(A) $a=4/7, r=3/7$ (B) $a=1, r=3/4$

(C) $a=3/2, r=1/2$ (D) $a=3, r=1/4$

Solution: By hypothesis

$$ar=\frac{3}{4} \quad \text{and} \quad \frac{a}{1-r}=4$$

Dividing the first equation by the second we get

$$r(1-r)=\frac{3}{4}\cdot\frac{1}{4}=\frac{3}{16}$$

$$16r^2-16r+3=0$$

$$(4r-1)(4r-3)=0$$

$$r=\frac{1}{4} \quad \text{or} \quad \frac{3}{4}$$

(i) $r=1/4 \Rightarrow a=3$

(ii) $r=3/4 \Rightarrow a=1$

Answers: (B), (D)

8. The sum of the first two terms of an infinite GP is equal to 5 and every term is three times the sum of all the terms that follow. If a and r are the first term and common ratio, respectively, then

(A) $a=4$ (B) $a=3$

(C) $r=1/4$ (D) $r=2/3$

Solution: By hypothesis

$$a+ar=5 \quad \text{and} \quad a=3\left(\frac{ar}{1-r}\right)$$

Therefore $1-r=3r \Rightarrow r=1/4$. This value of r gives $a=4$.

Answers: (A), (C)

9. If x, y, z are in GP and $x+y, y+z, z+x$ are in AP in that order, then

(A) common ratio of the GP is 2

(B) $x=y=z$

(C) common ratio of the GP is –2

(D) common ratio is 1/2

Solution: Let $y=xr$ and $z=xr^2$. Then

$$x+y=x(1+r), \quad y+z=xr(1+r), \quad z+x=x(r^2+1)$$

These are in AP. Therefore

$$2xr(1+r)=x(1+r)+x(r^2+1)=x(r^2+r+2)$$

$$2r(1+r)=2+r+r^2$$

$$r^2+r-2=0$$

$$(r+2)(r-1)=0$$

$$r=1 \quad \text{or} \quad -2$$

Answers: (B), (C)

10. The first two terms of an infinitely decreasing GP are $\sqrt{3}$ and $2/(\sqrt{3}+1)$. Then the

(A) common ratio is $(\sqrt{3}-1)/\sqrt{3}$

(B) sum to infinity of the GP is $3\sqrt{3}$

(C) common ratio is $1/\sqrt{3}$

(D) sum to infinity is 3

Solution: The common ratio is

$$\frac{2}{(\sqrt{3}+1)\sqrt{3}}=\frac{\sqrt{3}-1}{\sqrt{3}}$$

The sum to infinity is

$$\frac{\sqrt{3}}{1-[(\sqrt{3}-1)/\sqrt{3}]}=\frac{\sqrt{3}}{1/\sqrt{3}}=3$$

Answers: (A), (D)

11. If $d \neq 0$ and the sequence $a(a+d)$, $(a+d)(a+2d)$, $(a+2d)\,a$ forms a GP, then

(A) common ratio of the GP is -2

(B) $3a = -2d$

(C) $a = -2d$

(D) common ratio is 2

Solution: It is given that

$$\frac{(a+d)(a+2d)}{a(a+d)} = \frac{(a+2d)a}{(a+d)(a+2d)}$$

$$\frac{a+2d}{a} = \frac{a}{a+d}$$

$$a^2 + 3ad + 2d^2 = a^2$$

$$d(3a+2d) = 0$$

Therefore $3a = -2d$ (since $d \neq 0$). Hence, the common ratio is

$$\frac{a+2d}{a} = \frac{a-3a}{a} = -2$$

Answers: (A), (B)

***Note*:** If x, y, z are in AP with a non-zero common difference and xy, yz, zx are in GP, then common ratio of the GP is -2 and also $3x = -2(z-y)$.

12. The sum of the first three terms of a GP is 6 and the sum of its first three odd terms is 10.5. Then the first term and the common ratio are

(A) $8, -1/2$ (B) $8, 1/2$

(C) $24/19, 3/2$ (D) $24/29, 3/2$

Solution: Let $a, ar, ar^2, \ldots$ be the GP. It is given that

$$a(1+r+r^2) = 6 \tag{5.40}$$

$$a(1+r^2+r^4) = \frac{21}{2} \tag{5.41}$$

Dividing Eq. (5.41) by Eq. (5.40) we get

$$\frac{1+r^2+r^4}{1+r+r^2} = \frac{21}{12} = \frac{7}{4}$$

$$4 + 4r^2 + 4r^4 = 7 + 7r + 7r^2$$

$$4r^4 - 3r^2 - 7r - 3 = 0$$

Now $r = -1/2$ is a solution. We have

$$(2r+1)(2r^3 - r^2 - r - 3) = 0$$

$$(2r+1)(2r-3)(r^2+r+1) = 0$$

Therefore

$$r = \frac{-1}{2}, \frac{3}{2}$$

(i) When $r = -1/2$, $a = 8$

(ii) When $r = 3/2$, $a = 24/19$

Answers: (A), (C)

13. The ratio of the sum of the cubes of an infinitely decreasing GP to the sum of its squares is 12:13. The sum of the first and second terms is equal to 4/3. If a, r and s_∞ denote the first term, common ratio and sum to infinity of the GP, then

(A) $r = 1/3, a = 6/5, s_\infty = 9/5$

(B) $r = 1/3, a = 1$

(C) $r = -4/3, a = -1/3, s_\infty = -1/7$

(D) $s_\infty = 3/2$

Solution: By hypothesis, $|r| < 1$ and

$$\frac{a^3}{1-r^3} : \frac{a^2}{1-r^2} = 12:13$$

$$\frac{a(1-r^2)}{1-r^3} = \frac{12}{13}$$

Therefore

$$13a(1+r) = 12(1+r+r^2) \tag{5.42}$$

Also, given that

$$a + ar = \frac{4}{3} \tag{5.43}$$

From Eqs. (5.42) and (5.43), we get

$$13 \times \frac{4}{3} = 12(1+r+r^2)$$

$$13 = 9(1+r+r^2)$$

$$9r^2 + 9r - 4 = 0$$

$$(3r-1)(3r+4) = 0$$

The values of r thus obtained are

$$r = \frac{1}{3} \quad \text{or} \quad \frac{-4}{3}$$

(i) When $r = 1/3$, then $a = 1$ and

$$s_\infty = \frac{1}{1-(1/3)} = \frac{3}{2}$$

(ii) When $r = -4/3$, then $a = 4$.

Therefore (C) is not possible.

Answers: (B), (D)

14. Let

$$x = \log_{\sqrt{5}}\left(\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots \infty\right)$$

Then

(A) $x = -2\log_5 2$ (B) $5^x = 2$

(C) $(0.2)^x = 4$ (D) $(0.2)^x = 2$

Solution: We have

$$\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots=\frac{1/4}{1-(1/2)}=\frac{1}{2}$$

Therefore

$$x=\log_{\sqrt{5}}\left(\frac{1}{2}\right)=-2\log_5 2$$

$$(0.2)^x=\left(\frac{1}{5}\right)^{-2\log_5 2}=5^{\log_5 4}=4$$

$$5^x=5^{-2\log_5 2}=5^{\log_5(1/4)}=\frac{1}{4}$$

Answers: (A), (C)

15. If s_1, s_2 and s_3 are, respectively, equal to the sums of the first $n, 2n$ and $3n$ terms of a GP, then

(A) $s_1(s_3-s_2)=(s_2-s_1)^2$ (B) $s_1^2+s_2^2=s_1(s_2+s_3)$

(C) $s_1(s_2+s_3)=(s_1+s_2)^2$ (D) $s_2^2=s_1s_3$

Solution: Let the GP be $a, ar, ar^2, \ldots$. Given that

$$s_1=\frac{a(1-r^n)}{1-r};\quad s_2=\frac{a(1-r^{2n})}{1-r};\quad s_3=\frac{a(1-r^{3n})}{1-r}$$

Then

$$s_3-s_2=\frac{a}{1-r}(r^{2n}-r^{3n})=\frac{r^{2n}a(1-r^n)}{1-r}=r^{2n}s_1$$

$$s_1(s_3-s_2)=\left[\frac{a(1-r^n)}{1-r}r^n\right]^2$$

$$s_2-s_1=\frac{ar^n(1-r^n)}{1-r}$$

$$(s_2-s_1)^2=s_1(s_3-s_2)$$

Also,

$$(s_2-s_1)^2+2s_1s_2=s_1(s_3-s_2+2s_2)$$

and hence

$$s_1^2+s_2^2=s_1(s_3+s_2)$$

Answers: (A), (B)

16. If a, b, c and d are in GP, then

(A) $(a^2+b^2+c^2)(b^2+c^2+d^2)=(ab+bc+cd)^2$

(B) $(a-d)^2=(b-c)^2+(c-a)^2+(d-b)^2$

(C) $a^2-b^2, b^2-c^2, c^2-d^2$ are in GP

(D) $a^2+b^2, b^2+c^2, c^2+d^2$ are in GP

Solution: Let $b=ar, c=ar^2$ and $d=ar^3$.

(A) $(a^2+b^2+c^2)(b^2+c^2+d^2)$

$$=a^2(1+r^2+r^4)a^2(r^2+r^4+r^6)$$
$$=a^4r^2(1+r^2+r^4)^2$$
$$=(a^2r+a^2r^3+a^2r^5)^2$$
$$=(ab+bc+cd)^2$$

Therefore (A) is true.

(B) $(b-c)^2+(c-a)^2+(d-b)^2$

$$=(ar-ar^2)^2+(ar^2-a)^2+(ar^3-ar)^2$$
$$=a^2[(r^4-2r^3+r^2)+(r^4-2r^2+1)$$
$$+(r^6-2r^4+r^2)]$$
$$=a^2(r^6-2r^3+1)$$
$$=a^2(r^3-1)^2$$
$$=(d-a)^2$$

Therefore (B) is true.

(C) $a^2-b^2=a^2(1-r^2), b^2-c^2=a^2(r^2-r^4)=a^2r^2(1-r^2)$

and $\quad c^2-d^2=a^2(r^4-r^6)=a^2r^4(1-r^2)$

So,

$$(a^2-b^2)(c^2-d^2)=a^4r^4(1-r^2)^2=(b^2-c^2)^2$$

Therefore (C) is true.

(D) $a^2+b^2=a^2(1+r^2), b^2+c^2=a^2r^2(1+r^2)$

and $\quad c^2+d^2=a^2r^4(1+r^2)$

So,

$$(a^2+b^2)(c^2+d^2)=a^4r^4(1+r^2)^2=(b^2+c^2)^2$$

Therefore (D) is true.

Answers: (A), (B), (C), (D)

17. Let a, b, c be three distinct real numbers in GP. If x is real and $a+b+c=xb$, then

(A) $x<-1$ (B) $0<x<1$ (C) $2<x<3$ (D) $x>3$

Solution: Let $b=ar$ and $c=ar^2$, and $r\neq 1$. Then

$$a(1+r+r^2)=xar$$

Therefore

$$r^2+(1-x)r+1=0$$
$$(1-x)^2-4\geq 0$$
$$x^2-2x-3\geq 0$$
$$(x+1)(x-3)\geq 0$$
$$x\leq -1 \quad\text{or}\quad x\geq 3$$

(i) When $x = -1$, then $r^2 + 2r + 1 = 0$ and hence $r = -1$, so that $a = c$, a contradiction to the hypothesis.

(ii) When $x = 3$, then $r^2 - 2r + 1 = 0$ and hence $r = 1$, so that $a = b = c$.

Therefore $x < -1$ or $x > 3$.

Answers: (A), (D)

18. Let a and b be positive real numbers. If a, A_1, A_2, b are in AP, a, G_1, G_2, b are in GP and a, H_1, H_2, b are in HP, then

$$\frac{G_1G_2}{H_1H_2} =$$

(A) $(2a+b)(a+2b)/9ab$ (B) $(H_1H_2)/(A_1A_2)$

(C) $(A_1+A_2)/(H_1+H_2)$ (D) $(H_1+H_2)/(A_1+A_2)$

Solution: It is given that

$$A_1 = a + \frac{b-a}{3} = \frac{2a+b}{3}, \; A_2 = a + \frac{2(b-a)}{3} = \frac{a+2b}{3}$$

$$G_1 = a\left(\frac{b}{a}\right)^{1/3} = a^{2/3}\cdot b^{1/3}, \; G_2 = a\left(\frac{b}{a}\right)^{2/3} = a^{1/3}\cdot b^{2/3}$$

$$H_1 = \frac{3ab}{3b + 1(a-b)} = \frac{3ab}{a+2b}, \; H_2 = \frac{3ab}{3b+2(a-b)} = \frac{3ab}{2a+b}$$

Therefore

$$\frac{G_1G_2}{H_1H_2} = \frac{ab(a+2b)(2a+b)}{9a^2b^2} = \frac{(a+2b)(2a+b)}{9ab}$$

$$\frac{A_1+A_2}{H_1+H_2} = \left(\frac{2a+b}{3} + \frac{a+2b}{3}\right)\left(\frac{3ab}{a+2b} + \frac{3ab}{2a+b}\right)^{-1}$$

$$= \frac{(a+2b)(2a+b)}{9ab}$$

Answers: (A), (C)

19. Let s_n be the sum to n terms of the series

$$\frac{3}{1^2} + \frac{5}{1^2+2^2} + \frac{7}{1^2+2^2+3^2} + \frac{9}{1^2+2^2+3^2+4^2} + \cdots$$

Then

(A) $s_n = n/n+1$ (B) $s_n = 6n/n+1$

(C) $s_\infty = 1$ (D) $s_\infty = 6$

Solution: Let

$$u_K = \frac{2K+1}{1^2+2^2+\cdots+K^2} = \frac{6}{K(K+1)} = 6\left(\frac{1}{K} - \frac{1}{K+1}\right)$$

Therefore sum of n terms is

$$s_n = u_1 + u_2 + \cdots + u_n = 6\left(1 - \frac{1}{n+1}\right) = \frac{6n}{n+1}$$

and sum to infinity is

$$s_\infty = \lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(\frac{6}{1+(1/n)}\right) = 6$$

Answers: (B), (D)

20. Let $a_1, a_2, a_3, a_4, \ldots$ be in GP. If the HM of a_1 and a_2 is 12 and that of a_2 and a_3 is 36, then

(A) $a_1 = 8$ (B) $a_2 = 24$

(C) $a_3 = 72$ (D) $a_4 = 216$

Solution: Let $a_2 = a_1 r, a_3 = a_1 r^2, a_4 = a_1 r^3, \ldots$. Then

$$12 = \frac{2a_1a_2}{a_1+a_2} = \frac{2a_1^2 r}{a_1(1+r)} = \frac{2ra_1}{1+r} \tag{5.44}$$

$$36 = \frac{2a_2a_3}{a_2+a_3} = \frac{2a_1^2r^3}{a_1(r+r^2)} = \frac{2r^2a_1}{1+r} \tag{5.45}$$

From Eqs. (5.44) and (5.45), we get

$$\frac{12}{2ra_1} = \frac{36}{2r^2a_1}$$

and hence $r = 3$.

From Eq. (5.44),

$$12 = \frac{6a_1}{1+3} = \frac{3}{2}a_1$$

and hence $a_1 = 8$. Therefore

$$a_2 = a_1 r = 24, a_3 = a_1r^2 = 72, a_4 = a_1 r^3 = 216$$

Answers: (A), (B), (C), (D)

21. Let $A_1, A_2; G_1, G_2$ and H_1, H_2 be two AMs, GMs and HMs, respectively, between two positive real numbers a and b. Then

(A) $A_1H_2 = ab$ (B) $G_1G_2 = ab$

(C) $A_1H_2 = a^2b^2$ (D) $A_2H_1 = ab$

Solution: By Problem 18 (solution)

$$A_1 = \frac{2a+b}{3}, \quad A_2 = \frac{a+2b}{3}$$

$$G_1 = a^{2/3}b^{1/3}, \quad G_2 = a^{1/3}b^{2/3}$$

$$H_1 = \frac{3ab}{a+2b}, \quad H_2 = \frac{3ab}{2a+b}$$

Therefore

$$A_1H_2 = \frac{2a+b}{3}\cdot\frac{3ab}{2a+b} = ab$$

$$G_1G_2 = a^{2/3}b^{1/3}\cdot a^{1/3}b^{2/3} = ab$$

$$A_2H_1 = \frac{a+2b}{3}\cdot\frac{3ab}{a+2b} = ab$$

Answers: (A), (B), (D)

22. Consider four non-zero real numbers a, b, c, d (in this order). If a, b, c are in AP and b, c, d are in HP, then

(A) $ac = bd$ (B) $a/b = c/d$

(C) $a/c = b/d$ (D) $a+b/a-b = c+d/c-d$

Solution: Since a, b, c are in AP,

$$2b = a + c$$

Since b, c, d are in HP,

$$c = \frac{2bd}{b+d}$$

$$c(b+d) = 2bd = (a+c)d$$

therefore

$$cb = ad \quad \text{and} \quad \frac{a}{c} = \frac{b}{d}, \frac{a+b}{a-b} = \frac{c+d}{c-d}$$

Answers: (B), (C), (D)

23. Let $a_1, a_2, a_3, \ldots, a_n$ be in AP and $h_1, h_2, h_3, \ldots, h_n$ be in HP. If $a_1 = h_1$ and $a_n = h_n$, then

(A) $a_r h_{n-r+1} = a_1 a_n$ (B) $a_{n-r+1} h_r = a_1 a_n$

(C) $a_r h_{n-r+1}$ is independent of r (D) $a_r h_r = a_1 a_n$

Solution: We have

$$a_r = a_1 + (r-1)\left(\frac{a_n - a_1}{n-1}\right)$$

$$= \frac{a_1 n - a_1 + (r-1)a_n - a_1 r + a_1}{n-1} \tag{5.46}$$

$$= \frac{a_1(n-r) + a_n(r-1)}{n-1}$$

$$h_{n-r+1} = \frac{a_1 a_n (n-1)}{a_n(n-1) + (n-r)(a_1 - a_n)}$$

$$= \frac{a_1 a_n (n-1)}{a_n(r-1) + a_1(n-r)} \tag{5.47}$$

From Eqs. (5.46) and (5.47), we get

$$a_r h_{n-r+1} = a_1 a_n = a_{n-r+1} h_r$$

Answers: (A), (B), (C)

24. If a, b, c are in HP, then

(A) $a/(b+c), b/(c+a), c/(a+b)$ are in AP

(B) $a/(b+c), b/(c+a), c/(a+b)$ are in HP

(C) $a/(b+c-a), b/(c+a-b), c/(a+b-c)$ are in HP

(D) $a/(a+b+c), b/(a+b+c), c/(a+b+c)$ are in HP

Solution: Given that $1/a, 1/b, 1/c$ are in AP. Therefore

$$\frac{a+b+c}{a}, \frac{a+b+c}{b}, \frac{a+b+c}{c} \text{ are in AP}$$

$$\Rightarrow \frac{a+b+c}{a} - 1, \frac{a+b+c}{b} - 1, \frac{a+b+c}{c} - 1 \text{ are in AP}$$

$$\Rightarrow \frac{b+c}{a}, \frac{c+a}{b}, \frac{a+b}{c} \text{ are in AP}$$

$$\Rightarrow \frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b} \text{ are in HP}$$

Therefore (B) is true. Also,

$$\frac{b+c}{a} - 1, \frac{c+a}{b} - 1, \frac{a+b}{c} - 1 \text{ are in AP}$$

$$\Rightarrow \frac{a}{b+c-a}, \frac{b}{c+a-b}, \frac{c}{a+b-c} \text{ are in HP}$$

Therefore (C) is true. Also,

$$\frac{a}{a+b+c}, \frac{b}{a+b+c}, \frac{c}{a+b+c} \text{ are in HP}$$

Therefore (D) is true.

Answers: (B), (C), (D)

25. If a, b, c are in AP; b, c, d are in GP and c, d, e are in HP, then

(A) a, c, e are in GP (B) $e = (2b-a)^2/a$

(C) a, c, e are in HP (D) $e = ab^2/(2a-b)^2$

Solution: By hypothesis,

$$2b = a + c, c^2 = bd \quad \text{and} \quad d = \frac{2ce}{c+e}$$

Therefore

$$c^2 = bd = \frac{a+c}{2}\cdot\frac{2ce}{c+e}$$

and also

$$c(e+c) = e(a+e)$$

This gives $c^2 = ae$ and hence (A) is true. Also,

$$e = \frac{c^2}{a} = \frac{(2b-a)^2}{a}$$

and hence (B) is true.

Answers: (A), (B)

26. In Problem 25, if $a = 2$ and $e = 18$, then the possible values of b, c, d are respectively,

(A) 4, 6, 9
(B) −2, −6, −18
(C) 6, 4, 9
(D) 2, 6, 18

Solution: By Problem 25, $c^2 = ae = 2 \times 18$ and hence $c = \pm 6$. Now, $2b = a + c = 2 \pm 6 = 8$ or -4 ($b = 4$ or -2). Therefore

$$d = \frac{c^2}{b} = \frac{36}{4} \quad \text{or} \quad \frac{36}{-2} = 9 \quad \text{or} \quad -18$$

Hence

$$b = 4, c = 6, d = 9$$

or

$$b = -2, c = -6, d = -18$$

Answers: (A), (B)

27. Which of the following statement(s) is (are) true?

(A) If $a^x = b^y = c^z$ and a, b, c are in GP, then x, y, z are in HP.
(B) If $a^{1/x} = b^{1/y} = c^{1/z}$ and a, b, c are in GP, then x, y, z are in AP.
(C) If a, b, c are positive, each of them not equal to 1, and are in GP, then, for any positive $u \neq 1$, $\log_a u, \log_b u, \log_c u$ are in HP.
(D) If a, b, c are in AP and b, c, a are in HP, then c, a, b are in GP.

Solution: (A) $a = b^{y/x}, c = b^{y/z}$ and $b^2 = ac$ imply that

$$b^2 = b^{(y/x)+(y/z)}$$

and hence

$$2 = y\left(\frac{1}{x} + \frac{1}{z}\right)$$

Therefore x, y, z are in HP. Thus (A) is true.

(B) $a = b^{x/y}, c = b^{z/y}$ and $b^2 = ac$ imply that

$$b^2 = b^{(x+z)/y}$$

and hence

$$2y = x + z$$

Thus (B) is true.

(C) We have

$$b^2 = ac, \frac{1}{\log_a u} = \log_u a, \frac{1}{\log_b u} = \log_u b, \frac{1}{\log_c u} = \log_u c$$

Now $2\log_u b = \log_u a + \log_u c$. Therefore

$$\frac{2}{\log_b u} = \frac{1}{\log_a u} + \frac{1}{\log_c u}$$

Thus (C) is true.

(D) We have $2b = a + c$. Therefore

$$c = \frac{2ba}{a+b}$$

$$c = \frac{(a+c)a}{a+b}$$

$$bc + ca = a^2 + ac$$

$$a^2 = bc$$

Thus (D) is true.

Answers: (A), (B), (C), (D)

Matrix-Match Type Questions

1. Match the items in Column I with those in Column II.

Column I	*Column II*
(A) If the sum of n terms of the series 5(1/2), 6(3/4), 8, … is 238, then n is	(p) 5
	(q) −2(1/2)
(B) The first term of an AP is 5, the last term is 45 and the sum of the terms is 400. The number of terms and the common difference are, respectively,	(r) 2(2/3)
(C) The sum of three numbers which are in AP is 27 and sum of their squares is 293. Then the common difference is	(s) 16
(D) The fourth and 54th terms of an AP are, respectively, 64 and −61. The common difference is	(t) −5

Solution:

(A) We have $a = 11/2, d = 5/4$. Therefore

$$\frac{n}{2}\left[2\left(\frac{11}{2}\right) + (n-1)\frac{5}{4}\right] = 238$$

$$\frac{n}{2}\left[\frac{44 + 5(n-1)}{4}\right] = 238$$

$$5n^2 + 39n - 8 \times 238 = 0$$

$$5n^2 - 80n + 119n - 8 \times 238 = 0$$

$$5n(n-16) + 119(n-16) = 0$$

$$n = 16$$

Answer: (A) → (s)

(B) We have $a = 5$, d = common difference. Now

$$a + (n-1)d = 45 \Rightarrow (n-1)d = 40 \tag{5.48}$$

$$\frac{n}{2}[10 + (n-1)d] = 400$$

$$\frac{n}{2}(10 + 40) = 400$$

$$n = \frac{800}{50} = 16$$

Substituting this value of n in Eq. (5.48) we get

$$d = \frac{40}{15}$$

Finally,

$$n = 16, d = 2\frac{2}{3}$$

Answer: (B) → (r), (s)

(C) Let the three numbers be $(a - d)$, a, $(a + d)$. Then by hypothesis

$$(a-d) + a + (a+d) = 27$$

$$3a = 27$$

$$a = 9$$

Again by hypothesis, since sum of their squares is 293 we have

$$(9-d)^2 + 9^2 + (9+d)^2 = 293$$

$$243 + 2d^2 = 293$$

$$d^2 = 25$$

$$d = \pm 5$$

Answer: (C) → (p), (t)

(D) Since the fourth term is 64 we get

$$a + 3d = 64$$

Since the 54th term is –61 we get

$$a + 53d = -61$$

Solving the two equations we get

$$-50d = 125$$

$$d = -2\frac{1}{2}$$

Answer: (D) → (q)

2. Match the items in Column I with those in Column II.

Column I	*Column II*
(A) If the pth, qth and rth terms of an AP are a, b, c respectively, then the value of $a(q-r) + b(r-p) + c(p-q)$ is	(p) 15
(B) The sum of m terms of an AP is n and the sum of n terms is m, then the [sum of $(m+n)$ terms] + $(m+n)$ is	(q) 27
(C) If five arithmetic means are inserted between 2 and 4, then the sum of the five means are	(r) $-(m+n)$
(D) In an AP, if the sum of n terms is $3n^2$ and the sum of m terms is $3m^2$ $(m \neq n)$ then, the sum of the first three terms is	(s) $2(m+n)$
	(t) 0

Solution:

(A) Since a, b, c are pth, qth and rth terms of an AP, let

$$a = \alpha + (p-1)d \tag{5.49}$$

$$b = \alpha + (q-1)d \tag{5.50}$$

$$c = \alpha + (r-1)d \tag{5.51}$$

Solving Eqs. (5.49) and (5.50) we get

$$d(p-q) = a - b$$

$$d = \frac{a-b}{p-q}$$

Therefore

$$c(p-q) = \frac{(a-b)c}{d}$$

Similarly,

$$b(r-p) = \frac{(c-a)b}{d}$$

$$a(q-r) = \frac{a(b-c)}{d}$$

Therefore

$$\sum a(q-r) = \frac{1}{d}\sum a(b-c) = \frac{1}{d}(0) = 0$$

Answer: (A) → (t)

(B) By hypothesis we get

$$\frac{m}{2}[2a + (m-1)d] = n \tag{5.52}$$

$$\frac{n}{2}[2a + (n-1)d] = m \tag{5.53}$$

Subtracting Eq. (5.53) from Eq. (5.52) and solving we get

$$2a(m-n)+d(m^2-n^2)-d(m-n)=2(n-m)$$
$$2a+(m+n-1)d=-2$$

Therefore sum of the first $(m+n)$ terms is

$$\frac{m+n}{2}[2a+(m+n-1)d]=-(m+n)$$

Answer: (B) → (r)

(C) Sum of the n AM's between x and y is

$$n\left(\frac{x+y}{2}\right)$$

Therefore sum of the five AM's between 2 and 4 is

$$5\left(\frac{2+4}{2}\right)=15$$

Answer: (C) → (p)

(D) By hypothesis we have

$$\frac{n}{2}[2a+(n-1)d]=3n^2$$
$$2a+(n-1)d=6n \tag{5.54}$$

Similarly,

$$2a+(m-1)\,d=6m \tag{5.55}$$

Solving Eqs. (5.54) and (5.55) we get

$$d=6, a=3$$

Therefore, the sum of the first 3 terms is

$$\frac{3}{2}(6+2\times 6)=27$$

Answer: (D) → (q)

3. Match the items in Column I with those in Column II.

Column I	*Column II*
(A) The sum of three numbers which are in AP is 12 and the sum of their cubes is 288. The greater of the three numbers is	(p) 25
(B) Let s_n denote the sum of the first n terms of an AP. If $s_{2n}=3s_n$, then s_{3n}/s_n equals	(q) 26
(C) $4n^2$ is the sum of the first n terms of an AP whose common difference is	(r) 6
(D) The least value of n for which the sum $3+6+9+\cdots+n$ is greater than 1000 is	(s) 7
	(t) 8

Solution:

(A) Let the three numbers be $a-d, a, a+d$. By hypothesis

$$(a-d)+a+(a+d)=12$$
$$3a=12$$
$$a=4$$

Also it is given that the sum of their cubes is 288; therefore

$$(a-d)^3+a^3+(a+d)^3=288$$
$$3a^3+6ad^2=288$$
$$a^3+2ad^2=96$$
$$64+8d^2=96$$
$$d=\pm 2$$

Therefore the numbers are 2, 4, 6.

Answer: (A) → (r)

(B) We have

$$\frac{2n}{2}[2a+(2n-1)d]=3\left(\frac{n}{2}\right)[2a+(n-1)d]$$
$$2a=(n+1)d$$

Therefore

$$\frac{S_{3n}}{s_n}=\frac{3[2a+(3n-1)d]}{2a+(n-1)d}$$
$$=\frac{3[(n+1)d+(3n-1)d]}{(n+1)d+(n-1)d}$$
$$=3\left(\frac{4n}{2n}\right)=6$$

Answer: (B) → (r)

(C) The nth term is

$$4[n^2-(n-1)^2]=4(2n-1)$$

Substituting $n=1,2,3,\ldots$, we get the series $4, 12, 20, \ldots$.

Answer: (C) → (t)

(D) By hypothesis

$$\frac{n}{2}[6+(n-1)3]>1000$$
$$n(3n+3)>2000$$
$$\left(n+\frac{1}{2}\right)^2>\frac{2000}{3}+\frac{1}{4}$$

$$n + \frac{1}{2} > 25.8$$

$$n = 26$$

Answer: (D) → (q)

4. Match the items in Column I with those in Column II.

Column I	*Column II*
(A) If four GMs are inserted between 160 and 5, then the second mean is	(p) 7
(B) Sum to infinity of the series $1 + \frac{1}{2} + \frac{1}{2^2} + \cdots$	(q) 20
(C) If $5^2 \cdot 5^4 \cdot 5^6 \cdots 5^{2n} = (0.04)^{-28}$, then the value of the n is	(r) 40
(D) If $y > 0$ and $y^2 = (0.2)^x$ where $x = \log_{\sqrt{5}}\left(\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \infty\right)$ then the value of y is	(s) 10
	(t) 2

Solution:

(A) Let g_1, g_2, g_3, g_4 be the four GMs between 160 and 5. Then $g_i = 160r^i, 1 \le i \le 4$, and $5 = 160r^5$. Therefore

$$r^5 = \frac{5}{160} = \frac{1}{32} = \left(\frac{1}{2}\right)^5$$

and hence $r = 1/2$. Hence

$$g_2 = 160\left(\frac{1}{2}\right)^2 = 40$$

Answer: (A) → (r)

(B) $1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \infty = \frac{1}{1-(1/2)} = 2$

Answer: (B) → (t)

(C) We have

$$5^{2+4+6+\cdots+2n} = (0.04)^{-28}$$

$$5^{2n[(n+1)/2]} = \left(\frac{1}{25}\right)^{-28} = 5^{56}$$

$$n(n+1) = 56$$

$$n = 7$$

Answer: (C) → (p)

(D) We have

$$y^2 = (0.2)^x$$

where

$$x = \log_{\sqrt{5}}\left(\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \infty\right)$$

Now

$$x = \log_{\sqrt{5}}\left(\frac{1}{2}\right)$$

$$= -2\log_5 2$$

Therefore

$$y^2 = (0.2)^x$$

$$= (5^{-1})^{-2\log_5 2} = 4$$

Hence $y = 2$.

Answer: (D) → (t)

5. Match the items in Column I with those in Column II.

Column I	*Column II*
(A) A GP contains even number of terms. The sum of all terms is equal to five times the sum of all odd terms. Then the common ratio is	(p) 1
(B) In a GP, the terms are alternately positive and negative, beginning with a positive term. Any term is the AM of the next immediate two terms. Then the common ratio is	(q) −2
	(r) 2
(C) If $y = c^{(\sin^2 x + \sin^4 x + \sin^6 x + \cdots + \infty)\log_e 2}$ $(0 < x < \pi/2)$ satisfies the equation $x^2 - 17x + 16 = 0$, then the value of $\sin 2x/(1+\cos^2 x)$ is	(s) 4
(D) If the same y in (C) satisfies the same equation $x^2 - 17x + 16 = 0$, then the value of $6\sin x/(\sin x + \cos x)$ is equal to	(t) 2/3

Solution:

(A) Let $a, ar, ar^2, \ldots, ar^{2n-1}$ be the $2n$ terms of the GP. It is given that

$$a + ar + ar^2 + \cdots + ar^{2n-1} = 5(a + ar^2 + ar^4 + \cdots + ar^{2n-2})$$

Therefore

$$\frac{a(r^{2n}-1)}{r-1} = 5\frac{a[(r^2)^n - 1]}{r^2 - 1} = 5\frac{a(r^{2n}-1)}{(r-1)(r+1)}$$

Solving this we get $r + 1 = 5$ or $r = 4$.

Answer: (A) → (s)

(B) Let the numbers be $a, a(-r), a(-r)^2, a(-r)^3, \ldots$ where $a > 0$ and $r > 0$. Since any term is the AM of the immediate next two terms, therefore

$$a(-r) + ar^2 = 2a$$
$$r^2 - r - 2 = 0$$
$$(r-2)(r+1) = 0$$

which gives $r = 2$.

Answer: (B) → (r)

For parts (C) and (D)

$$0 < x < \frac{\pi}{2} \Rightarrow 0 < \sin x < 1$$

Therefore,

$$\sin^2 x + \sin^4 x + \sin^6 x + \cdots \infty = \frac{\sin^2 x}{1-\sin^2 x} = \tan^2 x$$

$$e^{(\sin^2 x + \sin^4 x + \cdots \infty)\log_e 2} = e^{\tan^2 x \cdot \log_e 2} = 2^{\tan^2 x}$$

The roots of $x^2 - 17x + 16 = 0$ are 1 and 16. Then

(i) $2^{\tan^2 x} = 1 \Rightarrow \tan x = 0$, which is false since $0 < x < \pi/2$
(ii) $2^{\tan^2 x} = 16 \Rightarrow \tan^2 x = 4$ or $\tan x = 2$

(C) $\dfrac{\sin 2x}{1+\cos^2 x} = \dfrac{2\sin x \cos x}{1+\cos^2 x} = \dfrac{2\tan x}{1+\sec^2 x} = \dfrac{2\tan x}{2+4} = \dfrac{4}{6} = \dfrac{2}{3}$

Answer: (C) → (t)

(D) $\dfrac{6\sin x}{\sin x + \cos x} = \dfrac{6\tan x}{\tan x + 1} = \dfrac{6(2)}{2+1} = 4$

Answer: (D) → (s)

6. Let $S_1, S_2, S_3, \ldots$ be squares such that the length of the side of S_n is equal to the length of the diagonal of S_{n+1}. Match the items in Column I with those in Column II, if the length of the side of S_1 is equal to 10 units.

Column I	*Column II*
(A) Length of the side of S_3 is	(p) 7
(B) Length of the diagonal of S_4	(q) 5
(C) The area of S_n is less than 1 if n is greater than	(r) 6
	(s) 200
(D) Sum of the areas of the squares is	(t) $10\sqrt{2}/(\sqrt{2}-1)$

Solution: Let a_n be the length of the side of S_n. It is given that

$$a_n = \sqrt{2}a_{n+1} \quad \text{or} \quad a_{n+1} = \frac{1}{\sqrt{2}}a_n$$

(A) Let $a_1 = 10$. Then

$$a_2 = \frac{1}{\sqrt{2}}10 \quad \text{and} \quad a_3 = \frac{1}{\sqrt{2}}a_2 = \frac{1}{2}10 = 5$$

Answer: (A) → (q)

(B) Length of the diagonal of

$$S_4 = \sqrt{2}a_4 = \sqrt{2}\cdot\frac{1}{\sqrt{2}}a_3 = 5$$

Answer: (B) → (q)

(C) $a_1, a_2, a_3, \ldots$ are in GP with $a_1 = 10$ and common ratio $r = 1/\sqrt{2}$. Therefore

$$a_n = 10\left(\frac{1}{\sqrt{2}}\right)^{n-1}$$

The area of S_n is

$$a_n^2 = 100\left(\frac{1}{2}\right)^{n-1}$$

and $\quad a_n^2 < 1 \Leftrightarrow \dfrac{100}{2^{n-1}} < 1 \Leftrightarrow 100 < 2^{n-1} \Leftrightarrow 7 < n$

Answer: (C) → (p)

(D) Sum of the areas of the squares is

$$\sum_{n=1}^{\infty} a_n^2 = \sum_{n=1}^{\infty}\frac{100}{2^{n-1}}$$
$$= 100\left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots\right)$$
$$= 100(2)$$
$$= 200 \text{ sq. units}$$

Answer: (D) → (s)

7. Match the items in Column I with those in Column II.

Column I	*Column II*
(A) If $n = 3$, then the numbers $2n$, $n(n-1)$ and $n(n-1)(n-2)$ are in	(p) GP
(B) If a, b, c are in AP, then $a + 1/bc$, $b + 1/ac$ and $c + 1/ab$ are in	(q) HP
(C) If $x > 1, y > 1$ and $z > 1$are three numbers in GP, then the numbers $\dfrac{1}{1+\log x}, \dfrac{1}{1+\log y}, \dfrac{1}{1+\log z}$ are in	(r) AGP
(D) If a, b, c are in HP, then the numbers $a - \dfrac{b}{2}, \dfrac{b}{2}, c - \dfrac{b}{2}$ are in	(s) AP

Solution:

(A) If $n = 3$, the three numbers are 6, 6, 6, which are in AP, GP, HP and AGP.

Answer: (A) → (p), (q), (r), (s)

(B) Let a, b, c be in AP. Then $b - a = c - b$. Therefore

$$\left(b + \frac{1}{ca}\right) - \left(a + \frac{1}{bc}\right) = (b - a) + \frac{b - a}{abc}$$
$$= (c - b) + \frac{c - b}{abc}$$
$$= \left(c + \frac{1}{ab}\right) - \left(b + \frac{1}{ca}\right)$$

Answer: (B) → (s)

(C) Let x, y, z be in GP; $x > 1, y > 1$ and $z > 1$. Then

$1 + \log x,\ 1 + \log y,\ 1 + \log z$ are in AP

and so

$$\frac{1}{1 + \log x}, \frac{1}{1 + \log y}, \frac{1}{1 + \log z} \text{ are in HP}$$

Answer: (C) → (q)

(D) Let a, b, c be in HP. Then

$$b = \frac{2ac}{a + c}$$

Now

$$\left(a - \frac{b}{2}\right)\left(c - \frac{b}{2}\right) = ac - \frac{b}{2}(a + c) + \frac{b^2}{4}$$
$$= ac - \frac{ac}{a + c}(a + c) + \frac{b^2}{4}$$
$$= \frac{b^2}{4}$$

Answer: (D) → (p)

8. Match the items in Column I with those in Column II.

Column I	*Column II*
(A) a, b, c and d are positive, each is not equal to 1 and $K \neq 1$. If $\frac{a + bK}{a - bK} = \frac{b + cK}{b - cK} = \frac{c + dK}{c - dK}$ then a, b, c, and d are in	(p) AP
(B) If a_1, a_2, a_3, and a_4 are four numbers such that $\frac{a_2 a_3}{a_1 a_4} = \frac{a_2 + a_3}{a_1 + a_4} = 3\left(\frac{a_2 - a_3}{a_1 - a_4}\right)$ then a_1, a_2, a_3, and a_4 are in	(q) HP
(C) If a_1, a_2, a_3 are in AP; a_2, a_3, a_4 are in GP and a_3, a_4, a_5 are in HP, then a_1, a_3, a_5 are in	(r) AGP
(D) If the sum to n terms of a series is pn^2, then the series is in	(s) GP

Solution:

(A) We have

$$\frac{a + bK}{a - bK} = \frac{b + cK}{b - cK} = \frac{c + dK}{c - dK}$$
$$\frac{a}{a - bK} = \frac{b}{b - cK} = \frac{c}{c - dK}$$
$$\frac{a - bK}{a} = \frac{b - cK}{b} = \frac{c - dK}{c}$$
$$\frac{b}{a} = \frac{c}{b} = \frac{d}{c}$$

This implies that a, b, c, d are in GP.

Answer: (A) → (s)

(B) We have

$$\frac{a_1 + a_4}{a_1 a_4} = \frac{a_2 + a_3}{a_2 a_3}$$
$$\frac{1}{a_1} + \frac{1}{a_4} = \frac{1}{a_2} + \frac{1}{a_3}$$
$$\frac{1}{a_2} - \frac{1}{a_1} = \frac{1}{a_4} - \frac{1}{a_3} \quad (5.56)$$

Also,

$$3\left(\frac{a_2 - a_3}{a_1 - a_4}\right) = \frac{a_2 a_3}{a_1 a_4}$$
$$3\left(\frac{1}{a_3} - \frac{1}{a_2}\right) = \frac{1}{a_4} - \frac{1}{a_1} \quad (5.57)$$

From Eqs. (5.56) and (5.57), we get that

$$\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3} \quad \text{and} \quad \frac{1}{a_4}$$

are in AP. Therefore a_1, a_2, a_3, a_4 are in HP.

Answer: (B) → (q)

(C) We have

$$2a_2 = a_1 + a_3$$
$$a_3^2 = a_2 a_4$$
$$a_4 = \frac{2a_3 a_5}{a_3 + a_5}$$

Therefore

$$a_3^2 = \left(\frac{a_1 + a_3}{2}\right)\left(\frac{2a_3 a_5}{a_3 + a_5}\right)$$
$$a_3(a_3 + a_5) = a_5(a_1 + a_3)$$

$$a_3^2 = a_1 a_5$$

Therefore a_1, a_2, a_3 are in GP.

Answer: (C) → (s)

(D) The nth term is given by

$$u_n = s_n - s_{n-1} = p(n^2) - p(n-1)^2 = p(2n-1)$$

The series is $p, 3p, 5p, 7p, \ldots$ which is an AP.

Answer: (D) → (p)

Comprehension-Type Questions

1. Passage: Let a be the first term and d the common difference of an AP. Then sum of the first n terms is

$$\frac{n}{2}[2a+(n-1)d]$$

If n AMs are inserted between a and b, then the kth AM is

$$a + k\frac{(b-a)}{n+1}(k = 1, 2, 3, \ldots, n)$$

Now, answer the following questions.

(i) If the sum of the first m terms of an AP is same as the sum of the first n terms, then sum of the first $(m+n)$ terms is equal to

(A) $mn\,(m+n)$ (B) $(mn+1)(m+n)$
(C) $(mn-1)(m+n)$ (D) 0

(ii) S_1, S_2, S_3 are sums of first n terms of three APs whose first terms are unity and the common difference are respectively 1, 2, 3. Then $S_1 + S_3$ is equal to

(A) S_2 (B) $3S_2$ (C) $2S_2$ (D) S_2^2

(iii) Let N be the natural number set and $f: N \to \mathbb{R}$ be a function defined by $f(n) = 3n - 1$. If

$$\sum_{k=1}^{n} f(k) = 155$$

then

(A) $n = 8$ (B) $n = 10$ (C) $n = 11$ (D) $n = 9$

Solution:

(i) We have

$$\frac{m}{2}[2a+(m-1)d] = s_m = s_n = \frac{n}{2}[2a+(n-1)d]$$

Therefore

$$2a(m-n) = d[n^2 - m^2 + m - n]$$

$$2a = d[-(m+n)+1] = -d(m+n-1)$$

$$2a + d(m+n-1) = 0$$

So, the sum of the first $(m+n)$ terms is

$$\frac{m+n}{2}[2a+(m+n-1)d] = 0$$

Answer: (D)

(ii) We have

$$S_1 = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

$$S_2 = \frac{n}{2}[2+(n-1)2] = n^2$$

$$S_3 = \frac{n}{2}[2+(n-1)3] = \frac{n(3n-1)}{2}$$

Now

$$S_1 + S_3 = \frac{n}{2}[n+1+3n-1] = 2n^2 = 2S_2$$

Answer: (C)

(iii) We have $f(n) = 3n - 1$ which implies that $f(1), f(2), f(3), \ldots$ are in AP with first term 2 and common difference 3. Therefore

$$155 = \frac{n}{2}[4+(n-1)3] = \frac{n}{2}[3n+1]$$

$$3n^2 + n - 310 = 0$$

$$(n-10)(3n+31) = 0$$

$$n = 10$$

Answer: (B)

2. Passage: Let $p < q < r < s$ and p, q, r, s be in AP. Further, let p and q be the roots of the equation $x^2 - 2x + A = 0$ while r and s be the roots of $x^2 - 18x + B = 0$. Answer the following questions.

(i) $|A+B|$ is equal to

(A) 80 (B) 74 (C) 84 (D) 77

(ii) If an AP is formed with A as first term and 8 as common difference then B appears at

(A) 11th place (B) 12th place
(C) 10th place (D) no place

(iii) In the above question, the sum of the nine AMs inserted between A and B is

(A) 233 (B) 323
(C) 333 (D) 222

Solution:

(i) Let $p = a - 3d$, $q = a - d$, $r = a + d$ and $s = a + 3d$. Now $p < q < r < s \Rightarrow d > 0$ Since p and q are the roots of the equation $x^2 - 2x + A = 0$ we have

$$\begin{aligned} p + q &= 2 \\ a - 3d + a - d &= 2 \\ 2a - 4d &= 2 \\ a - 2d &= 1 \end{aligned} \tag{5.58}$$

Since r and s are the roots of $x^2 - 18x + B = 0$ we have

$$\begin{aligned} r + s &= 18 \\ a + d + a + 3d &= 18 \\ 2a + 4d &= 18 \\ a + 2d &= 9 \end{aligned} \tag{5.59}$$

Solving Eqs. (5.58) and (5.59) we get $a = 5$. Substituting the value of a in Eq. (5.58) we get

$$\begin{aligned} a - 2d &= 1 \\ 5 - 2d &= 1 \\ 4 &= 2d \\ 2 &= d \end{aligned}$$

Now

$$\begin{aligned} A &= pq = (5-6)(5-2) = -3 \\ B &= rs = (5+2)(5+6) = 77 \end{aligned}$$

Therefore

$$|A + B| = 74$$

Answer: (B)

(ii) We have

$$\begin{aligned} 77 &= -3 + 8(n-1) \\ n &= 11 \end{aligned}$$

Answer: (A)

(iii) Sum of the nine means is

$$\frac{9}{2}(-3 + 77) = 37 \times 9 = 333$$

Answer: (C)

3. Passage: Let a, b, c be in GP. Answer the following three questions.

(i) $\dfrac{a^2 + ab + b^2}{ab + bc + ca}$ is equal to

(A) $(a+b)/(b+c)$ (B) $(b+c)/(c+a)$
(C) $(c+a)/(a+b)$ (D) $(a+b)/2(b+c)$

(ii) If $ab + bc + ca = 156$ and $abc = 216$ and the numbers are in the descending order, then the common ratio is

(A) 1/2 (B) 2 (C) 3 (D) 1/3

(iii) If $a + b + c = 14$ and $a + 1$, $b + 1$ and $c - 1$ are in AP, then the sum to infinity of the GP whose first three terms are a, b and c (in the descending order) is

(A) 8 (B) 16 (C) 4 (D) 2

Solution:

(i) Let $b = ar$ and $c = ar^2$. Then

$$\begin{aligned} \frac{a^2 + ab + b^2}{ab + bc + ca} &= \frac{a^2(1 + r + r^2)}{a^2r + a^2r^3 + a^2r^2} \\ &= \frac{a^2(1 + r + r^2)}{a^2r(1 + r + r^2)} = \frac{1}{r} \\ &= \frac{a(r+1)}{r(r+1)a} = \frac{ar + a}{ar^2 + ar} = \frac{a+b}{b+c} \end{aligned}$$

Answer: (A)

(ii) Let the numbers a, b and c be x/r, x and xr, respectively. Then

$$\begin{aligned} \frac{x}{r} \cdot x \cdot xr &= 216 \\ x^3 &= 216 \\ x &= 6 \end{aligned}$$

Also,

$$\begin{aligned} \left(\frac{x}{r}\right)x + x(xr) + (xr)\left(\frac{x}{r}\right) &= 156 \\ x^2\left(\frac{1}{r} + r + 1\right) &= 156 \\ \frac{1 + r + r^2}{r} = \frac{156}{x^2} &= \frac{156}{36} = \frac{13}{3} \\ 3r^2 - 10r + 3 &= 0 \end{aligned}$$

$$(3r-1)(r-3)=0$$
$$r=\frac{1}{3} \quad \text{or} \quad 3$$

The numbers are in descending order. Therefore $r = 1/3$.

Answer: (D)

(iii) Let $a = x/r$, $b = x$ and $c = xr$. Then

$$x\left(\frac{1}{r}+1+r\right)=14 \tag{5.60}$$

Also, given that $(x/r)+1$, $x+1$ and $xr-1$ are in AP. Therefore

$$\left(\frac{x}{r}+1\right)+(xr-1)=2(x+1)$$
$$x\left(\frac{r^2+1}{r}-2\right)=2 \tag{5.61}$$

From Eqs. (5.60) and (5.61),

$$\frac{r^2+r+1}{r^2-2r+1}=\frac{14}{2}=7$$
$$6r^2-15r+6=0$$
$$2r^2-5r+2=0$$
$$(2r-1)(r-2)=0$$
$$r=\frac{1}{2} \quad \text{or} \quad 2$$

Then

$$r=\frac{1}{2} \quad \text{and} \quad x=4$$
$$a=8, b=4, c=2$$

The sum to infinity is

$$s_\infty=\frac{8}{1-(1/2)}=16$$

Answer: (B)

4. Passage: Let x and y be real numbers such that x, $x+2y$ and $2x+y$ are in AP and $(y+1)^2$, $xy+5$ and $(x+1)^2$ are in GP. With this information, answer the following three questions.

(i) The common difference of the AP is

(A) 2 (B) 3 (C) 4 (D) 3/2

(ii) The common ratio of the GP is

(A) 2 (B) 1/2 (C) 3 (D) 1/3

(iii) The sum of four AM's between x and y and product of four GMs between x and y is

(A) $4+4\sqrt{3}$

(B) $4(2+\sqrt{3})$

(C) $8(1+\sqrt{3})$

(D) 17

Solution:

(i) Since x, $x+2y$ and $2x+y$ are in AP, we have

$$x+(2x+y)=2(x+2y)$$
$$x=3y$$

Again, since $(y+1)^2$, $xy+5$ and $(x+1)^2$ are in GP, we have

$$(x+1)^2(y+1)^2=(xy+5)^2$$

By substituting $x = 3y$ in this, we get that

$$(3y^2+5)^2=(3y+1)^2(y+1)^2=(3y^2+4y+1)^2$$
$$(3y^2+5+3y^2+4y+1)[3y^2+5-(3y^2+4y+1)]=0$$
$$(6y^2+4y+6)(-4y+4)=0$$
$$(3y^2+2y+3)(y-1)=0$$

Now $3y^2+2y+3=0$ has no real roots and hence $y=1$ and $x=3y=3$. Therefore, the AP is 3, 5 and 7. The common difference is 2.

Answer: (A)

(ii) We have

$$(y+1)^2=2^2$$
$$xy+5=3+5=8$$
$$(x+1)^2=4^2=16$$

Therefore the common ratio is 2.

Answer: (A)

(iii) We have $x = 3$ and $y = 1$. The sum of four AM's between 3 and 1 is

$$4\frac{(3+1)}{2}=8$$

The product of four GMs between 3 and 1 is $(\sqrt{3})^4 = 9$. Therefore the required sum is $8 + 9 = 17$.

Answer: (D)

Assertion–Reasoning Type Questions

In each of the following, two statements, I and II, are given and one of the following four alternatives has to be chosen.

(A) Both I and II are correct and II is a correct reasoning for I.

(B) Both I and II are correct but II is not a correct reasoning for I.

(C) I is true, but II is not true.

(D) I is not true, but II is true.

1. **Statement I:** If $\log a, \log b, \log c$ are in AP and $\log a - \log 2b, \log 2b - \log 3c, \log 3c - \log a$ are also in AP, then a, b, c form the sides of a triangle.

   Statement II: Three positive real numbers form the sides of a triangle, if the sum of any two of them is greater than the third.

Solution: Clearly a, b, c are positive and

$$\log a + \log c = 2\log b \Rightarrow b^2 = ac \qquad (5.62)$$

Also

$$(\log a - \log 2b) + (\log 3c - \log a) = 2(\log 2b - \log 3c)$$

$$\log 3c - \log 2b = 2(\log 2b - \log 3c)$$

$$3(\log 3c - \log 2b) = 0$$

$$2b = 3c \qquad (5.63)$$

From Eqs. (5.62) and (5.63) we get

$$a = \frac{9c}{4}, b = \frac{3c}{2}$$

Now

$$a + b = \frac{15c}{4} > c$$

$$b + c = \frac{3c}{2} + c = \frac{5c}{2} = \frac{5}{2}\left(\frac{4a}{9}\right) > a$$

$$c + a = c + \frac{9c}{4} = \frac{13c}{4} > \frac{3c}{2} = b$$

Therefore a, b, c form the sides of a triangle. In other words, Statements I and II are true and II is a correct reason for I.

Answer: (A)

2. **Statement I:** Let $a_1, a_2, a_3, \ldots, a_{24}$ be in AP. If $a_1 + a_5 + a_{10} + a_{15} + a_{20} + a_{24} = 225$, then $a_1 + a_2 + a_3 + \cdots + a_{24}$ is equal to 800.

   Statement II: $a_5 + a_{20} = a_{10} + a_{15} = a_1 + a_{24}$.

Solution: Let d be the common difference. Then

$$a_5 + a_{20} = (a + 4d) + (a + 19d) = 2a + 23d = a_1 + a_{24}$$

$$a_{10} + a_{15} = (a + 9d) + (a + 14d) = 2a + 23d = a_1 + a_{24}$$

Therefore Statement II is true. Also

$$225 = a_1 + a_5 + a_{10} + a_{15} + a_{20} + a_{24} = 3(a_1 + a_{24})$$

$$a_1 + a_{24} = 75$$

Hence

$$\sum_{k=1}^{24} a_k = \frac{24}{2}(a_1 + a_{24}) = 12 \times 75 = 900$$

Therefore Statement I is false.

Answer: (D)

3. **Statement I:** If a, b, c, are real numbers satisfying the relation $25(9a^2 + b^2) + 9c^2 - 15(5ab + bc + 3ca) = 0$, then a, b, c are in AP.

   Statement II: If x, y, z are any real numbers such that $x^2 + y^2 + z^2 - xy - yz - zx = 0$, then $x = y = z$.

Solution: We have

$$x^2 + y^2 + z^2 - xy - yz - zx = 0$$

$$\Leftrightarrow \frac{1}{2}[(x - y)^2 + (y - z)^2 + (z - x)^2] = 0$$

$$\Leftrightarrow x = y = z$$

Therefore, Statement II is true. For Statement I, take $x = 15a, y = 5b$ and $z = 3c$ so that

$$x^2 + y^2 + z^2 - xy - yz - zx = 0$$

$$\Rightarrow x = y = z$$

$$\Rightarrow 15a = 5b = 3c$$

$$\Rightarrow b = 3a \quad \text{and} \quad c = 5a$$

Therefore, a, b, c are in AP.

Answer: (A)

4. **Statement I:** A ball is dropped from a height of 8 feet. Each time the ball hits the ground, it rebounds half the height. The total distance travelled by the ball when it comes to rest is 16 feet.

   Statement II: Sum to infinity of GP with common ratio $r(|r| < 1)$ and first term a is $a/(1 - r)$.

Solution: Statement II is clearly true. But the distance travelled by the ball follows an infinite GP after hitting the ground. Therefore, distance travelled by the ball equals

$$8+2\left[4+2+1+\frac{1}{2}+\frac{1}{4}+\cdots\right]=8+2\left(\frac{4}{1-(1/2)}\right)$$
$$=8+16=24 \text{ feet}$$

I is false and II is correct.

Answer: (D)

5. **Statement I:** The sum to infinity of the series $1(0.1)+3(0.01)+5(0.001)+\cdots$ 29/81.

Statement II: Sum to infinity of AGP $a+(a+d)r+(a+2d)r^2+\cdots$ is

$$\frac{a}{1-r}+\frac{dr}{(1-r)^2} \quad \text{when } |r|<1$$

Solution: The series given in Statement I is an AGP with $a=1/10$, $d=2$ and $r=1/10$. Therefore the sum to infinity is given by

$$\frac{a}{1-r}+\frac{dr}{(1-r)^2}$$

Substituting the values we get

$$\frac{1/10}{1-(1/10)}+\frac{2(1/10)}{[1-(1/10)]^2}=\frac{1}{9}+\frac{20}{81}=\frac{29}{81}$$

Statements I and II are both correct and II is the correct explanation for I.

Answer: (A)

6. **Statement I:** In a GP, if the common ratio is positive and less than 1/2 and the first term is positive, then each term of the GP is greater than sum to infinity of all the terms of the GP that follow it.

Statement II: If the common ratio r of a GP is such that $-1<r<1$ and the first term is a, then the sum to infinity of the GP is

$$\frac{a}{1-r}$$

Solution: Statement II is clearly correct. Let the GP be $a, ar, ar^2, ar^3, \ldots$ and $0<r<1/2$. The nth term is ar^{n-1} ($=t_n$, say). Now,

$$ar^n+ar^{n+1}+ar^{n+2}+\cdots=ar^n(1+r+r^2+\cdots)=\frac{ar^n}{1-r}$$

Therefore

$$t_n>t_{n+1}+t_{n+2}+\cdots\infty \Leftrightarrow ar^{n-1}>\frac{ar^{n+1}}{1-r}$$
$$\Leftrightarrow ar^{n-1}(1-r)>ar^{n+1}$$
$$\Leftrightarrow 1-r>r^2$$

which is true because $0<r<1/2$. Therefore both Statements I and II are true and II is a correct explanation of I.

Answer: (A)

7. **Statement I:** In a GP, the sum of the first n terms is 255, the nth term is 128 and the common ratio is 2. Then the value of n is 8.

Statement II: The sum of the first n terms of a GP whose first term is a and common ratio r is

$$\frac{a(1-r^n)}{1-r}$$

Solution: The nth term is $ar^{n-1}=128$ and $r=2$. Therefore

$$a\cdot 2^{n-1}=128$$

Now

$$255=\frac{a(1-r^n)}{1-r}=\frac{a-ar^n}{1-r}=\frac{a-256}{1-2}$$
$$a=1$$

Also,

$$ar^{n-1}=128$$
$$2^{n-1}=128$$
$$n-1=7 \quad \text{or} \quad n=8$$

Answer: (A)

8. **Statement I:** ABC is an equilateral triangle with side 24. $\Delta A_1B_1C_1$ is formed from ΔABC joining the midpoints of its sides. Again $\Delta A_2B_2C_2$ is formed by joining the midpoints of the sides of $\Delta A_1B_1C_1$. The process is continued infinitely. Then the sum of the perimeters of all the triangles including ΔABC is 144.

Statement II: The area of an equilateral triangle of side 'a' units is $(\sqrt{3}/4)a^2$ square units.

Solution: Sum to infinity of the perimeters is given by

$$3a+3\left(\frac{a}{2}\right)+3\left(\frac{a}{2^2}\right)+\cdots$$

Substituting $a=24$ we get

$$3a\left(1+\frac{1}{2}+\frac{1}{2^2}+\cdots\right)=3\times 24\times\frac{1}{1-(1/2)}=144$$

Therefore Statements I and II are correct. However, Statement II is not the correct explanation for I.

Answer: (B)

9. Statement I: If a, b, c and x are real and $(a^2 + b^2)x^2 - 2b(a+c)x + (b^2+c^2) = 0$, then a, b, c are in GP with x as common ratio.

Statement II: For any real numbers p and q, $p^2 + q^2 = 0 \Leftrightarrow p = 0 = q$.

Solution: The given equation can be written as

$$(a^2x^2 - 2abx + b^2) + (b^2x^2 - 2bcx + c^2) = 0$$

$$(ax-b)^2 + (bx-c)^2 = 0$$

$$ax - b = 0 \quad \text{and} \quad bx - c = 0$$

Solving for x we get

$$x = \frac{b}{a} = \frac{c}{b}$$

Therefore a, b, c are in GP with common ratio x.

Answer: (A)

10. Statement I: If a, b, c are, respectively, the pth, qth, rth terms of an HP, then $(q-r)bc + (r-p)ca + (p-q)ab = (p+q+r)abc$.

Statement II: The nth term of an HP is of the form

$$\frac{1}{a+(n-1)d}$$

Solution: Since a is the pth term of an HP, let

$$a = \frac{1}{a_1 + (p-1)d}$$

Therefore

$$\frac{1}{a} = a_1 + (p-1)d$$

Similarly,

$$\frac{1}{b} = a_1 + (q-1)d$$

$$\frac{1}{c} = a_1 + (r-1)d$$

Now

$$\frac{1}{c} - \frac{1}{b} = (r-q)d$$

Therefore

$$b - c = -(q-r)bcd$$

Similarly,

$$c - a = -(r-p)cad$$
$$a - b = -(p-q)abd$$

Hence

$$-d\sum bc(q-r) = \sum(b-c) = 0$$
$$\sum bc(q-r) = 0$$

Statement II is correct and I is false.

Answer: (D)

11. Statement I: If

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

then

$$H_n = n - \left(\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{n-1}{n}\right)$$

Statement II: If $K > 1$ is an integer, then

$$\frac{1}{K} + \frac{K-1}{K} = 1$$

Solution: H_n can be written as

$$H_n = 1 + \left(1 - \frac{1}{2}\right) + \left(1 - \frac{2}{3}\right) + \left(1 - \frac{3}{4}\right) + \cdots + \left(1 - \frac{n-1}{n}\right)$$

$$= n - \left(\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{n-1}{n}\right)$$

Both Statements I and II are correct and II is a correct explanation of I.

Answer: (A)

12. Statement I: If a, b, c are in AP and b, c, d are in HP, then $a : b = c : d$.

Statement II: AM of x and y is $(x+y)/2$ and HM of x and y is $2xy/(x+y)$.

Solution: We have

$$2b = a + c \quad \text{and} \quad c = \frac{2bd}{b+d}$$

Therefore

$$c = \frac{(a+c)d}{b+d}$$

$$bc + cd = ad + cd$$

$$bc = ad \quad \text{or} \quad a : b = c : d$$

Both Statements I and II are correct and II is a correct explanation of I.

Answer: (A)

13. Statement I: If a, b and c are positive real numbers, then

$$(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq 9$$

Statement II: AM of three positive real numbers ≥ their GM.

Solution: We have

$$\frac{a+b+c}{3} \geq (abc)^{1/3}$$

$$\frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{3} \geq \left(\frac{1}{a}\cdot\frac{1}{b}\cdot\frac{1}{c}\right)^{1/3}$$

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq (abc)^{1/3}\left(\frac{1}{abc}\right)^{1/3}\cdot 9 = 9$$

Statements I and II are both correct and II is a correct explanation for I.

Answer: (A)

14. Statement I: If

$$\frac{1}{a_1}, \frac{1}{a_2}, \ldots, \frac{1}{a_n}, \ldots$$

are in HP, then

$$a_1^2 - a_2^2 + a_3^2 - a_4^2 + \cdots + a_{2K-1}^2 - a_{2K}^2 = \frac{K}{2K-1}(a_1^2 - a_{2K}^2)$$

Statement II: If $x_1, x_2, \ldots, x_n, \ldots$ are in AP, then $x_2 - x_1 = x_3 - x_2 = x_4 - x_3 = \cdots$

Solution: Let

$$\frac{1}{a_1}, \frac{1}{a_2}, \ldots$$

be in HP. Then $a_1, a_2, \ldots$ are in AP, with common difference, say d. Now,

$$d = a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = \cdots$$

Therefore

$$a_1^2 - a_2^2 = (a_1 - a_2)(a_1 + a_2) = -d(a_1 + a_2)$$

$$a_3^2 - a_4^2 = -d(a_3 + a_4)$$

$$a_{2K-1}^2 - a_{2K}^2 = -d(a_{2K-1} + a_{2K})$$

Adding we get

$$\begin{aligned} a_1^2 - a_2^2 + a_3^2 - a_4^2 + \cdots + a_{2K-1}^2 - a_{2K}^2 &= -d(a_1 + a_2 + \cdots + a_{2K}) \\ &= -d\cdot\frac{2K}{2}[a_1 + a_{2K}] \\ &= -dK(a_1 + a_{2K}) \end{aligned}$$

Now, $a_{2K} = a_1 + (2K-1)d$. Therefore

$$d = \frac{a_{2K} - a_1}{2K-1}$$

and hence the sum equals

$$-\frac{(a_{2K} - a_1)}{2K-1}\cdot K\cdot(a_1 + a_{2K}) = \frac{K}{2K-1}(a_1^2 - a_{2K}^2)$$

Statements I and II are both correct and II is a correct explanation of I.

Answer: (A)

15. Statement I: Sum to infinity of the series

$$\frac{3}{1!} + \frac{5}{2!} + \frac{9}{3!} + \frac{15}{4!} + \frac{23}{5!} + \cdots$$

is $4e$.

Statement II: The nth term of the series in Statement I is

$$\frac{n^2 - n + 3}{n!} \quad \text{and} \quad e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots\infty$$

Solution: The nth term is

$$u_n = \frac{n^2 - n + 3}{n!} = \frac{n(n-1)+3}{n!} = \frac{1}{(n-2)!} + \frac{3}{n!} \quad \text{for all } n \geq 2$$

Therefore

$$\begin{aligned} \sum_{n=2}^{\infty} u_n &= \left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots\right) + 3\left(\frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots\right) \\ &= e + 3(e-2) = 4e - 6 \end{aligned}$$

The required sum is $3 + (4e - 6) = 4e - 3$. Statement II is correct and I is not correct.

Answer: (D)

16. Statement I: $\displaystyle\sum_{K=1}^{\infty} \frac{K}{K^4 + K^2 + 1} = \frac{1}{2}$

Statement II:
$$\frac{K}{K^4 + K^2 + 1} = \frac{1}{2}\left(\frac{1}{K^2 - K + 1} - \frac{1}{K^2 + K + 1}\right)$$

Solution: Statement II is correct and

$$\sum_{K=1}^{\infty} \frac{K}{K^4 + K^2 + 1} = \sum_{K=1}^{\infty} \frac{1}{2}\left[\frac{1}{K^2 - K + 1} - \frac{1}{K^2 + K + 1}\right] = \lim_{n\to\infty} s_n$$

where

$$\begin{aligned} s_n &= \sum_{K=1}^{n} \frac{K}{K^4 + K^2 + 1} \\ &= \frac{1}{2}\left(1 - \frac{1}{n^2 + n + 1}\right) \end{aligned}$$

Therefore

$$\lim_{n\to\infty} s_n = \frac{1}{2}$$

Answer: (A)

SUMMARY

5.1 ***Sequence:*** Let $\mathbb{Z}^+$ be the set of all positive integers and X any set. Then a mapping $a : \mathbb{Z}^+ \to X$ is called a sequence in x. For any $n \in \mathbb{Z}^+$, we prefer to write a_n for the imag $a(n)$ and the sequence is denoted by $\{a_n\}$.

5.2 ***Finite and infinite sequences:*** A sequence is said to be finite if its range is finite. A sequence which is not finite is said to be infinite sequence.

5.3 ***Constant and ultimately constant sequences:*** A sequence $\{a_n\}$ is called a constant sequence if $a_n = a_m$ for all positive integers n and m. Sequence $\{a_n\}$ in called ultimately constant, if there is a positive integer m such that a_n is constant for $n > m$, that is $a_{m+1} = a_{m+2} = a_{m+3} = \cdots$

5.4 ***Series:*** If $\{a_n\}$ is a sequence of real or complex numbers, then an expression of the form $a_1 + a_2 + a_3 + \cdots$ is called series. If s_n is the sum of the first n terms of the sequence $\{a_n\}$, then again $\{s_n\}$ is a sequence called nth partial sum of the series or simply the sequence of partial sums of the series.

5.5 ***Limit of a sequence:*** Let $\{a_n\}$ be a sequence of real numbers and l a real number. Then l is said to be limit of the sequence $\{a_n\}$ if, for each positive real number $\in$ (epsilon) there exists a positive integer n_0 (depending on $\in$) such that $|a_n - l| < \in$ for all $n \geq n_0$.

5.6 ***Uniqueness of a limit:*** If a sequence has a limit, then the limit is unique.

5.7 ***Notation:*** If l is the limit of a sequence $\{a_n\}$, then we write $\lim\limits_{n\to\infty} a_n = l$ (or $\operatorname{lt}\limits_{n\to\infty} a_n = l$) and some times we write $a_n \to l$.

5.8 ***Sum of an infinite series:*** Let $\{a_n\}$ be a sequence of real numbers and $s_n = a_1 + a_2 + \cdots + a_n$. If the sequence $\{s_n\}$ of partial sums has limit s, then we write $\sum_{n=1}^{\infty} a_n = s$. If $\{s_n\}$ has no finite limit, then the series is said to be divergent.

5.9 ***Arithmetic progression (AP):*** A sequence $\{a_n\}$ of real numbers is called an arithmetic progression (AP) if $a_{n+1} - a_n$ is constant for all positive integers $n \geq 1$, and this constant number is called the common difference of the AP.

5.10 ***General form of AP:*** The terms of an AP with first term 'a' and common difference d are $a, a + d, a + 2d, a + 3d, \ldots$, and the nth term being $a + (n - 1)d$.

QUICK LOOK

1. If $\{a_n\}$ is an AP and K is any real number, then $\{a_n + \mathrm{K}\}$ is also an AP with same common difference.
2. $\{\mathrm{K}a_n\}$ is also an AP.
3. If $\{a_n\}$ and $\{b_n\}$ are arithmetic progressions, then $\{a_n + b_n\}$ is also an AP.

5.11 ***Product of two AP's:*** Product of two arithmetic progressions is also an AP if and only if one of them must be a constant sequence.

5.12 ***Arithmetic mean (AM):*** If three real numbers a, b, c are in AP, then b is called AM between a and c and in this case $2b = a + c$.

5.13 ***Arithmetic means (AM's):*** If $a, A_1, A_2, \ldots A_n$ and b are in AP, the $A_1, A_2, \ldots A_n$ are called n AM's between a and b.

5.14 ***Formula for n AM's between a and b:*** If $A_1, A_2, \ldots A_n$ are n AM's between a and b, then the Kth mean A_K is given by

$$A_K = a + K\frac{(b-a)}{n+1} \quad \text{for } K = 1, 2, \ldots, n$$

5.15 ***Sum to first n terms of an AP:*** Let s_n be the sum to first n terms of an AP with first term 'a' and common difference 'd'. Then

$$s_n = \frac{n}{2}[2a + (n-1)d] \quad \text{or} \quad s_n = \frac{n}{2} \quad [\text{first term} + n\text{th term}]$$

QUICK LOOK

If $A_1, A_2, \ldots, A_n$ are n AM's between a and b then

$$A_1 + A_2 + \cdots + A_n = \frac{n(a+b)}{2}$$

5.16 ***Ratio of nth terms of two AP's:*** Let t_n be the nth term of an AP whose first term is a and common difference d and s_n is its sum to first n terms. Let t'_n be the nth term of another AP with first term b and common difference e whose sum to first n terms is s'_n. Then

$$\frac{t_n}{t'_n} = \frac{s_{2n-1}}{s'_{2n-1}}$$

5.17 ***Characterization of an AP:*** A sequence of real numbers is an arithmetic progression if and only if its sum of the first n terms is a quadratic expression in n with constant term zero.

5.18 ***Helping points:***

(1) Three numbers in AP can be taken as $a-d$, a, $a+d$.

(2) Four numbers in AP can be taken as $a-3d$, $a-d, a+d, a+3d$.

(3) Five numbers in AP can be taken as $a-2d$, $a-d, a, a+d, a+2d$.

5.19 ***Geometric progression (GP):*** A sequence $\{a_n\}$ of non-zero real numbers is called GP if $a_n/a_{n-1} = a_{n+1}/a_n$ for $n \geq 2$. That is the ratio a_{n+1}/a_n is constant for $n \geq 1$ and this constant ratio is called the common ratio of the GP and is generally denoted by r.

5.20 ***General form:*** GP with first term a and common ratio r can be expressed as $a, ar, ar^2, \ldots$ whose nth term is ar^{n-1}.

QUICK LOOK

1. If three numbers are in GP, then they can be taken as $a/r, a, ar$.
2. If four numbers are in GP, then they can be taken as $a/r^2, a/r, ar, ar^2$.

5.21 ***Sum to first n-terms of a GP:*** The sum of the first n-terms of a GP with first term 'a' and common ratio $r \neq 1$ is

$$\frac{a(1-r^n)}{1-r}$$

5.22 ***Sum to infinity of a GP:*** If $-1 < r < l$ is the common ratio of a GP whose first term is a, then $a/1-r$ is called sum to infinity of the GP.

5.23 ***Geometric mean and geometric means:*** If three numbers a, b and c are in GP, then b is called the Geometric mean (GM) between a and c and $b^2 = ac$. If x and y are positive real numbers, then x, $\sqrt{xy}$, y are in GP.

If $a, g_1, g_2, \ldots g_n, b$ are in GP, then $g_1, g_2, \ldots, g_n$ are called n geometric means between a and b.

5.24 ***Formula for GM's:*** If $g_1, g_2, g_3, \ldots, g_n$ are n GM's between a and b, then kth GM g_k is given by $g_k = a(b/a)^{k/n+1}$ for $k = 1, 2, \ldots, n$.

5.25 ***Product of n GM's:*** The product of n GM's between a and b is $(\sqrt{ab})^n$.

5.26 ***Arithmetic geometric progression (AGP):*** Sequence of numbers of the form $a, (a+d)r, (a+2d)r^2, + \cdots$ is called AGP and sum to n terms of an AGP is

$$\frac{a}{1-r} + \frac{dr(1-r^{n-1})}{(1-r)^2} - \frac{(a+(n-1)d)r^n}{1-r}$$

and
$$\frac{a}{1-r} + \frac{dr}{(1-r)^2}$$

is the sum to infinity.

5.27 ***AM–GM inequality:*** Let $a_1, a_2, \ldots, a_n$ be positive reals. Then

$$\frac{a_1 + a_2 + \cdots + a_n}{n}$$

is called AM of $a_1, a_2, \ldots, a_n$ and $(a_1 a_2 \cdots a_n)^{1/n}$ is called their GM. Further

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq (a_1 a_2 \cdots a_n)^{1/n}$$

and equality holds if and only if $a_1 = a_2 = a_3 = \cdots = a_n$.

5.28 ***Harmonic progression (HP):*** A sequence of non-zero reals is said to be in HP, if their reciprocals are in AP.

5.29 ***General form of an HP:*** Sequence of real numbers

$$\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \cdots \frac{1}{a+(n-1)d}, \cdots$$

can be taken as general form of an HP.

5.30 ***Harmonic mean and Harmonic means:***

(1) If a, b, c are in HP, then b is called the Harmonic mean (HM) between a and c and in this case $b = 2ac/a + c$.

(2) If $a, h_1, h_2, \ldots, h_n, b$ are in HP then $h_1, h_2, \ldots, h_n$ are called n HM's between a and b and further

$$h_K = \frac{ab(n+1)}{b(n+1) + K(a-b)} \quad \text{for } K = 1, 2, \ldots, n$$

5.31 ***Theorem:*** Let $a_1, a_2, \ldots, a_n$ be positive reals and A, G be AM and GM of the given numbers. Let

$$H = \frac{n}{1/a_1 + 1/a_2 + \cdots + 1/a_n}$$

which is called harmonic mean of $a_1, a_2, \ldots, a_n$. Then $A \geq G \geq H$ and equality holds if and only if $a_1 = a_2 = a_3 = \cdots = a_n$.

EXERCISES

Single Correct Choice Type Questions

1. If a, b and c are in AP, then $(a-c)^2$ is equal to
(A) $2(b^2-ac)$ (B) $4b^2-ac$
(C) b^2-4ac (D) $4(b^2-ac)$

2. Let $a_1, a_2, a_3, \ldots$ be an AP. If $a_3 = 7$ and $a_7 = 3a_3 + 2$, then the common difference is
(A) 1 (B) 4 (C) –1 (D) –4

3. Let a and b be positive real numbers. Then the sum of the first 10 terms of the series
$$\log a + \log\left(\frac{a^2}{b}\right) + \log\left(\frac{a^3}{b^2}\right) + \log\left(\frac{a^4}{b^3}\right) + \cdots$$
is
(A) $5(11\log a - 9\log b)$ (B) $5(10\log a - 9\log b)$
(C) $10(11\log a - 9\log b)$ (D) $50\log\left(\frac{a}{b}\right)$

4. The fourth power of the common difference of an arithmetic progression with integer entries is added to the product of any four consecutive terms of it. Then the resultant is
(A) (perfect square of an integer) + 1
(B) cube of an integer
(C) perfect square of an integer
(D) (cube of an integer) + 1

5. If
$$\log_3 2, \log_3(2^x - 5), \log_3\left(2^x - \frac{7}{2}\right)$$
are AP, then the value of x is
(A) 2 (B) 3 (C) 4 (D) 2 or 3

6. Let a and b be two positive integers and
$$\frac{a+b}{2} = x, \quad \sqrt{ab} = y \quad \text{and} \quad 2x + y^2 = 27$$
Then the numbers a and b are
(A) 6, 3 (B) 5, 10 (C) 6, 10 (D) 6, 12

7. Let $a_1, a_2, a_3, \ldots$ be an arithmetic progression of positive real numbers. Then
$$\frac{1}{\sqrt{a_1}+\sqrt{a_2}} + \frac{1}{\sqrt{a_2}+\sqrt{a_3}} + \cdots + \frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}} =$$
(A) $\dfrac{n+1}{\sqrt{a_1}+\sqrt{a_n}}$ (B) $\dfrac{n-1}{\sqrt{a_1}+\sqrt{a_n}}$
(C) $\dfrac{n}{\sqrt{a_1}+\sqrt{a_n}}$ (D) $\dfrac{n}{\sqrt{a_n}-\sqrt{a_1}}$

8. Let a and b be positive real numbers such that $a > b$ and $a + b = 4\sqrt{ab}$. Then $a:b$ is equal to
(A) $2+\sqrt{3} : 2-\sqrt{3}$ (B) $3+\sqrt{2} : 3-\sqrt{2}$
(C) $4+\sqrt{3} : 4-\sqrt{3}$ (D) $3:2$

9. The sum of the first n terms of two sequences in AP are in the ratio $(3n-13):(5n+21)$. Then the ratio of their 24th terms is
(A) 2 : 3 (B) 3 : 2 (C) 1 : 2 (D) 3 : 4

10. In an AP, the first term, the $(n-1)$th term and the nth term are a, b and c, respectively. Then the sum of the first n terms is
(A) $\dfrac{(a+b+2c)(a+b)}{2(b-c)}$ (B) $\dfrac{(a+b-2c)(a+b)}{2(b-c)}$
(C) $\dfrac{(a+b+2c)(a+c)}{2(c-b)}$ (D) $\dfrac{(2c-a-b)(a+c)}{2(c-b)}$

11. In an AP, if the mth term is $1/n$ and the nth term is $1/m$, then the (mn)th term is
(A) $\dfrac{mn(mn+1)}{mn-1}$ (B) $\dfrac{mn(mn-1)}{mn+1}$
(C) $\dfrac{mn(m+n)}{m-n}$ (D) independent of m and n

12. The sums of the first $n, 2n$ and $3n$ terms of an AP are s_1, s_2 and s_3, respectively. Then s_3 is equal to
(A) $2(s_2 - s_1)$ (B) $3(s_2 - s_1)$
(C) $4(s_2 - s_1)$ (D) $2s_1s_2$

13. The ages of boys in a certain class of a school follow an AP with the common difference 4 months. If the youngest boy is of 8 years and the sum of the ages of all the boys in the class is 168 years, then the number of boys in the class is
(A) 16 (B) 17 (C) 18 (D) 19

14. Let s_n be the first n terms of an AP with first term a and common difference d. If s_{kn}/s_n is independent of n, then
(A) $Kn = 6$ (B) $d = 2a$
(C) $a = 2d$ (D) $Kn = 8$

15. If the sum of the first m terms of an AP is equal to the sum of either the next n terms or to the next p terms, then
(A) $(m+n)\left(\dfrac{1}{m}-\dfrac{1}{p}\right) = (m+p)\left(\dfrac{1}{m}-\dfrac{1}{n}\right)$

(B) $(m+n)\left(\frac{1}{m}+\frac{1}{p}\right)=(m+p)\left(\frac{1}{m}+\frac{1}{n}\right)$

(C) $(m-n)\left(\frac{1}{m}-\frac{1}{p}\right)=(m-p)\left(\frac{1}{m}-\frac{1}{n}\right)$

(D) $(m-n)\left(\frac{1}{m}+\frac{1}{p}\right)=(m-p)\left(\frac{1}{m}+\frac{1}{n}\right)$

16. Suppose a, b, c are in AP and a^2, b^2, c^2 are in GP. If $a<b<c$ and $a+b+c=3/2$, then the value of a is

(A) $1/2\sqrt{2}$ (B) $1/2\sqrt{3}$

(C) $(1/2)-(1/\sqrt{3})$ (D) $(1/2)-(1/\sqrt{2})$

17. The sum of an infinite geometric series is 162 and the sum of its first n terms is 160. The inverse of the common ratio is a positive integer. Then a possible value of the common ratio is

(A) $-1/3$ (B) $1/3$ (C) $1/2$ (D) $-1/2$

18. Suppose that a, b, c are in GP and $a^x=b^y=c^z$. Then

(A) x, y, z are in GP (B) x, y, z are in AP

(C) x, y, z are in HP (D) xy, yz, zx are in HP

19. The distances passed over by a pendulum bob in successive swings are 16, 12, 9, 6.75, $\ldots$. Then the total distance traversed by the bob before it comes to rest is

(A) 60 (B) 64 (C) 65 (D) 67

20. $x_1, x_2, x_3, \ldots$ is an infinite sequence of positive integers in the ascending order are in GP such that $x_1 \cdot x_2 \cdot x_3 \cdot x_4 = 64$. Then x_5 is equal to

(A) 4 (B) 64 (C) 128 (D) 16

21. If $s_n = 1+2+3+\cdots+n$ and $S_n = 1^3+2^3+3^3+\cdots+n^3$, then

(A) $S_n = 2s_n$ (B) $S_n = s_n^2$

(C) $2S_n = 3s_n^2$ (D) $2S_n = s_n^2$

22. If $x = 1+3a+6a^2+10a^3+\cdots\infty$ and $y = 1+4b+10b^2+20b^3+\cdots\infty$ where $-1<a, b<1$, then $1+3ab+5(ab)^2+\cdots+\infty$ is $(1+ab)/(1-ab)^2$, where

(A) $a=\dfrac{x^{1/3}-1}{x^{1/3}},\ b=\dfrac{y^{1/4}-1}{y^{1/4}}$

(B) $a=\dfrac{x^{1/3}+1}{x^{1/3}},\ b=\dfrac{y^{1/3}+1}{y^{1/3}}$

(C) $a=\dfrac{x^{1/3}+1}{x^{1/3}},\ b=\dfrac{y^{1/4}+1}{y^{1/4}}$

(D) $a=\dfrac{(xy)^{1/12}-1}{(xy)^{1/12}}$

23. Let a and b be distinct positive real numbers. If $a, A_1, A_2, \ldots A_{2n-1}, b$ are in AP; $a, G_1, G_2, \ldots, G_{2n-1}, b$ are in GP and $a, H_1, H_2, \ldots, H_{2n-1}, b$ are in HP, then the roots of the equation $A_n x^2 - G_n x + H_n = 0$ are

(A) real and unequal (B) real and equal

(C) imaginary (D) rational

24. If S_r denotes the sum of the first r terms of a GP, then S_n, $S_{2n}-S_n$ and $S_{3n}-S_{2n}$ are in

(A) AP (B) GP (C) HP (D) AGP

25. Let a be the first term and r the common ratio of a GP. If A and H are the AM and HM of the first n terms of the GP, then the product $A \cdot H$ is equal to

(A) a^2r^{n-1} (B) ar^n (C) a^2r^n (D) a^2r^{2n}

26. In a GP of alternately positive and negative terms, any term is the AM of the next two terms. Then the common ratio $(\neq -1)$ is

(A) $-1/3$ (B) -3 (C) -2 (D) $-1/2$

27. If x, y and z are positive real numbers, then

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x}$$

belongs to the interval

(A) $[2, +\infty)$ (B) $[3, +\infty)$

(C) $(3, +\infty)$ (D) $(-\infty, 3)$

28. a, b, c be positive numbers in AP. Let A_1 and G_1 be AM and GM, respectively, between a and b, while A_2 and G_2 are AM and GM, respectively, between b and c. Then

(A) $A_1^2+A_2^2=G_1^2+G_2^2$ (B) $A_1 A_2 = G_1 G_2$

(C) $A_1^2-A_2^2=G_1^2-G_2^2$ (D) $A_1 G_2 = A_2 G_1$

29. Let a, b, c be positive and

$$P = a^2b+ab^2-ac^2-a^2c$$
$$Q = b^2c+bc^2-a^2b-ab^2$$

and $$R = c^2a+ca^2-b^2c-bc^2$$

If the quadratic equation $Px^2+Qx+R=0$ has equal roots, then a, b and c are in

(A) AP (B) GP (C) HP (D) AGP

30. If $a_1, a_2, \ldots, a_n$ are in HP and

$$a_1a_2+a_2a_3+a_3a_4+\cdots+a_{n-1}a_n = Ka_1a_n$$

then K is equal to

(A) n (B) $n-1$ (C) $n+1$ (D) $n+2$

31. If $H_1, H_2, \ldots, H_n$ are n HMs between a and b, then $\frac{H_1+a}{H_1-a}+\frac{H_n+b}{H_n-b}$ is equal to

(A) $n/2$ (B) n (C) $3n$ (D) $2n$

32. If S_1, S_2, S_3 are sums of the first n terms of three APs whose first terms are unity and their common differences are in HP, then

$$\frac{2S_3S_1 - S_1S_2 - S_2S_3}{(S_1 - 2S_2 + S_3)} =$$

(A) n (B) $2n$ (C) $2(n-1)$ (D) $3n$

33. If $H_1, H_2, \ldots, H_n$ are n HMs between a and b and n is a root of the equation $(1-ab)x^2-(a^2+b^2)x-(1+ab)=0$, then $H_1 - H_n$ is equal to

(A) $ab(a-b)$ (B) $ab(a+b)$
(C) $ab(a+b)^2$ (D) $a^2b^2(a+b)$

34. Sum to first n terms of the series

$$1+2\left(1+\frac{1}{n}\right)+3\left(1+\frac{1}{n}\right)^2+4\left(1+\frac{1}{n}\right)^3+\cdots$$

is

(A) n^2+1 (B) $(n-1)^2$ (C) n^2 (D) $(n+1)^2$

35. If a, b, c and d are positive such that $a+b+c+d=2$, then $M=(a+b)(c+d)$ satisfies

(A) $0 < M \le 1$ (B) $1 \le M \le 2$
(C) $2 \le M \le 3$ (D) $3 \le M \le 4$

36. Sum to first n terms of the series $1(1!)+2(2!)+3(3!)+4(4!)+\cdots$ is

(A) $n!+1$ (B) $(n+1)!+1$
(C) $(n+1)!$ (D) $(n+1)!-1$

37. Sum to infinity of the series

$$\frac{1}{(1+x)(1+2x)}+\frac{1}{(1+2x)(1+3x)}+\frac{1}{(1+3x)(1+4x)}+\cdots\infty$$

where $x \ne 0$ is

(A) $1/x$ (B) $1/x+1$
(C) $1/x(1+x)$ (D) $1/(x+1)^2$

38. Sum to n terms of the series $8+4+2+8+28+68+154+\cdots$ is

(A) $\frac{n}{12}[3n^3-14n^2-3n+100]$

(B) $\frac{n}{12}[3n^3-14n^2-3n+110]$

(C) $\frac{n^2}{12}[3n^2-14n+110]$

(D) $\frac{n}{12}[3n^3+14n^2-3n+100]$

39. Sum to n terms of the series $6+3+2+3+6+11+\cdots$ is

(A) $\frac{n}{6}(2n^2-15n+49)$ (B) $\frac{n}{6}(n^2+15n+49)$

(C) $\frac{n}{6}(2n^2-10n+49)$ (D) $\frac{n}{6}(2n^2+10n+49)$

40. Sum to n terms of the series $7+10+14+20+30+48+82+\cdots$ is

(A) $2^{n-1}+n^2+5n$ (B) 2^n+n^2+5n-1
(C) $2^{n-1}+n^2+5n-1$ (D) 2^n+n^2+5n+1

41. Given that α, β, a, b are in AP; α, β, c, d are in GP and α, β, e, f are in HP. If b, d, f are in GP, then

$$\frac{\beta^6-\alpha^6}{\alpha\beta(\beta^4-\alpha^4)} =$$

(A) 2/3 (B) 3/2 (C) 4/3 (D) 3/4

42. The sum of first 10 terms of an AP is 155, and the sum of the first 2 terms of a GP is 9. If the first term of the AP is equal to the common ratio of GP and the first term of the GP is equal to the common difference of AP, then the sum of the common difference of AP and the common ratio of GP maybe

(A) 8 (B) 5 (C) 4 (D) 16

43. The sum of an infinite GP is 2 and the sum of their cubes is 24. Then, values of the first term and the common ratio are, respectively,

(A) 3, –1/2 (B) –3, –1/2
(C) 2, –1/3 (D) –2, 1/3

44. Let s_n represent the sum of the first n terms of a GP with first term a and common ratio r. Then $s_1+s_2+s_3+\cdots+s_n$ is equal to

(A) $\frac{na}{1-r}-\frac{ar(1-r^n)}{(1-r)^2}$ (B) $\frac{na}{1-r}-\frac{ar(1-r^n)}{1-r^2}$

(C) $\frac{na}{1-r}+\frac{ar(1-r^n)}{(1-r)^2}$ (D) $\frac{na}{1-r}+\frac{ar(1-r^n)}{1-r^2}$

45. In a GP, the $(m+n)$th term is p and $(m-n)$th term $(m>n)$ is q. Then the mth term is

(A) pq (B) $(pq)^{n/m}$
(C) $\sqrt{pq}$ (D) $(pq)^{(m+n)/(m-n)}$

46. The sides of a right-angled triangle are in GP. If A and C are acute angles of the given triangle, then the values of $\tan A$ and $\tan C$ are

(A) $\frac{\sqrt{5}+1}{2}, \frac{\sqrt{5}-1}{2}$ (B) $\sqrt{\frac{\sqrt{5}+1}{2}}, \sqrt{\frac{\sqrt{5}-1}{2}}$

(C) $\sqrt{5}, \frac{1}{\sqrt{5}}$ (D) $\frac{\sqrt{3}+1}{2}, \frac{\sqrt{3}-1}{2}$

47. The length of the side of a square is 'a' units. A second square is formed by joining the midpoints of the sides. A third square is formed by joining the midpoints of the sides of the second square and this process is continued so on. Then the sum of the areas of all these squares is

(A) $a^2\sqrt{2}$ (B) $2a^2$ (C) $3/2a^2$ (D) $4a^2$

Multiple Correct Choice Type Questions

1. Let s_n and s'_n be sums of first n terms of two AP's with first terms a and b and common differences d and e respectively. If

$$\frac{s_n}{s'_n} = 2 \quad \text{and} \quad \frac{a+(n-1)d}{b} = \frac{b+(n-1)e}{a} = 4$$

then

(A) $\frac{d}{e} = 26$ (B) $\frac{a+(n-1)d}{b+(n-1)e} = \frac{7}{2}$

(C) $\frac{d}{e} = \frac{2}{7}$ (D) $\frac{a+(n-1)d}{b+(n-1)e} = 2$

2. The ratio of sums to first n terms of two AP's is $(7n+1):(4n+27)$. Then

(A) the ratio of their nth terms is $(14n-6):(8n+23)$

(B) the ratio of their mth terms is $(14m+6):(8m+23)$

(C) the ratio of their first terms is $8:31$

(D) the ratio of the first terms is $20:31$

3. If the mth term of an AP is $1/n$ and the nth term is $1/m$, then

(A) the first term is $1/mn$

(B) common differences $1/mn$

(C) (mn)th term is 1

(D) sum to mn terms is $(mn+1)/2$

4. Three numbers form an AP. The sum of the three terms is 3 and the sum of their cubes is 4. Then

(A) common difference is $\pm 1/\sqrt{6}$

(B) common difference is $1/6$

(C) product of the numbers is $5/6$

(D) product of the numbers is 6

5. Let $1, a_1, a_2, a_3, a_4, a_5$ and 0.3 be in AP. Then

(A) the common difference is $7/60$

(B) $a_3 = 13/20$

(C) $a_1 + a_2 + a_3 + a_4 + a_5 = 0.65$

(D) $a_1 = 53/60$

6. If α, β are the roots of the equation $x^2 - 4x + \gamma = 0$ and γ, δ are the roots of the equation $x^2 - 64x + \mu = 0$ and $\alpha < \beta < \gamma < \delta$ are n GPs, then

(A) $\lambda = 64/25$ (B) $\mu = 4^7/25$

(C) $\lambda = 8/5$ (D) $\mu = 64/25$

7. The sum of three numbers in GP is 70; if the two extreme terms be multiplied each with 4 and the middle by 5, the products are in AP. Then possible values of the common ratio are

(A) 2 (B) 3 (C) 1/2 (D) 1/3

8. The first term of an infinite GP is unity and any term is equal to the sum of all the succeeding terms. If the common ratio is r, then

(A) $r = 1/2$

(B) $r = 1/3$

(C) $1 + r + r^2 + \cdots + \infty$ is 2

(D) $1 + 3r + 5r^2 + 7r^3 + \cdots + \infty$ is 6

9. If $1 + (x-1) + (x-1)^2 + (x-1)^3 + \cdots + \infty$ exists, thus x may lie in the interval

(A) $0 < x < 2$ (B) $0 < x < 1$

(C) $-1 < x < 0$ (D) $-1 < x < -1/2$

10. Let n be a positive integer. If $\sum_{k=1}^{n} k, \sqrt{10}/3 \sum_{k=1}^{n} k^2$ and $\sum_{k=1}^{n} k^3$ are in GP, then

(A) $n = 4$

(B) $n = 5$

(C) The sum of the given terms is 10

(D) The common ratio of the GP is $\sqrt{10}$

11. If $a_1, a_2, a_3, \ldots$ are in GP such that

$$a_4 : a_6 = 1:4 \quad \text{and} \quad a_2 + a_5 = 216$$

then

(A) common ratio is ± 2

(B) $a_1 = 12$ or $108/7$

(C) sum of the first five terms is 200

(D) common ratio is $\pm 1/2$ and $a_1 = 6$ or $7/108$

12. For $0 < x < \pi/2$, if $\sin x$, $\sqrt{2}(\sin x + 1)$ and $6(\sin x + 1)$ are in GP, then
(A) common ratio is 1/2 (B) first term is 1/2
(C) common ratio is $3\sqrt{2}$ (D) fifth term is 162

13. For $0 < \theta < \pi/2$, let

$$x = \sum_{n=0}^{\infty} \cos^{2n}\theta,\ y = \sum_{n=0}^{\infty} \sin^{2n}\theta \quad \text{and} \quad z = \sum_{n=0}^{\infty} \cos^{2n}\theta \sin^{2n}\theta$$

then
(A) $xyz = xz + y$ (B) $xyz = xy + y$
(C) $xyz = x + y + z$ (D) $xyz = yz + x$

14. x, y, z are greater than 1 and are in GP. Let

$$a = \frac{1}{1+\log x},\ b = \frac{1}{1+\log y} \quad \text{and} \quad c = \frac{1}{1+\log z}$$

Then
(A) $(1-a)/a, (1-b)/b, (1-c)/c$ are in AP
(B) a, b, c are in GP
(C) $1/a, 1/b, 1/c$ are in AP
(D) $b = (2ac)/(a+c)$

15. Let a, x, b be in AP; a, y, b in GP and a, z, b in HP where a and b are distinct positive real numbers. If $x = y + 2$ and $a = 5z$, then
(A) $y^2 = zx$ (B) $x > y > z$
(C) $a = 9, b = 1$ (D) $a = 1/4, b - 9/4$

16. Let $a, b, c,$ be three real numbers. Then
(A) a, b, c are in AP if $(a-b)/(b-c) = 1$
(B) a, b, c are in GP if $(a-b)/(b-c) = a/b$
(C) a, b, c are in HP if $(a-b)/(b-c) = a/c$
(D) a, b, c are in HP if $(a-b)/(b-c) = c/a$

17. Let a_1, a_2, a_3 and a_4 be four positive real numbers. Then
(A) $a_2a_3 - a_1a_4 > 0$ if a_1, a_2, a_3, a_4 are in AP
(B) $a_2a_3 - a_1a_4 = 0$ if a_1, a_2, a_3, a_4 are in GP
(C) $a_2a_3 - a_1a_4 < 0$ if a_1, a_2, a_3, a_4 are in HP
(D) a_1, a_2, a_3 and a_4 are positive numbers not all equal, then

$$\frac{1}{4}(a_1 + a_2 + a_3 + a_4) > (a_1 a_2 a_3 a_4)^{1/4}$$

$$> \frac{4}{(1/a_1) + (1/a_2) + (1/a_3) + (1/a_4)}$$

18. If a, x, y, z, b are in AP, then the value of $x + y + z$ is 15, when a, x, y, z, b are in HP, then

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{5}{3}$$

In such case
(A) $a = 1, b = 9$ (B) $a = 2, b = 3$
(C) $a = 9, b = 1$ (D) $a = 3, b = 2$

19. If a, b, c are in AP and a^2, b^2, c^2 are in HP, then which of the following is true?
(A) $a = b = c$ (B) $-a/2, b, c$ are in GP
(C) $a, b, -c/2$ are in GP (D) $a/2, b, c$ are in HP

20. Assume d is the GM between ca and ab, e is the GM between ab and bc and f is the GM between bc and ca. If a, b, c are in AP, then
(A) d^2, e^2, f^2 are in AP
(B) d^2, e^2, f^2 are in GP
(C) $e + f, f + d, d + e$ are in GP
(D) $e + f, f + d, d + e$ are in HP

21. If a, b, c are in HP; b, c, d are in GP and c, d, e are in AP, then
(A) a, c, e are in GP (B) a, d, e are in GP
(C) b, c, e are in GP (D) $e = (ab^2)/(2a - b)^2$

22. If a, b, c and d are distinct positive real numbers and are in HP, then
(A) $ad < bc$ (B) $ad > bd$
(C) $(a + d) > (b + c)$ (D) $(a + d) < (b + c)$

23. Let s_n and s_∞ be, respectively, sum to n terms and sum to infinity of the series

$$\frac{1\cdot 2}{3!} + \frac{2\cdot 2^2}{4!} + \frac{3\cdot 2^3}{5!} + \frac{4\cdot 2^4}{6!} + \cdots$$

Then
(A) the nth term is $n \cdot 2^n/(n+2)!$
(B) $s_n = 1 - 2^{n+1}/(n+2)!$
(C) $s_n = 1/2 - 2^n/(n+1)!$
(D) $s_\infty = 1$

24. Let s_n be the sum to n terms of the series

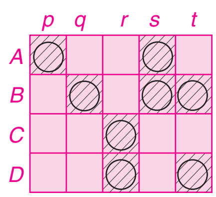
$$\frac{3}{1\cdot 2}\left(\frac{1}{2}\right) + \frac{4}{2\cdot 3}\left(\frac{1}{2}\right)^2 + \frac{5}{3\cdot 4}\left(\frac{1}{2}\right)^3 + \cdots$$

then
(A) $s_n = 1 - 1/(n+1)2^n$ (B) $s_n = 1/2 - 1/n\cdot 2^n$
(C) $\lim_{n\to\infty} s_n = \frac{1}{2}$ (D) $\lim_{n\to\infty} s_n = 1$

Matrix-Match Type Questions

In each of the following questions, statements are given in two columns, which have to be matched. The statements in Column I are labeled as **(A)**, **(B)**, **(C)** and **(D)**, while those in Column II are labeled as **(p)**, **(q)**, **(r)**, **(s)** and **(t)**. Any given statement in Column I can have correct matching with *one or more* statements in Column II. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following example.

Example: If the correct matches are (A) → (p), (s); (B) → (q), (s), (t); (C) → (r); (D) → (r), (t); that is if the matches are (A) → (p) and (s); (B) → (q), (s) and (t); (C) → (r); and (D) → (r), (t); then the correct darkening of bubbles will look as follows:



1. Match the items in Column I with those in Column II.

Column I	*Column II*
(A) The sum of all integers between 250 and 1000 which are divisible by 3 is	(p) 3050
(B) The sum of all odd numbers between 1 and 1000 that are divisible by 3 is	(q) 156375
(C) The sum of all integers from 1 to 100 which are divisible by exactly one of 2 and 5 is	(r) 3550
(D) If 7100 AMs are inserted between $\sin^2\theta$ and $\cos^2\theta$, then their sum is	(s) 83667
	(t) 83666

2. Match the items of Column I with those of Column II.

Column I	*Column II*
(A) $100\left(\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{99\cdot 100}\right)=$	(p) 7
(B) If x is the AM between two real numbers a and b, $y=a^{2/3}\cdot b^{1/3}$ and $z=a^{1/3}\cdot b^{2/3}$, then $y^3+z^3/xyz=$	(q) 9
	(r) 99
(C) If 198 AMs are inserted between 1/4 and 3/4, then the sum of these AM's is	(s) 100
(D) If n is a positive integer such that $n, [n(n-1)]/2$ and $[n(n-1)(n-2)]/6$ are in AP, then the value of n is	(t) 2

3. Match the items of Column I with those of Column II.

Column I	*Column II*
(A) If $1/a(b+c), 1/b(c+a), 1/c(a+b)$ are in HP, then a, b and c are in	(p) AP
(B) If $b+c, c+a, a+b$ are in HP, then $a/(b+c), b/(c+a), c/(a+b)$ are in	(q) GP
(C) If a, b, c are in HP, then $(1/a)+(1/bc)$, $(1/b)+(1/ca)$, $(1/c)+(1/ab)$ are in	(r) HP
(D) If a, b, c are in AP, then $(bc)/a(b+c)$, $(ca)/b(c+a)$, $(ab)/c(a+b)$	(s) Not in AP/GP/HP

4. In Column I some series are given and in Column II their nth terms are given. Match them.

Column I	*Column II*
(A) $3/4+5/36+7/144+9/400+\cdots$	(p) $3^{n-1}+n$
(B) $2+5+12+31+86+249+\cdots$	(q) $1/6(n^3+6n^2+11n+6)$
(C) $4+10+20+35+56+84+120+\cdots$	(r) $1/2(n^2-n+2)$
(D) $1+2+4+7+11+\cdots$	(s) $(2n+1)/n^2(n+1)^2$

5. Match the items of Column I to those of Column II.

Column I	*Column II*
(A) If a, b, c are positive real numbers such that sum of any two is greater than the third, then $$\sum\frac{\sqrt{a}}{\sqrt{b}+\sqrt{c}-\sqrt{a}}$$ is greater than or equal to	(p) 8
(B) If a, b, c are positive and $a+b+c=1$, then the minimum value of $(1+a)(1+b)(1+c)/(1-a)(1-b)(1-c)=$	(q) 2
(C) If a, b, c are positive reals then the minimum value of $(a+b+c)$ $(1/a)+(1/b)+(1/c)$ is K^2 where K is	(r) 3
(D) P is a point interior to the ΔABC. The lines AP, BP and CP meet the opposite sides in D, E and F, respectively. Then, the minimum value of $AP/PD+BP/PE+CP/PF=$	(s) 6

Comprehension-Type Questions

1. Passage: The terms $1, \log_y^x, \log_z^y$ and $-15\log_x^z$ are in AP. Based on this information, answer the following three questions.

(i) The common difference of the AP is

(A) 2 (B) –2 (C) 1/2 (D) –1/2

(ii) The value of xy is

(A) 1 (B) –1 (C) z^2 (D) z^3

(iii) yz is equal to

(A) x (B) x^2 (C) z^{-2} (D) z^{-3}

2. Passage: $a_1, a_2, a_3, \ldots, a_n, \ldots$ are in AP with common difference d. Further $\sin(A - B) = \sin A \cos B - \cos A \sin B$. Based on its information, answer the following questions.

(i) $\sec a_1 \sec a_2 + \sec a_2 \sec a_3 + \cdots + \sec a_{n-1} \sec a_n$ is equal to

(A) $\tan(a_{n+1}) - \tan a_1/\sin d$

(B) $\cot(a_{n+1}) - \cot a_1/\sin d$

(C) $\tan(a_{n+1}) + \tan a_1/\sin d$

(D) $\cot(a_{n+1}) + \cot a_1/\sin d$

(ii) $\operatorname{cosec} a_1 \operatorname{cosec} a_2 + \operatorname{cosec} a_2 \operatorname{cosec} a_3 + \cdots + \operatorname{cosec} a_{n-1} \operatorname{cosec} a_n$ is

(A) $\cot(a_{n+1}) + \cot a_1/\sin d$

(B) $\cot(a_{n+1}) - \cot a_1/\sin d$

(C) $\cot a_1 - \cot(a_{n+1})/\sin d$

(D) $\tan(a_{n+1}) - \cot a_n/\sin d$

(iii) If $a_1 = 0$, then

$$\left(\frac{a_3}{a_2}+\frac{a_4}{a_3}+\frac{a_5}{a_4}+\cdots+\frac{a_n}{a_{n-1}}\right)-a_2\left(\frac{1}{a_2}+\frac{1}{a_3}+\cdots+\frac{1}{a_{n-2}}\right)=$$

(A) $a_{n-1}/a_2 + a_2/a_{n-1}$ (B) $a_n/a_2 + a_2/a_n$

(C) $a_2/a_{n-1} - a_{n-1}/a_2$ (D) $a_2/a_n - a_n/a_2$

3. Passage: Let v_r denote the sum of the first r terms of an AP whose first term is r and the common difference $(2r-1)$. Let $T_r = v_{r+1} - v_r - 2$ and $Q_r = T_{r+1} - T_r$ for $r = 1, 2, 3, \ldots$. Then answer the following questions:

(i) The sum $v_1 + v_2 + \cdots + v_n$ is equal to

(A) $1/12n(n+1)(3n^2 - n + 1)$

(B) $1/12n(n+1)(3n^2 + n + 2)$

(C) $n/2(2n^2 - n + 1)$

(D) $1/3(2n^3 - 2n + 3)$

(ii) T_r is always

(A) an odd number

(B) an even number

(C) a prime number

(D) a composite number

(iii) Which of the following is a correct statement?

(A) $Q_1, Q_2, Q_3, \ldots$ are in AP with common difference 5

(B) $Q_1, Q_2, Q_3, \ldots$ are in AP with common difference 6

(C) $Q_1, Q_2, Q_3, \ldots$ are in AP with common difference 11

(D) $Q_1 = Q_2 = Q_3 = \cdots$

4. Passage: Let A_1, G_1, H_1 denote the AM, GM, and HM, respectively, of two distinct positive reals. For $n \ge 2$, let A_{n-1} and H_{n-1} have AM, GM and HM as A_n, G_n and H_n, respectively. Answer the following questions:

(i) Which one of the following statement is correct?

(A) $G_1 > G_2 > G_3 > \cdots$

(B) $G_1 < G_2 < G_3 < \cdots$

(C) $G_1 = G_2 = G_3 = \cdots$

(D) $G_1 < G_3 < G_5 < \cdots$ and $G_2 > G_4 > G_6 > \cdots$

(ii) Which of the following statements is correct?

(A) $A_1 < A_2 < A_3 < \cdots$

(B) $A_1 > A_2 > A_3 > \cdots$

(C) $A_1 > A_3 > A_5 > \cdots$

(D) $A_1 < A_3 < A_5 < \cdots$ and $A_2 > A_4 > A_6 > \cdots$

(iii) Which of the following statements is correct?

(A) $H_1 > H_2 > H_3 > \cdots$

(B) $H_1 > H_3 > H_5 > \cdots$ and $H_2 < H_4 < H_6 < \cdots$

(C) $H_1 < H_2 < H_3 < \cdots$

(D) $H_1 < H_3 < H_5 < \cdots$ and $H_2 > H_4 > H_6 > \cdots$

5. Passage: Let a and b distinct positive real numbers and

$$A = \frac{a+b}{2}, \quad G = \sqrt{ab} \quad \text{and} \quad H = \frac{2ab}{a+b}$$

Answer the following questions:

(i) If a, b are roots of the equation $x^2 - \lambda x + \mu = 0$, then

(A) $\lambda = 2A, \mu = G^2$ (B) $\lambda = A, \mu = G$

(C) $\lambda = -2A, \mu = G^2$ (D) $\lambda = -A, \mu = G$

(ii) If $a(b-c)x^2 + b(c-a)x + c(a-b) = 0$ has equal roots, then a, b, c are in

(A) AP (B) GP

(C) HP (D) Not in AP/GP/HP

(iii) A relation between A, G, H is

(A) $2H = A + G$ (B) $2G = A + H$

(C) $G^2 = A + H$ (D) $G^2 = AH$

Assertion–Reasoning Type Questions

In each of the following, two statements, I and II, are given and one of the following four alternatives has to be chosen.

(A) Both I and II are correct and II is a correct reasoning for I.

(B) Both I and II are correct but II is not a correct reasoning for I.

(C) I is true, but II is not true.

(D) I is not true, but II is true.

1. **Statement I:** x, y, z are positive and each is different from 1. If $2x^4 = y^4 + z^4$, $xyz = 8$ and $\log_y x$, $\log_z y$ and $\log_x z$ and $\log_y x$, $\log_z y$ and $\log_x z$ are in GP, then $x = y = z = 2$.

   Statement II: If a, b are positive and each is different from 1, then

$$\log_b^a = \frac{\log a}{\log b}$$

2. **Statement I:**
$$\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\frac{1}{3\cdot 4\cdot 5}+\cdots+\frac{1}{(n-2)(n-1)n}$$ is equal to $\frac{1}{2}\left[\frac{1}{2}-\frac{1}{(n-1)n}\right]$ for $n \geq 3$.

   Statement II: If K is a positive integer, then

$$\frac{1}{K(K+1)(K+2)} = \frac{1}{2}\left[\frac{1}{K(K+1)} - \frac{1}{(K+1)(K+2)}\right]$$

3. **Statement I:**
$$\left(1+\frac{1}{3}\right)\left(1+\frac{1}{9}\right)\left(1+\frac{1}{81}\right)\left(1+\frac{1}{3^8}\right)\cdots\left(1+\frac{1}{3^{2^n}}\right)$$ is equal to $\frac{3}{2}\left(1-\frac{1}{3^{2^{n+1}}}\right)$

   Statement II: $(a-b)(a+b) = a^2 - b^2$

4. **Statement I:** If the third term of a GP is 4, then the product of its first 5 terms is 4^4.

   Statement II: In a GP with first term a and common ratio r, the product of the first five terms is the fifth power of the third term.

5. **Statement I:** If non-zero numbers a, b, c, d are in AP, then the numbers abc, abd, acd, bcd are in HP.

   Statement II: If $a_1, a_2, a_3, \ldots, a_n \ldots$ are in AP and $k \neq 0$, then

$$\frac{a_1}{K}, \frac{a_2}{K}, \frac{a_3}{K}, \ldots, \frac{a_n}{K}, \ldots$$

   are also in AP.

6. **Statement I:** In ΔABC, if

$$\tan A \tan B + \tan B \tan C + \tan C \tan A = 9$$

   then the triangle is equilateral.

   Statement II: The arithmetic mean of finite set of positive real numbers is greater than or equal to their geometric (equality occurs if and only if all the numbers are equal).

7. **Statement I:** One can eliminate some terms of an AP of positive integers in such a way that the remaining terms form a GP.

   Statement II: If "a" is the first term and d be the common difference where both a and d are positive integer of an AP, then $a + ad$, $a + (2a + ad)d$, …, $a(1 + d)^n$, $n \geq 1$ belong to the AP.

8. **Statement I:** The sum to n terms of the series

$$\frac{3}{12}+\frac{5}{1^2+2^2}+\frac{7}{1^2+2^2+3^2}+\frac{9}{1^2+2^2+3^2+4^2}+\cdots \text{ is } \frac{6n}{n+1}$$

   Statement II: The nth term of the series

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots$$

   is

$$\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

9. **Statement I:** If d, e, f are in GP and the quadratic equations $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ and $dx^2 + ex + f = 0$ have a common root, then $d/a, e/b, f/c$ are in HP.

   Statement II: If α is a common root of the quadratic equations $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$, then

$$\alpha = \frac{c_1 a_2 - c_2 a_1}{a_1 b_2 - a_2 b_1}$$

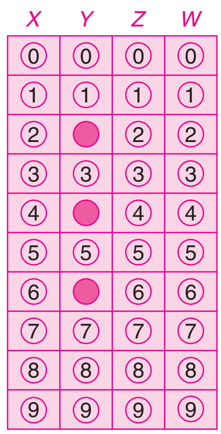
10. **Statement I:** If $\log_2(5.2^x + 1)$, $\log_4(2^{1-x} + 1)$ and 1 are in AP, then the value of x is $\log_2(0.4)$.

    Statement II: If a, b are positive and equal to 1, then

$$\log_b^a = \frac{\log_a}{\log_b}$$

Integer Answer Type Questions

The answer to each of the questions in this section is a non-negative integer. The appropriate bubbles below the respective question numbers have to be darkened. For example, as shown in the figure, if the correct answer to the question number Y is 246, then the bubbles under Y labeled as 2, 4, 6 are to be darkened.



1. Sum of one hundred AM's inserted between the numbers $\log_{10} 2$ and $\log_{10} 5$ is ______.

2. $f : \mathbb{R} \to \mathbb{R}$ is a function satisfying the relation $f(x+y) = f(x) + f(y)$ for all rational numbers x and y and $f(1) = 1$. If $\sum_{K=1}^{n} f(K) = 45$, then n value is ______.

3. For positive integer n, if $f(x) = (2 - x^n)^{1/n}$ and $g(x) = f(f(x))$, then $g(1)$, $g(2)$, $g(3)$, … form and AP with common differences ______.

4. Consider the equation
$$\left[\frac{x}{2}\right] + \left[\frac{2x}{3}\right] + \left[\frac{y}{4}\right] + \left[\frac{4y}{5}\right] = \frac{7x}{6} + \frac{21y}{20}$$
where $0 < x, y < 30$ and $[\cdot]$ denotes the integer part of a real number. $(x_1, y_1), (x_2, y_2), (x_3, y_3) \cdots$ are solutions of the above equation and $x_1 < x_2 < x_3 < \cdots$ are in AP whose common difference is ______.

5. x, y, z are real such that $x + y + z = 3$, $x^2 + y^2 + z^2 = 5$ and $x^3 + y^3 + z^3 = 7$. If $xy + yz + zx$, $x^3 + y^3 + z^3 - 3xyz$ and $x^4 + y^4 + z^4 + K$ are in AP ($K > 0$), then the values of K is ______.

6. If a, b, c are positive reals, then the maximum value of K such that $(1+a)(1+b)(1+c) > K(abc)^{4/7}$ is ______.

7. If x, y are positive real numbers and $3x + 4y = 5$, then the greatest value of $16x^2y^3$ is ______.

8. If a, b, c are positive and a + b + c = 1. Then the minimum value of
$$\left(\frac{1}{a} - 1\right)\left(\frac{1}{b} - 1\right)\left(\frac{1}{c} - 1\right)$$
is ______.

9. If x, y, z are positive real numbers such that $x^3 y^2 z^4 = 7$, then
$$2x + 5y + 3z \geq 9\left(\frac{525}{2^K}\right)^{1/9}$$
where K is equal to ______.

10. Three HMs are inserted between 1 and 3. Then 5[(first mean)/(third mean)] is equal to ______.

11. $a > b$ are positive real numbers and A, G are, respectively, their AM and GM. If $A = 2G$, then the ratio $a/b = K + 4\sqrt{3}$ where K is ______.

12. The second term of an infinite GP is 2 and its sum to infinity is 8. The first term is ______.

ANSWERS

Single Correct Choice Type Questions

1. (D)
2. (B)
3. (A)
4. (C)
5. (B)
6. (A)
7. (B)
8. (A)
9. (C)
10. (D)
11. (D)
12. (B)
13. (A)
14. (B)
15. (A)
16. (D)

17. (B)
18. (C)
19. (B)
20. (D)
21. (B)
22. (A)
23. (C)
24. (B)
25. (A)
26. (C)
27. (B)
28. (C)
29. (C)
30. (B)
31. (D)
32. (A)
33. (A)
34. (C)
35. (A)
36. (D)
37. (C)
38. (B)
39. (A)
40. (B)
41. (B)
42. (B)
43. (A)
44. (A)
45. (C)
46. (B)
47. (B)

Multiple Correct Choice Type Questions

1. (A), (B)
2. (A), (C)
3. (A), (B), (C), (D)
4. (A), (C)
5. (B), (D)
6. (A), (B)
7. (A), (C)
8. (A), (C), (D)
9. (A), (B)
10. (A), (D)
11. (A), (B)
12. (B), (C), (D)
13. (B), (C)
14. (A), (C), (D)
15. (A), (B)
16. (A), (B), (C)
17. (A), (B), (C), (D)
18. (A), (C)
19. (A), (B), (C)
20. (A), (D)
21. (A), (D)
22. (B), (C)
23. (A), (B), (D)
24. (A), (D)

Matrix-Match Type Questions

1. (A) → (q), (B) → (s), (C) → (p), (D) → (r)
2. (A) → (r), (B) → (t), (C) → (r), (D) → (p), (t)
3. (A) → (r), (B) → (p), (C) → (r), (D) → (r)
4. (A) → (s), (B) → (p), (C) → (q), (D) → (r)
5. (A) → (q), (r) (B) → (p), (C) → (r), (D) → (s)

Comprehension-Type Questions

1. (i) (B) (ii) (A) (iii) (C)
2. (i) (A) (ii) (C) (iii) (A)
3. (i) (B) (ii) (D) (iii) (B)
4. (i) (C) (ii) (B) (iii) (C)
5. (i) (A) (ii) (C) (iii) (D)

Assertion–Reasoning Type Questions

1. (A)
2. (A)
3. (A)
4. (D)
5. (A)
6. (A)
7. (A)
8. (A)
9. (B)
10. (A)

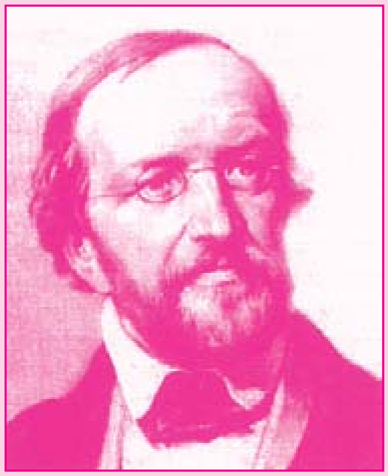
Integer Answer Type Questions

1. 50
2. 9
3. 1
4. 6
5. 7
6. 7
7. 3
8. 8
9. 7
10. 3
11. 7
12. 4

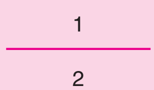
Permutations and Combinations

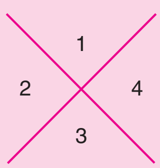
6

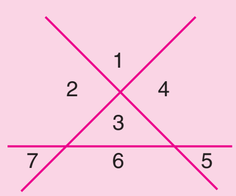
Permutations and Combinations



Johann Peter Gustav Lejeune **Dirichlet** was a German mathematician credited with the modern formal definition of a function. Dirichlet's brain is preserved in the anatomical collection of the University of Göttingen







Contents

Permutation: A **permutation** of a set of values is an arrangement of those values into a particular order. The arrangement can be linear or circular.

Combination: A **combination** is selection of objects from a set.

The concepts of permutations and combinations are important in view of several applications in day-to-day life and in the theory of probability (Probability is covered in Vol. II). A *combination* is only a *selection* while *permutation* is *selection* as well as *arrangement*.

Example

Forming a four-letter word using the letters of the word CHAPTER is a permutation, since it involves two steps, namely selection of four letters from among C, H, A, P, T, E and R and arrangement of these four letters. Suppose we select C, H, A and T. We can arrange them to form a four-letter word such as CHAT, CAHT, TCHA, so on. Forming a set with four letters is a combination which involves only one step, namely selection of four letters, say A, C, H, T. Then the four-element set formed is {A, C, H, T} which is same as {C, H, A, T}, {C, A, H, T}, {T, C, H, A}, etc.

In simpler terms, whenever there is importance to the arrangement or order in which the objects are placed, it is a permutation and, if there is no importance to the arrangement or order and only selection is required, it is a combination. One should be in a position to clearly see whether the concept of permutation or the concept of combination is applicable in a given situation. These concepts and methods we are going to develop in this chapter help us to determine the number of permutations or combinations without actually counting them.

6.1 | Factorial Notation

First, we introduce the factorial notation which is crucial in determining the number of permutations or combinations. For any positive integer n, we define $n!$ or $\lfloor n$ (read as *n factorial* or *factorial n*) recursively as follows:

$$n! = \begin{cases} 1 & \text{if } n = 1 \\ (n-1)!\cdot n & \text{if } n > 1 \end{cases}$$

Examples

(1) $1! = 1,\ 2! = 1!\cdot 2 = 2$

(2) $3! = 2!\cdot 3 = 2\cdot 3 = 6$

(3) $4! = 3!\cdot 4 = 6\cdot 4 = 24$

(4) $5! = 4!\cdot 5 = 24\cdot 5 = 120$

(5) Also, for convenience, we define $0! = 1$.

6.2 | Permutations

Before going to formal definitions and derivations we introduce the "*Fundamental Principle*" which plays a major role in the theory of permutations and combinations.

FUNDAMENTAL PRINCIPLE If a work W_1 can be performed in m different ways and another work W_2 in n different ways, then the two works can be performed simultaneously in mn different ways.

Example 6.1

A person has to travel from Chennai to Mumbai via Hyderabad. There are four different modes of travel from Chennai to Hyderabad, namely, car, bus, train and aeroplane (we denote these by A_1, A_2, A_3 and A_4, respectively) and that there are three different modes of travel from Hyderabad to Mumbai, namely bus, train and aeroplane (we denote these by B_1, B_2 and B_3, respectively). Then how many different modes of travel are available to that person to travel from Chennai to Mumbai via Hyderabad?

Solution: By the fundamental principle, there are $4\times 3 = 12$ different ways of travel from Chennai to Mumbai via Hyderabad. These are

$$\begin{array}{ccc} A_1 B_1 & A_1 B_2 & A_1 B_3 \\ A_2 B_1 & A_2 B_2 & A_2 B_3 \\ A_3 B_1 & A_3 B_2 & A_3 B_3 \\ A_4 B_1 & A_4 B_2 & A_4 B_3 \end{array}$$

Here, $A_1 \ B_1$ means travelling from Chennai to Hyderabad by car and from Hyderabad to Mumbai by bus; $A_4 \ B_2$ means travelling from Chennai to Hyderabad by aeroplane and from Hyderabad to Mumbai by train, etc.

The following is an abstraction of the fundamental principle.

THEOREM 6.1 If A is a set with m elements and B is a set with n elements, then $A \times B$ is a set with mn elements.

PROOF Let $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$. Then

$$A \times B = \{(a_i, b_j) \mid 1 \le i \le m \quad \text{and} \quad 1 \le j \le n\}$$

For each $a_i \in A$, there are n number of pairs whose first coordinate is a_i. These are

$$(a_i, b_1), (a_i, b_2), \ldots, (a_i, b_n)$$

The number of $a_i s$ is m. Therefore, the total number of elements in $A \times B$ is

$$n + n + \cdots + n \ \ (m \text{ times}) = mn$$

■

DEFINITION 6.1 For any finite set X, a bijection of X onto itself is called a ***permutation*** of X.

In other words, suppose X has n elements, say

$$X = \{x_1, x_2, \ldots, x_n\}$$

and suppose we have to keep elements of X in n different boxes, one in each box, as shown.

$$\begin{array}{ccccc} x_n & x_{n-1} & x_{n-2} & \cdots & x_1 \quad (n \text{ boxes}) \\ \downarrow & \downarrow & \downarrow & & \downarrow \\ \square & \square & \square & & \square \end{array}$$

An arrangement of this type is called a *permutation*. In the following, all permutations of a three-element set $\{a_1, a_2, a_3\}$ are given.

a_1	a_2	a_3
a_1	a_3	a_2
a_2	a_3	a_1
a_2	a_1	a_3
a_3	a_1	a_2
a_3	a_2	a_1

There are six permutations of a three-element set. If the number of elements of a given set X is large, it is not easy, as above, to enumerate the permutations of X. In the following, we develop a formula to find the number of such permutations.

Linear Permutations

In this section we would discuss linear permutations (i.e., arrangements of given objects in a line) with or without repetitions.

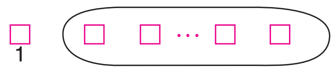
THEOREM 6.2 The number of permutations of an n-element set, taken all at a time, is $n!$

PROOF We will use induction on n.
If $n = 1$, then clearly there is only one permutation of a one-element set and $1! = 1$.

Let $n > 1$ and suppose that the number of permutations of any $(n-1)$-element set is $(n-1)!$.

Let X be an n-element set, say $X = \{x_1, x_2, \ldots, x_n\}$. Note that a permutation is a way of filling n blanks using the n elements of X with one element in each blank. Consider n blanks as given below:

To fill the first blank, we can use any of the n elements $x_1, x_2, \ldots, x_n$ in X. After filling the first blank, we are left with $n-1$ elements of X and these are to be used to fill up the remaining $n-1$ blanks. By induction hypothesis, the number of such permutations (filling the $n-1$ blanks with $n-1$ elements) is $(n-1)!$.



Now, we have two works W_1 and W_2. Work W_1 is filling up the first blank and work W_2 is filling up the remaining $(n-1)$ blanks. There are n ways of doing work W_1 and $(n-1)!$ ways of doing work W_2. Therefore, by the fundamental principle, the number of ways of doing works W_1 and W_2 simultaneously is $n \cdot (n-1)! = n!$. Thus there are $n!$ number of permutations of X. ■

Try it out Consider a five-element set. Choose any three elements and arrange them in three blanks. How many such arrangements can be made?

In the following we desire a formula for such a situation which generalizes the above theory.

THEOREM 6.3 Let n and r be positive integers and $r \le n$. Then the number of permutations of n (dissimilar) objects taken r at a time is equal to

$$n(n-1)(n-2)\cdots(n-r+1) = \prod_{s=0}^{r-1}(n-s)$$

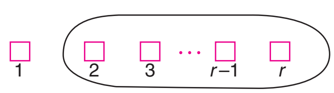
PROOF Note that the number of required permutations is equal to the number of ways of filling r blanks using the given n objects with one object in each blank. Again, we will use induction on n.

If $n = 1$, then $r = 1$ and the theorem is trivial.

Let $n > 1$ and assume the theorem is true for $n-1$; that is, for any k, $1 \le k \le n-1$, the number of permutations of $n-1$ dissimilar objects taken k at a time is

$$\prod_{s=0}^{k-1}[(n-1)-s] = (n-1)(n-2)\cdots[(n-1)-(k-1)]$$
$$= (n-1)(n-2)\cdots(n-k)$$

Diagrammatically it can be represented as follows:



To fill the first blank, we can use any one of the given n objects. Therefore, the first blank can be filled in n different ways. After filling up the first blank, we are left with $(n-1)$ objects and the left over $(r-1)$ blanks can be filled up with these $(n-1)$ objects. The number of different ways of filling the $(r-1)$ blanks using $(n-1)$ objects is

$$\prod_{s=0}^{(r-1)-1}[(n-1)-s] = (n-1)(n-2)\cdots(n-r+1)$$

Thus, by the fundamental principle, the number of ways of filling up the r places using n dissimilar objects is

$$n\cdot(n-1)(n-2)\cdots(n-r+1)$$ ■

DEFINITION 6.2 The number of permutations of n dissimilar things taken r at a time is denoted by ${}^n\mathrm{P}_r$ or $\mathrm{P}(n, r)$. However, ${}^n\mathrm{P}_r$ is more familiar and so we use this notation only. Therefore

$${}^n\mathrm{P}_r = n(n-1)(n-2)\cdots(n-r+1)$$

Note: As a convention, we define ${}^n\mathrm{P}_0 = 1$.

THEOREM 6.4 The following hold good for any positive integers n and r such that $r \le n$.

1. ${}^n\mathrm{P}_r = \dfrac{n!}{(n-r)!}$
2. ${}^n\mathrm{P}_r = n \cdot {}^{(n-1)}\mathrm{P}_{(r-1)}$
3. ${}^n\mathrm{P}_r = {}^{(n-1)}\mathrm{P}_r + r \cdot {}^{(n-1)}\mathrm{P}_{(r-1)}$

PROOF **1.** We have

$$\begin{aligned}{}^n\mathrm{P}_r &= n(n-1)(n-2)\cdots(n-r+1)\\ &= \frac{n(n-1)\cdots(n-r+1)(n-r)(n-r-1)\cdots 3\cdot 2\cdot 1}{(n-r)(n-r-1)\cdots 3\cdot 2\cdot 1}\\ &= \frac{n!}{(n-r)!}\end{aligned}$$

2. We have

$${}^n\mathrm{P}_r = \frac{n!}{(n-r)!} = \frac{n\cdot(n-1)!}{[(n-1)-(r-1)]!} = n\cdot {}^{(n-1)}\mathrm{P}_{(r-1)}$$

3. We have

$$\begin{aligned}{}^{(n-1)}\mathrm{P}_r + r\cdot {}^{(n-1)}\mathrm{P}_{r-1} &= \frac{(n-1)!}{(n-1-r)!} + \frac{(n-1)!r}{[(n-1)-(r-1)]!}\\ &= \frac{(n-1)!}{(n-r-1)!} + \frac{(n-1)!}{(n-r)!}r\\ &= \frac{(n-1)!}{(n-r-1)!}\left(1+\frac{r}{n-r}\right)\\ &= \frac{(n-1)!}{(n-r-1)!}\,\frac{n-r+r}{(n-r)}\\ &= \frac{n!}{(n-r)!} = {}^n\mathrm{P}_r\end{aligned}$$

■

Example 6.2

Find the number of permutations of four dissimilar things taken three at a time.

Solution: The number of permutations is

$${}^4\mathrm{P}_3 = \frac{4!}{(4-3)!} = \frac{4!}{1} = 24$$

Example 6.3

Find the number of four-letter words that can be formed using the letters of the word CHEMISTRY such that

(i) each word begins with letter T
(ii) each word ends with letter Y
(iii) each word begins with letter T and ends with letter Y

Solution: There are nine letters in the word CHEMISTRY. The total number of four-letter words using these nine letters is

$$^9P_4 = \frac{9!}{(9-4)!} = 9\cdot 8\cdot 7\cdot 6 = 3024$$

(i) If a four-letter word is to begin with T, the next three letters in the word can be chosen from the remaining $9-1=8$ letters. Therefore, the number of four-letter words each beginning with T is

$$^8P_3 = \frac{8!}{(8-3)!} = 8\cdot 7\cdot 6 = 336$$

(ii) A four-letter word is to end with Y means, we can choose the first three letters from among the remaining $9-1=8$ letters. Therefore, the number of four-letter words each ending with Y is $^8P_3 = 336$.

(iii) If a four-letter word is to begin with T and end with Y then the middle two letters can be chosen from among the remaining $9-2=7$ s. Therefore, the number of such words of is

$$^7P_2 = \frac{7!}{(7-2)!} = 42$$

Example 6.4

Find the number of ways of arranging 5 boys and 4 girls in a line so that there will be a boy in the beginning and at the ending.

Solution: There are 9 (5 boys + 4 girls) persons altogether. The first place and the last place are to be filled up by two boys from among 5 boys. The number of ways of doing is $^5P_2 = 20$.

B								*B*

All the 7 places in the middle are to be filled up by $7\,(9-2)$ persons (3 boys and 4 girls). The number of ways of doing this is 7!. Therefore, the total number of the required arrangements is $20\times 7! = 100800$.

Example 6.5

Find the number of four-letter words that can be formed using the letters of the word FRIENDS which contain the letter S and those which do not contain S.

Solution: There are 7 letters in the word FRIENDS. The total number of 4-letter words using these 7 letters is $^7P_4 = 840$.

Consider the four blanks given above. If the first blank is filled with S, then the remaining 3 blanks are to be filled using the remaining 6 letters. This can be done in $^6P_3$ ways. Similarly, the number of 4-letter words with S in the second place is $^6P_3$ and so are the numbers of words with S in each of third and fourth places. Therefore, the total number of 4-letter words containing S that can be formed with the letters in the word FRIENDS is

$$4\times {}^6P_3 = 4\times 120 = 480$$

The number of words not containing S is $840-480=360$. [Note that this is same as the number of 4-letter words using the 6 letters (other than S); that is $^6P_4$.]

Example 6.6

Find the number of ways of arranging the letters of the word KRISHNA such that all the vowels come together.

Solution: The number of letters in KRISHNA is 7 and among them there are two vowels, I and A. The vowels coming together means we have to treat the two vowels as one single unit.

Then we have 5 consonants +1 unit of vowels = 6 objects. These can be arranged in 6! ways.

The vowels can be permuted among themselves in 2! ways.

Therefore, the total number of arrangements in which the two vowels come together is $6!\times 2! = 1440$.

DEFINITION 6.3 If the words in a given set of words are arranged in the alphabetical order (as in a dictionary) and if a particular word is in the nth place in the list, then n is called the ***rank*** of that word.

Example 6.7

If the letters of the word PRISON are permuted in all possible ways and the words thus formed are arranged in dictionary order, then find the rank of the word SIPRON.

Solution: The sequence of the letters of the word PRISON in alphabetical order is INOPRS. In dictionary the words starting with I come first and the number of these is $5! = 120$. Next come the words starting with N and so on. The number of words starting with

I	is	$5! = 120$
N	is	$5! = 120$
O	is	$5! = 120$
P	is	$5! = 120$
R	is	$5! = 120$
SIN	is	$3! = 6$
SIO	is	$3! = 6$
SIPN	is	$2! = 2$
SIPO	is	$2! = 2$
SIPRN	is	$1! = 1$
SIPRON	is	$0! = 1$

Therefore, the rank of SIPRON is

$$(5 \times 120) + 6 + 6 + 2 + 2 + 1 + 1 = 618$$

Now, we derive a formula for the number of permutations of n dissimilar things taken r at a time when each thing can be repeated any number of times. First, we have below a natural generalization of the fundamental principle.

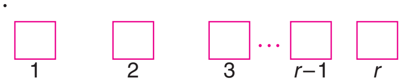
THEOREM 6.5 Let $A_1, A_2, \ldots, A_r$ be finite sets with $n_1, n_2, \ldots, n_r$ elements, respectively. Then the number of elements in $A_1 \times A_2 \times \cdots \times A_r$ is the product $n_1 n_2 \cdots n_r$.

PROOF We will use the fundamental principle and apply induction on r. If $r = 1$, the theorem is clean. Suppose that $r > 1$ and assume the theorem for $r - 1$. That is, the number of elements in $A_1 \times A_2 \times \cdots \times A_{r-1}$ is the product $n_1 n_2 \cdots n_{r-1}$. Since $A_1 \times A_2 \times \cdots \times A_{r-1} \times A_r$ and $(A_1 \times A_2 \times \cdots \times A_{r-1}) \times A_r$ are bijective and are finite sets, they have the same number of elements. By the fundamental principle and the induction hypothesis, the number of elements in $(A_1 \times A_2 \times \cdots \times A_{r-1}) \times A_r$ is $(n_1 n_2 \cdots n_{r-1}) n_r$ and hence the number of elements in $A_1 \times A_2 \times \cdots \times A_{r-1} \times A_r$ is $n_1 n_2 \cdots n_{r-1} n_r$. ■

COROLLARY 6.1 Let $W_1, W_2, \ldots, W_r$ be certain works. Suppose that W_i can be performed in n_i number of ways. Then the number of ways in which $W_1, W_2, \ldots, W_r$ can simultaneously be performed is $n_1 n_2 \cdots n_r$.

COROLLARY 6.2 Let n and r be positive integers such that $r \le n$. Then the number of permutations of n dissimilar things taken r at a time, when repetition of things is allowed any number of times, is n^r.

PROOF The number of required permutations is equal to the number of ways of filling up r blanks using the given n dissimilar things. If W_i is the work of filling the ith blank using the n things, then W_i can be performed in n number of ways. Therefore, $W_1, W_2, \ldots, W_r$ can be performed simultaneously in $n \cdot n \cdots n (r \text{ times}) = n^r$.



■

COROLLARY 6.3 The number of permutations of n dissimilar things taken r at a time, with atleast one repetition, is $n^r - {}^nP_r$.

PROOF The total number of permutations of n dissimilar things taken r at a time, without repetitions, is ${}^nP_r$. Therefore, the number of required permutations, with atleast one repetition, is $n^r - {}^nP_r$. ■

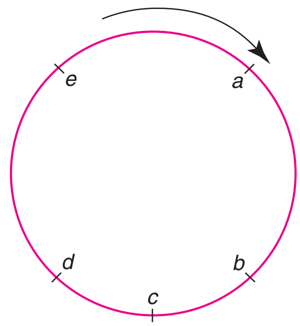
Example 6.8

A number lock has four rings and each ring has 10 digits, $0, 1, 2, \ldots, 9$. Find the maximum number of unsuccessful attempts that can be made by a thief who tries to open the lock without knowing the key code.

Solution: Each ring can be rotated in 10 different ways to get a digit on the top. Therefore, the total number of ways in which the four rings can be rotated is 10^4. Out of these, only one is a successful attempt and all the others are unsuccessful. Therefore, the maximum number of unsuccessful attempts is $10^4 - 1 = 9999$.

Circular Permutations

We now turn our attention on the arrangements of the given objects around a circle, that is, *circular permutations*. In this context, we come across two types of circular permutations. One is clockwise arrangement and the other is anticlockwise arrangement as shown in Figure 6.1. These two are same, but for the direction. In general, the direction is also important in circular permutations and hence we regard the two permutations shown in the figures below as two different circular permutations.



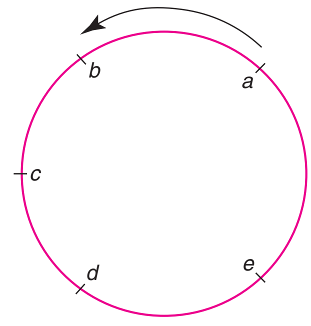


FIGURE 6.1 (a) Clockwise arrangement and (b) anticlockwise arrangement.

THEOREM 6.6 The number of circular permutations of n dissimilar things taken all at a time is $(n-1)!$.

PROOF Let N be the number of circular permutation of n things taken all at a time. If we take one such permutation it looks like as in Figure 6.2.

Starting at some point and reading in either clockwise or anticlockwise direction, but not both, we get n linear permutations from each circular permutation as shown for the one given in Figure 6.2.

$$\begin{array}{l} a_2\ a_3\ a_4 \cdots a_{n-1}\ a_n\ a_1 \\ a_3\ a_4\ a_5 \cdots a_n\ a_1\ a_2 \\ a_4\ a_5\ a_6 \cdots a_n\ a_1\ a_2\ a_3 \\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots \\ a_n\ a_1\ a_2 \cdots a_{n-2}\ a_{n-1} \\ a_1\ a_2\ a_3 \cdots a_{n-1}\ a_n \end{array}$$

Thus, each circular permutation gives rise to n linear permutations. Therefore, N circular permutations give rise to $N \times n$ linear permutations. But, we know that the number of linear permutations of n things, taken all at a time, is $n!$. Therefore

$$N \times n = n! = n \times (n-1)!$$

$$N = (n-1)!$$

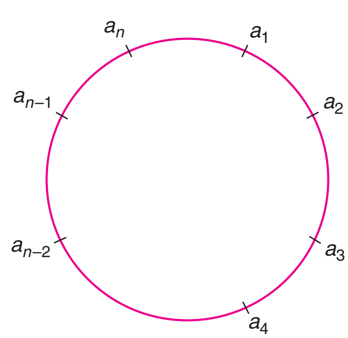


FIGURE 6.2 Theorem 6.6. ■

QUICK LOOK 1

Suppose we are to prepare a garland using n given flowers or a chain using n beeds. Any hanging type circular permutation looks like clockwise arrangement from one side and anticlockwise arrangement from the opposite side (in the same order of things). Hence, we should treat them as identical. Therefore, in such cases, the number of circular permutations of n things is half of the actual number of circular permutations; that is,

$$\frac{1}{2}(n-1)!$$

Example 6.9

Find the number of ways of arranging 5 boys and 8 girls around a circular table.

Solution: Total number of persons = 5 + 8 = 13. The total number of circular permutations is

$$(13-1)! = (12)!$$

Example 6.10

Suppose we are given 7 red roses and 4 yellow roses (no two of these are identical). Find the number of different ways of preparing a garland using all the given roses such that no two yellow roses come together.

Solution: First arrange the 7 red roses in a circular form in $(7-1)! = 6!$ ways.

Now, imagine a gap between two successive red roses. There are 7 such gaps and 4 yellow roses can be arranged in these 7 gaps in ${}^7P_4$ ways. Therefore, the total number of circular permutations is

$$6! \times {}^7P_4$$

But, in the case of garlands, clockwise and anticlockwise arrangements look alike. Thus, the number of possible distinct garlands is

$$\frac{1}{2}(6! \times {}^7P_4)$$

Next we consider permutations of things in which some are alike and the rest are different. For example, we may want to find the number of ways of permuting the letters of the word MATHEMATICS, in which there are 2 Ms, 2 As, 2 Ts and the rest are different. We derive formulae that can be used in such cases.

THEOREM 6.7 The number of linear permutations of n things, in which p things are alike and the rest are different, is $n!/p!$.

PROOF Suppose that we are given n things in which p are alike and the remaining are different. Let N be the number of permutations of these n things.

When we take one such permutation, it contains p like things. If we replace these p like things by p dissimilar things, then we can arrange these p things among themselves (without disturbing the relative positions of other things) in $p!$ ways. Therefore, each permutation when p things are alike gives rise to $p!$ permutations when all are different. Therefore, from the N such permutations we get $N \times p!$ permutations. But we know that the number of permutations of n different things is $n!$. Thus

$$N \times p! = n!$$

$$N = \frac{n!}{p!}$$ ■

We can extend, using induction, the above theorem for the case of having more than one set of like things in the given n things, by using Theorem 6.7 repeatedly.

COROLLARY 6.4 The number of linear permutations of n things, in which there are p alike things of one kind, q alike things of second kind and r alike things of third kind and the rest are different, is

$$\frac{n!}{p!q!r!}$$

Example 6.11

Find the number of ways of arranging the letters of the word ASSOCIATIONS.

Solution: There are 12 letters in the given word, among which there are 2 As, 3 Ss, 2 Os, 2 Is and the others (C, T, N) are different. Therefore, the required number of permutations is

$$\frac{(12)!}{2!\times 3!\times 2!\times 2!}=\frac{(12)!}{(2!)^3\times(3!)}$$

6.3 | Combinations

A combination is only a selection. There is no importance, as in the case of a permutation, to the order or arrangement of things in a combination. Thus, a combination of n things taken r at a time can be regarded as a subset with r elements of a set containing n elements.

QUICK LOOK 2

The number of combination of n dissimilar things taken r at a time is denoted by

$$^nC_r \quad \text{or} \quad \binom{n}{r} \quad \text{or} \quad C\,(n, r)$$

Note: ^nC_r is precisely the number of r-element subsets of an n-element set.

In the following theorem, we derive the formula for ^nC_r.

THEOREM 6.8 The combination of n dissimilar things taken r at a time is given by

$$^nC_r=\frac{^nP_r}{r!}=\frac{n!}{(n-r)!\,r!}=\frac{n(n-1)(n-2)\cdots(n-r+1)}{1\cdot 2\cdot 3\cdots(r-1)r}$$

That is, the number of combinations of n dissimilar things taken r at a time is

$$\frac{n!}{(n-r)!\,r!}$$

PROOF Any combination of r elements from among n dissimilar things can be treated as an r-element subset of an n-element set. Let us select one such combination of r elements and these r elements can be arranged in a line in $r!$ ways. Therefore, each combination of r elements gives rise to $r!$ number of permutations of r elements. So, the total number of permutations of n dissimilar things, taken r at a time, is equal to $^nC_r\times r!$. Therefore

$$^nP_r={}^nC_r\times r!$$

Thus

$$\begin{aligned}^nC_r&=\frac{^nP_r}{r!}=\frac{n!}{(n-r)!\,r!}\\&=\frac{n(n-1)\cdots(n-r+1)(n-r)\cdots 3\cdot 2\cdot 1}{(n-r)(n-r-1)\cdots 3\cdot 2\cdot 1\cdot r(r-1)\cdots 2}\\&=\frac{n(n-1)\cdots(n-r+1)}{1\cdot 2\cdot 3\cdots(r-1)r}\end{aligned}$$

■

COROLLARY 6.5 The number of r-element subsets of an n-element set is

$$^{n}C_{r} = \frac{n!}{(n-r)!r!}$$

PROOF The r-element subsets of an n-element set are precisely combinations of n dissimilar things taken r at a time. ■

Examples

(1) The number of subsets with exactly 4 elements in a set of 6 elements is

$$\frac{6!}{(6-4)!4!} = \frac{6\cdot 5}{1\cdot 2} = 15$$

(2) The number of ways of constituting a committee of 5 members from a group of 20 persons is

$$\frac{20!}{(20-5)!5!} = \frac{20\cdot 19\cdot 18\cdot 17\cdot 16}{1\cdot 2\cdot 3\cdot 4\cdot 5} = 15504$$

When we select r elements from n elements, we will be left with $n - r$ elements. Therefore, the number of ways of selecting r elements from the given n elements is equal to the number of ways of leaving $n - r$ elements. This is formally proved in the following theorem.

THEOREM 6.9 For any positive integers n and r with $r \le n$,

$$^{n}C_{r} = {}^{n}C_{n-r}$$

PROOF We have

$$\begin{aligned} ^{n}C_{r} &= \frac{n!}{(n-r)!\,r!} \\ &= \frac{n!}{r!\,(n-r)!} \\ &= \frac{n!}{[n-(n-r)]!\,(n-r)!} = {}^{n}C_{n-r} \end{aligned}$$ ■

COROLLARY 6.6 For any positive integer n, $^{n}C_{n} = {}^{n}C_{0} = 1$.

THEOREM 6.10 Let m and n be distinct positive integers. Then the number of ways of dividing $m + n$ things into two groups containing m things and n things is

$$\frac{(m+n)!}{m!\,n!}$$

PROOF When we select m things out of the given $(m + n)$ things, then n things will be left out. Therefore, the required number is precisely the number of ways of selecting m things from the $m + n$ things, $^{(m+n)}C_{m}$, equals

$$\frac{(m+n)!}{[(m+n)-m]!\,m!} = \frac{(m+n)!}{n!\,m!}$$ ■

COROLLARY 6.7 Let m, n and k be distinct positive integers. Then the number of ways of dividing $m + n + k$ things into three groups containing m things, n things and k things is

$$\frac{(m+n+k)!}{m!\,n!\,k!}$$

Proof First we select m things from $m + n + k$ things and then select n things from the remaining $n + k$ things and finally k things will be left out and they form the third group. The number of ways of selecting m things from the given $m + n + k$ is

$$\frac{(m+n+k)!}{[(m+n+k)-m]!\,m!} = \frac{(m+n+k)!}{(n+k)!\,n}$$

The number of ways of selecting n things from $n + k$ things is

$$\frac{(n+k)!}{n!\,k!}$$

By the fundamental principle, the number of ways of dividing $m + n + k$ things into three groups of m things, n things and k things is

$$\frac{(m+n+k)!}{(n+k)!\,m!} \times \frac{(n+k)!}{n!\,k!} = \frac{(m+n+k)!}{m!\,n!\,k!}$$

■

Corollary 6.8 The number of ways of dividing $2n$ things into two equal groups of n things each is

$$\frac{(2n)!}{2!\,n!\,n!}$$

Proof By Theorem 6.10, we can divide $2n$ things into two groups in $(2n!)/n!\,n!$ ways. Since the groups have equal number of elements, we can interchange them in 2! ways. They give rise to the same division. Therefore, $2n$ things can be divided into two equal groups of n things each in

$$\frac{(2n)!}{2!\,n!\,n!} \quad \text{ways}$$

■

The above can be generalized for the division of mn things into equal m groups, as given in the following.

Corollary 6.9 Let m and n be positive integers. Then the number of ways of dividing mn things into m groups, each containing n things, is

$$\frac{(mn)!}{m!\,(n!)^m}$$

Corollary 6.10 Let m and n be positive integers. The number of ways of distributing mn things equally among m persons is

$$\frac{(mn)!}{(n!)^m}$$

Proof By Corollary 6.9, we can divide mn things into m groups, each containing n things, in

$$\frac{(mn)!}{m!\,(n!)^m} \quad \text{ways}$$

In each such division, there are m groups, which are not identical but only contain equal number of things. We have to distribute m groups to m persons in $m!$ ways. Therefore, the total number of required distributions is

$$\frac{(mn)!}{m!\,(n!)^m} \times m! = \frac{(mn)!}{(n!)^m}$$

■

Example 6.12

Find the number of ways of selecting 11 member cricket team from a group of players consisting 7 batsmen, 5 bowlers and 3 wicket keepers such that there must be atleast 3 bowlers and 2 wicket keepers in the team.

Solution: The teams can be selected with the compositions given in Table 6.1. The last column of the table gives the number of ways of selecting the team.

Therefore, the total number of ways of selecting the 11-member cricket team is 35 + 105 + 175 + 315 + 210 + 210 which is equal to 1050.

Table 6.1 Example 6.12

Bowlers	*Wicket keepers*	*Batsman*	*Number of ways of selecting the team*
5	3	3	${}^5C_5 \times {}^3C_3 \times {}^7C_3 = 35$
5	2	4	${}^5C_5 \times {}^3C_2 \times {}^7C_4 = 105$
4	3	4	${}^5C_4 \times {}^3C_3 \times {}^7C_4 = 175$
4	2	5	${}^5C_4 \times {}^3C_2 \times {}^7C_5 = 315$
3	3	5	${}^5C_3 \times {}^3C_3 \times {}^7C_5 = 210$
3	2	6	${}^5C_3 \times {}^3C_2 \times {}^7C_6 = 210$

Example 6.13

Suppose that a set of m parallel lines intersect another set of n parallel lines. Then, find the number of parallelograms formed by these lines.

Solution: To form a parallelogram, we have to select two lines from the first set and two lines from the second set. The number of such selections is ${}^mC_2 \times {}^nC_2$.

We have proved earlier in Theorem 6.9 that, for any positive integers n and r such that $r \le n$, ${}^nC_r = {}^nC_{n-r}$. Converse of this is proved in the following.

THEOREM 6.11 Let r, s and n be positive integers such that $r \le n$ and $s \le n$. Then ${}^nC_r = {}^nC_s$ if and only if $r = s$ or $r = n - s$.

PROOF Suppose that ${}^nC_r = {}^nC_s$ and $r \ne s$. We can assume that $r < s$. Then $n - s < n - r$. Consider

$$
\begin{gathered}
{}^nC_r = {}^nC_s \\
\frac{n!}{(n-r)!\,r!} = \frac{n!}{(n-s)!s!} \\
(n-r)!\,r! = (n-s)!s! \\
(n-r)(n-r-1)\cdots(n-s+1)(n-s)!r! = (n-s)!s(s-1)(s-2)\cdots(r+1)r! \; (\text{since } n-s < n-r \text{ and } r < s) \\
(n-r)(n-r-1)\cdots(n-s+1) = s(s-1)\cdots(r+1) \\
(a+1)(a+2)\cdots(a+K) = (r+1)(r+2)\cdots(r+K)
\end{gathered}
$$

where $a = n - s$ and $K = s - r$. This gives $a = r$ and hence $n - s = r$. [Otherwise, if $a < r$, then $a + i < r + i$ for all $1 \le i \le K$ and hence $(a+1)(a+2)\cdots(a+K) < (r+1)(r+2)\cdots(r+K)$, since all these are positive integers.] Thus, if ${}^nC_r = {}^nC_s$, then $r = s$ or $r = n - s$ (i.e., $r + s = n$). ■

THEOREM 6.12 Let r and n be positive integers such that $r \le n$. Then,

$$ {}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r $$

PROOF We have

$$
\begin{aligned}
{}^nC_{r-1} + {}^nC_r &= \frac{n!}{[n-(r-1)]!\,(r-1)!} + \frac{n!}{(n-r)!\,r!} \\
&= \frac{n!}{(n-r)!\,(r-1)!}\left[\frac{1}{n-r+1} + \frac{1}{r}\right]
\end{aligned}
$$

$$= \frac{n!}{(n-r)!\,(r-1)!}\left[\frac{n+1}{(n-r+1)r}\right]$$

$$= \frac{(n+1)!}{[(n+1)-r]!\; r!} = {}^{n+1}\mathrm{C}_r$$ ■

COROLLARY 6.11 For $2 \le r \le n$,

$${}^n\mathrm{C}_{r-2} + 2\cdot{}^n\mathrm{C}_{r-1} + {}^n\mathrm{C}_r = {}^{n+2}\mathrm{C}_r$$

PROOF We have

$$\begin{aligned}{}^n\mathrm{C}_{r-2} + 2\cdot{}^n\mathrm{C}_{r-1} + {}^n\mathrm{C}_r &= ({}^n\mathrm{C}_{r-2} + {}^n\mathrm{C}_{r-1}) + ({}^n\mathrm{C}_r + {}^n\mathrm{C}_{r-1})\\ &= {}^{n+1}\mathrm{C}_{r-1} + {}^{n+1}\mathrm{C}_r = {}^{n+2}\mathrm{C}_r \quad \text{(from Theorem 6.12)}\end{aligned}$$ ■

Example 6.14

Find the value of ${}^{25}\mathrm{C}_4 + \sum_{r=0}^{4} {}^{(29-r)}\mathrm{C}_3$.

Solution: We have

$$\begin{aligned}{}^{25}\mathrm{C}_4 + \sum_{r=0}^{4} {}^{(29-r)}\mathrm{C}_3 &= {}^{25}\mathrm{C}_4 + {}^{25}\mathrm{C}_3 + {}^{26}\mathrm{C}_3 + {}^{27}\mathrm{C}_3 + {}^{28}\mathrm{C}_3 + {}^{29}\mathrm{C}_3\\ &= {}^{26}\mathrm{C}_4 + {}^{26}\mathrm{C}_3 + {}^{27}\mathrm{C}_3 + {}^{28}\mathrm{C}_3 + {}^{29}\mathrm{C}_3\\ &= {}^{27}\mathrm{C}_4 + {}^{27}\mathrm{C}_3 + {}^{28}\mathrm{C}_3 + {}^{29}\mathrm{C}_3\\ &= {}^{28}\mathrm{C}_4 + {}^{28}\mathrm{C}_3 + {}^{29}\mathrm{C}_3\\ &= {}^{29}\mathrm{C}_4 + {}^{29}\mathrm{C}_3 = {}^{30}\mathrm{C}_4\end{aligned}$$

Example 6.15

If ${}^{12}\mathrm{C}_{s+1} = {}^{12}\mathrm{C}_{2s-5}$, then find the value of s.

Solution:

$${}^{12}\mathrm{C}_{s+1} = {}^{12}\mathrm{C}_{2s-5} \Rightarrow s+1 = 2s-5 \quad\text{or}\quad s+1 = 12-(2s-5)$$

$$\Rightarrow s = 6 \quad\text{or}\quad 3s = 16$$

We take $s = 6$ (since s is an integer).

THEOREM 6.13 If p alike things are of one kind, q alike things are of second kind and r alike things are of third kind, then the number of ways of selecting any number of things (one or more) out of them is

$$(p+1)(q+1)(r+1)-1$$

PROOF From the first p things, we can select 0 or 1 or 2 or … or p things. Since all the p things are alike, we have to decide only the number of things to be selected. This can be done in $p + 1$ ways. Similarly, we can select any number of things from the second kind in $q + 1$ ways and from the third kind in $r + 1$ ways. Hence by the fundamental principle, we can select any number of things from the three groups in $(p+1)(q+1)(r+1)$ ways. But this includes the selection of 0 from each group. Since, we have to select one or more things, the number of required ways is

$$(p+1)(q+1)(r+1)-1$$ ■

COROLLARY 6.12 If $p_1, p_2, \ldots, p_r$ are distinct primes and $a_1, a_2, \ldots, a_r$ are positive integers, then the number of positive integers that divide $p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ is $(a_1+1)(a_2+1)\cdots(a_r+1)$.

PROOF Let $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$. Then any positive integer that divides n must be of the form $p_1^{b_1} p_2^{b_2} \cdots p_r^{b_r}$, where $0 \le b_i \le a_i$ for all $1 \le i \le r$ and b_is are integers. Now, each b_i can take $a_i + 1$ values that is $0, 1, 2, \ldots, a_i$. Therefore, as in Theorem 6.13, the number of positive integral divisors of n is

$$(a_1+1)(a_2+1)\cdots(a_r+1)$$ ■

Examples

(1) If there are 6 red beads of same type, 8 blue beads of same type and 10 yellow beads of same type, then the number of ways of selecting any number of beads (one or more) is

$$(6+1)(8+1)(10+1)-1=693$$

(2) Any divisor of n other than 1 and n is called a *proper divisor* of n. The number of proper positive integral divisors of 10800 $(=2^4\times 3^3\times 5^2)$ is

$$(4+1)(3+1)(2+1)-2=58$$

THEOREM 6.14 Let n be a positive integer. Then the number of ways in which n can be written as a sum of (atleast two) positive integers, considering the same set of integers in a different order as being different, is $2^{n-1}-1$.

PROOF Write n number of 1s on a line and put the symbol "(" on the left of the first 1 and the symbol ")" on the right of the last 1, as shown below:

$$(1-1-1-\cdots-1-1)$$

Consider the $n-1$ spaces between the two consecutive 1s. By filling each of these $n-1$ spaces with one of the two symbols "+" and ") + (", we get an expression of n as a sum of positive integers and vice versa. For example,

$$(1+1)+(1+1+1)+(1+1)+(1+1+1)$$

corresponds $2+3+2+3=10$ and

$$(1+1)+(1)+(1+1)+(1+1)+(1+1+1)$$

corresponds to $2+1+2+2+3$.

The number of ways of filling $n-1$ spaces each with one of the two symbols is 2^{n-1}. Among these we have to exclude one expression, namely,

$$(1+1+1+\cdots+1)=n$$

Since, we are interested only in sums with atleast two summands, thus, the number of required ways in which n can be written as a sum of (atleast two) positive integers is

$$2^{n-1}-1$$ ■

The proofs of the following two results are similar to that of Theorem 6.14.

THEOREM 6.15 Let m and n be positive integers such that $m\le n$. Then the number of m-tuples $(x_1, x_2, \ldots, x_m)$ of positive integers satisfying the equation $x_1+x_2+\cdots+x_m=n$ is ${}^{n-1}C_{m-1}$.

PROOF As in the proof of Theorem 6.14, write n number of 1s on a line and put the symbol "(" on the left of the first 1 and the symbol ")" on the right of the last 1 as shown below:

$$(1-1-1-1-\cdots-1-1)$$

Consider the $n-1$ gaps between the two consecutive 1s. Choose any $m-1$ of these gaps and fill them with the symbol ") + (" and the remaining gaps be filled with the symbol +. Then, we get an m-tuple $(x_1, x_2, \ldots, x_m)$ such that

$$x_1+x_2+\cdots+x_m=n$$

and vice-versa. For example, for $n = 10$ and $m = 4$

$$(1+1+1)+(1)+(1+1)+(1+1+1+1)$$

gives a 4-tuple (3, 1, 2, 4) with $3+1+2+4=10$ and conversely the tuple (2, 2, 3, 3) is obtained by the expression

$$(1+1)+(1+1)+(1+1+1)+(1+1+1)$$

Therefore, each choice of $m-1$ gaps from $n-1$ gaps gives rise to an m-tuple $(x_1, x_2, \ldots, x_m)$ of positive integers satisfying the equation

$$x_1 + x_2 + \cdots + x_m = n$$

and vice-versa. Thus the number of required m-tuples is ${}^{n-1}\mathrm{C}_{m-1}$. ■

THEOREM 6.16 Let m and n be positive integers. Then the number of m-tuples $(x_1, x_2, \ldots, x_m)$ of non-negative integers satisfying the equation $x_1 + x_2 + \cdots + x_m = n$ is ${}^{(n+m-1)}\mathrm{C}_{(m-1)}$.

PROOF We will slightly modify the proof of Theorem 6.15. Here x_is can be 0 also. Consider $n+m-1$ boxes in a row as shown:

$$\square \qquad \square \qquad \square \cdots \square \qquad \square$$

Let us choose any $m-1$ of these and label these chosen boxes as $b_1, b_2, \ldots, b_{m-1}$ from left to right. For $1 \le i \le m-1$, let x_{i+1} be the number of boxes that are not chosen between b_i and b_{i+1}. Let x_1 be the number of boxes to the left of b_1 and let x_m be the number of boxes to the right of b_{m-1}. It can be easily seen that this is a one-to-one correspondence between the $(m-1)$-element subsets of the $(n+m-1)$-element set of boxes onto the m-tuples $(x_1, x_2, \ldots, x_m)$ of non-negative integers satisfying the equation $x_1 + x_2 + \cdots + x_m = n$. For example, for $n = 4$ and $m = 6$, the 6-tuple $(1, 0, 0, 1, 0, 2)$ corresponds to the choice of b_1, b_2, b_3, b_4, b_5 given below.

$$\square \ \underset{b_1}{\blacksquare} \ \underset{b_2}{\blacksquare} \ \underset{b_3}{\blacksquare} \ \square \ \underset{b_4}{\blacksquare} \ \underset{b_5}{\blacksquare} \ \square \ \square$$

Thus, the number of required m-tuples is ${}^{(m+n-1)}\mathrm{C}_{(m-1)}$. ■

Examples

(1) The number of 6-tuples $(x_1, x_2, x_3, x_4, x_5, x_6)$ of positive integers satisfying the equation $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 12$ is

$${}^{(12-1)}\mathrm{C}_{(6-1)} = {}^{11}\mathrm{C}_5 = 462$$

(2) The number of 6-tuples $(x_1, x_2, x_3, x_4, x_5, x_6)$ of non-negative integers satisfying the equation $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 12$ is

$${}^{(12+6-1)}\mathrm{C}_{(6-1)} = {}^{17}\mathrm{C}_5 = 6188$$

THEOREM 6.17 The maximum number of parts into which a plane is cut by n lines is

$$\frac{n^2 + n + 2}{2}$$

PROOF Let $\psi(n)$ denote the maximum number of parts into which a plane is cut by n lines. We shall prove that

$$\psi(n) = \frac{n^2 + n + 2}{2}$$

by using induction on n. Clearly

$$\psi(1) = 2 = \frac{1^2 + 1 + 2}{2}$$

Note that the number of parts cut by n lines is maximum only when any two of these lines intersect. We can see from the adjoining figure that

$$\psi(2) = 4 = \frac{2^2 + 2 + 2}{2}$$

When we draw another line, intersecting these two, we get three more parts, as shown in Figure 6.3. In general, we can get n more parts by considering the nth line in addition to $\psi(n-1)$. That is,

$$\psi(n) = \psi(n-1) + n$$

Figure 6.3 shows the same for $n = 4$. By induction, we have

$$\begin{aligned}\psi(n) &= \psi(n-1) + n \\ &= \frac{(n-1)^2 + (n-1) + 2}{2} + n \\ &= \frac{n^2 + n + 2}{2}\end{aligned}$$

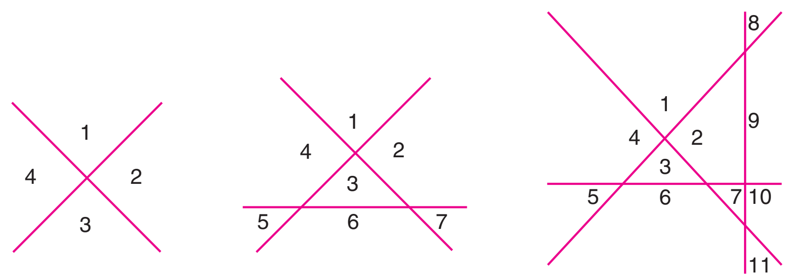


FIGURE 6.3 Theorem 6.17. ■

Example

The maximum number of parts into which a plane is cut by 8 lines is $\frac{8^2 + 8 + 2}{2} = 37$.

Note: The minimum number of parts into which a plane is cut by n lines is $n + 1$, since n parallel lines give us $n + 1$ parts. Figure 6.4 shows the same for 4 parallel lines. Any pair of intersecting lines gives us more number of parts.

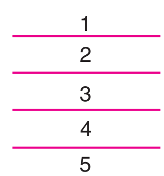


FIGURE 6.4 A plane cut by 4 lines into 5 parts.

Now, we will turn our attention to the number of various types of functions from a finite set into another finite set. We first prove following simple theorem.

THEOREM 6.18 Let X and Y be non-empty finite sets, $|X| = m$ and $|Y| = n$. Then

1. The number of functions from Y into X is m^n.
2. The number of injections (one-one functions) from Y into X is zero if $m < n$, and ${}^m\mathrm{C}_n \cdot n!$ $(= {}^m\mathrm{P}_n)$ if $m \geq n$.
3. The number of bijections of Y onto X is zero if $m \neq n$, and $m!$ if $m = n$.

PROOF

1. With each function $f: Y \to X$, each element of Y is to be mapped onto one element in the m-element set X. Since Y has n elements, by the fundamental principle, the number of functions of Y into X is m^n.
2. If there is an injection of Y into X, then $|Y| \leq |X|$. Therefore, if $n > m$, then there are no injections of Y into X. Suppose that $n \leq m$. If $f: Y \to X$ is an injection, then $|Y| = |f(Y)|$, that is, $f(Y)$ is an n-element subset of X. On the other hand, with each n-element subset Z of X, we can get

$n!$ number of bijections from Y onto Z, each of which can be treated as an injection of Y into X. Since $|X| = m$, the number of n-element subsets of X is ${}^m\mathrm{C}_n$. Thus, the number of injections of Y into X is

$$ {}^m\mathrm{C}_n \cdot n! = \frac{m!}{(m-n)!\,n!} n! = {}^m\mathrm{P}_n $$

3. We already have this (from part 2). ■

THEOREM 6.19 For any positive integers m and r such that $m \geq r$, let $\alpha_m(r)$ be the number of surjections of an m-element set onto an r-element set. Then

$$ \sum_{s=1}^{r} {}^r\mathrm{C}_s \alpha_m(s) = r^m $$

PROOF Let $m \geq r > 0$, A be an m-element set and B be an r-element set. The total number of mappings of A into B is r^m. Each mapping $f : A \to B$ can be regarded as a surjection of A onto $f(A)$; also $1 \leq |f(A)| \leq r$. On the other hand, with each s-element subset $(1 \leq s \leq r)$ C of B, any surjection of A onto C can be regarded as a mapping of A into B. Therefore, the total number of mappings of A into B is equal to the total number of surjections of A onto non-empty subsets of B. For each $1 \leq s \leq r$, there are ${}^r\mathrm{C}_s$ number of subsets of B and hence the number of mappings $f : A \to B$ such that $|f(A)| = s$ is ${}^r\mathrm{C}_s \alpha_m(s)$. Therefore,

$$ \sum_{s=1}^{r} {}^r\mathrm{C}_s \alpha_m(s) = r^m $$

■

COROLLARY 6.13 For any integers $m \geq r > 0$, the number $\alpha_m(r)$ of surjections of an m-element set onto an r-element set is given by a recursive formula

$$ \alpha_m(r) = r^m - \sum_{s=1}^{r-1} {}^r\mathrm{C}_s \alpha_m(s) $$

and

$$ \alpha_m(1) = 1 $$

Try it out Prove that $\alpha_m(r)$ is also equal to

$$ \sum_{s=0}^{r-1} (-1)^s \; {}^r\mathrm{C}_{(r-s)} (r-s)^m $$

Example 6.16

Let A be a 4-element set and B a 3-element set. Then evaluate the number $\alpha_4(3)$ of surjection of A onto B.

Solution: We have

$$ \alpha_4(1) = 1 $$

$$ \alpha_4(2) = 2^4 - {}^2\mathrm{C}_1 \cdot \alpha_4(1) = 14 $$

$$ \begin{aligned} \alpha_4(3) &= 3^4 - {}^3\mathrm{C}_1\alpha_4(1) - {}^3\mathrm{C}_2\alpha_4(2) \\ &= 81 - 3 \cdot 1 - 3 \cdot 14 \\ &= 36 \end{aligned} $$

Thus, there are 36 surjections of a 4-element set onto a 3-element set.

DEFINITION 6.4 Let X be a non-empty set. A bijection of X onto itself is called a *permutation* on X. A permutation f on X is called a *derangement* of X if $f(x) \neq x$ for all $x \in X$.

We will derive a recursive formula for the number of derangements of a finite set.

THEOREM 6.20 For any positive integer r, let d_r be the number of derangements of an r-element set. Then

$$1+\sum_{r=1}^{n} {}^{n}C_r d_r = n!$$

for any integer $n > 0$ or

$$\sum_{r=0}^{n} {}^{n}C_r d_r = n! \quad (\text{where } d_0 = 1)$$

and

$$d_n = n! - \sum_{r=0}^{n-1} {}^{n}C_r d_r$$

PROOF Let X be an n-element set, $n > 0$, and $P(X)$ be the set of all permutations on X. It is well-known that $P(X)$ has $n!$ elements. For any subset A of X, let

$$D(A) = \{f \in P(X) \mid f(a) \neq a \text{ for all } a \in A; \quad f(x) = x \text{ for all } x \in X - A\}$$

That is,

$$D(A) = \{f \in P(X) \mid f(x) \neq x \Leftrightarrow x \in A\}$$

Then, clearly $D(0)$ has only one element, namely the identity map and $D(X)$ is precisely the set of all derangements of X. For any $f \in P(X)$, we set that $f \in D(A)$, where $A = \{x \in X \mid f(x) \neq x\}$. Therefore, we get that

$$P(X) = \bigcup_{A \subseteq X} D(A)$$

It can be easily verified that

$$D(A) \cap D(B) = 0 \quad \text{whenever } A \neq B$$

and hence $P(X)$ is the disjoint union of $D(A)$s, $A \subseteq X$. Since there are ${}^{n}C_r$ number of subsets of X, each with r elements, it follows that

$$n! = |P(X)| = \sum_{A \subseteq X} |D(A)| = 1 + \sum_{r=1}^{n} {}^{n}C_r d_r \quad (\text{since } |D(0)| = 1)$$

where d_r is the number of derangements of an r-element set [since the members of $D(A)$ are in one-to-one correspondence with the derangements of A; $f \mapsto f/A$ is that one-to-one correspondence]. Thus

$$n! = 1 + \sum_{r=1}^{n} {}^{n}C_r d_r$$

This also can be expressed as

$$n! = \sum_{r=0}^{n} {}^{n}C_r d_r$$

where $d_0 = 1$. Note that $d_1 = 0$ and hence

$$d_n = n! - 1 - \sum_{r=2}^{n-1} {}^{n}C_r d_r$$

■

Try it out Prove that the number of derangements of an n-element set is

$$n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

Examples

We have

(1) $d_0 = 1$, by definition.

(2) $d_1 = 0$, since a singleton set cannot have derangement.

(3) $d_2 = 2! - 1 = 1$

(4) $d_3 = 3! - 1 - {}^3C_2 d_2 = 6 - 1 - 3 \cdot 1 = 2$

(5) $d_4 = 4! - 1 - {}^4C_2 d_2 - {}^4C_3 d_3 = 24 - 1 - 6 \cdot 1 - 4 \cdot 2 = 9$

(6) $d_5 = 5! - 1 - {}^5C_2 d_2 - {}^5C_3 d_3 - {}^5C_4 d_4$
$= 120 - 1 - 10 \cdot 1 - 10 \cdot 2 - 5 \cdot 9 = 44$

Example 6.17

List all the derangements of the 4-element set $\{1, 2, 3, 4\}$.

Solution: The derangements are as follows:

$1 \to 2$	$1 \to 2$	$1 \to 2$
$2 \to 1$	$2 \to 3$	$2 \to 4$
$3 \to 4$	$3 \to 4$	$3 \to 1$
$4 \to 3$	$4 \to 1$	$4 \to 3$

$1 \to 3$	$1 \to 3$	$1 \to 3$
$2 \to 1$	$2 \to 4$	$2 \to 4$
$3 \to 4$	$3 \to 1$	$3 \to 2$
$4 \to 2$	$4 \to 2$	$4 \to 1$

$1 \to 4$	$1 \to 4$	$1 \to 4$
$2 \to 1$	$2 \to 3$	$2 \to 3$
$3 \to 2$	$3 \to 1$	$3 \to 2$
$4 \to 3$	$4 \to 2$	$4 \to 1$

Try it out Show that there are 44 derangements of a 5-element set and there are 265 of a 6-element set.

THEOREM 6.21 Let n be a positive integer and $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be a prime decomposition of n. Then the number of distinct ordered pairs of positive integers (p, q), such that the least common multiple of p and q is n, is

$$(2\alpha_1 + 1)(2\alpha_2 + 1) \cdots (2\alpha_k + 1)$$

PROOF Since both p and q are factors n, we can suppose

$$p = p_1^{x_1} p_2^{x_2} \cdots p_k^{x_k} \quad \text{and} \quad q = p_1^{y_1} p_2^{y_2} \cdots p_k^{y_k}$$

where x_i and y_i $(i = 1, 2, \ldots, k)$ are non-negative integers. As n is the least common multiple of p and q, we have

$$\max\{x_i, y_i\} = \alpha_i$$

Hence, (x_i, y_i) can be equal to $(0, \alpha_i), (1, \alpha_i), (2, \alpha_i), \ldots, (\alpha_I, \alpha_I)$ and $(\alpha_I, 0), (\alpha_I, 1), (\alpha_I, 2), \ldots, (\alpha_I, \alpha_{I-1})$ whose number is $2\alpha_I + 1$. By multiplication principle, there are $(2\alpha_1 + 1)(2\alpha_2 + 1) \cdots (2\alpha_k + 1)$ ordered pairs of positive integers (p, q) whose least common multiple is

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$$

■

Example

Consider $n = 2^3 \times 5^2 \times 7^5$. Then the number of distinct ordered pairs of positive integers (p, q), whose least common multiple is $n = 2^3 \times 3^2 \times 7^5$, is

$$(6 + 1)(4 + 1)(10 + 1) = 7 \times 5 \times 11$$
$$= 385$$

WORKED-OUT PROBLEMS

Single Correct Choice Type Questions

1. If m and n are positive integers such that ${}^{m+n}P_2 = 90$ and ${}^{m-n}P_2 = 30$, then the number of ordered pairs (m, n) of such integers is

(A) 4 (B) 3 (C) 2 (D) 1

Solution: Given that

$$90 = {}^{m+n}P_2 = (m+n)(m+n-1)$$

and

$$30 = {}^{m-n}P_2 = (m-n)(m-n-1)$$

Therefore

$$(m+n)^2 - (m+n) - 90 = 0$$
$$(m+n-10)(m+n+9) = 0$$

We take $m+n=10$ (since m and n are positive). Similarly, $m-n=6$ (we have to consider $m > n$ only). Therefore $m=8$ and $n-2$ or

$$(m, n) = (8, 2)$$

Answer: (D)

2. If ${}^{2n+1}P_{n-1} : {}^{2n-1}P_n = 3:5$, then the value of n is

(A) 5 (B) 4 (C) 6 (D) 7

Solution: Given that

$$5 \cdot {}^{2n+1}P_{n-1} = 3 \cdot {}^{2n-1}P_n$$
$$\frac{5 \cdot (2n+1)!}{[(2n+1)-(n-1)]!} = \frac{3 \cdot (2n-1)!}{(2n-1-n)!}$$
$$\frac{5(2n+1)!}{(n+2)!} = \frac{3 \cdot (2n-1)!}{(n-1)!}$$
$$\frac{5(2n+1) \cdot 2n}{(n+2)(n+1)n} = 3$$
$$10(2n+1) = 3(n+2)(n+1)$$
$$3n^2 - 11n - 4 = 0$$
$$(n-4)(3n+1) = 0$$
$$n = 4$$

Answer: (B)

3. If four times the number of permutations of n distinct objects taken three at a time is equal to five times the number of permutations of $n-1$ distinct objects taken three at a time, then n is equal to

(A) 20 (B) 15 (C) 10 (D) 25

Solution: By hypothesis,

$$4\,{}^{n}P_3 = 5\,{}^{n-1}P_3$$
$$\frac{4 \cdot n!}{(n-3)!} = \frac{5 \cdot (n-1)!}{(n-1-3)!}$$
$$\frac{4n}{n-3} = 5$$
$$4n = 5(n-3)$$
$$n = 15$$

Answer: (B)

4. If ${}^{56}P_{r+6} : {}^{54}P_{r+3} = 30800:1$, then r is equal to

(A) 41 (B) 31 (C) 21 (D) 39

Solution: By hypothesis

$$\frac{(56)!}{(56-(r+6))!} = 30800 \cdot \frac{(54)!}{[54-(r+3)]!}$$
$$\frac{(56)!}{(50-r)!} = \frac{30800 \cdot (54)!}{(51-r)!}$$
$$56 \cdot 55 = \frac{30800}{(51-r)}$$
$$51 - r = \frac{30800}{56 \cdot 55} = 10$$
$$r = 41$$

Answer: (A)

5. If ${}^{9}P_5 + 5 \cdot {}^{9}P_4 = {}^{10}P_r$, then r is equal to

(A) 6 (B) 5 (C) 4 (D) 3

Solution: By hypothesis,

$$\frac{9!}{(9-5)!} + \frac{5 \cdot 9!}{(9-4)!} = \frac{(10)!}{(10-r)!}$$
$$9!\left(\frac{1}{4!} + \frac{5}{5!}\right) = \frac{(10)!}{(10-r)!}$$
$$\frac{(10)!}{(10-r)!} = 2 \cdot \frac{9!}{4!} = \frac{10!}{5!}$$
$$10 - r = 5$$
$$r = 5$$

Answer: (B)

6. There are finite number of distinct objects. If the arrangements of 4 objects (in a row) is 12 times the number of arrangements of 2 objects, then the number of objects is

(A) 10 (B) 8 (C) 4 (D) 6

Solution: Let the number of objects be n. Then, by hypothesis,

$$^{n}\mathrm{P}_4 = 12 \times {}^{n}\mathrm{P}_2$$

$$\frac{n!}{(n-4)!} = 12 \times \frac{n!}{(n-2)!}$$

$$1 = \frac{12}{(n-2)(n-3)}$$

$$n^2 - 5n - 6 = 0$$

$$(n-6)(n+1) = 0$$

Now $n \neq -1$, therefore $n = 6$.

Answer: (D)

7. The value of $\sum_{K=1}^{20} K \cdot {}^{K}\mathrm{P}_K$ is

(A) 20! + 1 (B) 21! – 1
(C) 20! – 1 (D) 21! – 20

Solution: We have

$$\sum_{K=1}^{20} K \cdot {}^{K}\mathrm{P}_K = \sum_{K=1}^{20} K \cdot \frac{K!}{1}$$

$$= \sum_{K=1}^{20} [(K+1)-1]K!$$

$$= \sum_{K=1}^{20} [(K+1)! - K!]$$

$$= (2! - 1!) + (3! - 2!) + \cdots + [(21)! - (20)!]$$

$$= (21)! - 1!$$

Answer: (B)

8. The number of 6-digit numbers that can be formed by using the numerals 0, 1, 2, 3, 4 and 5 (without repetition of the digits) such that even numbers occupy odd places is

(A) 48 (B) 24 (C) 36 (D) 72

Solution: We can arrange 0, 2, 4 in the odd places in 3! ways. After filling the odd places, the remaining 3 places can be filled by the remaining numbers (1,3 and 5) in 3! ways. But among these numbers, there are $2! \times 3!$ numbers in which 0 occupies the first place from the left. Therefore, the required number is

$$3! \cdot 3! - 2!\,3! = 36 - 12 = 24$$

Answer: (B)

9. The first 7 letters of the English alphabet are arranged in a row. The number of arrangements in which A, B and C are never separated is

(A) 5! (B) $3 \times 5!$ (C) $4! \times 5!$ (D) $3! \times 5!$

Solution: Consider A, B, C as one single object so that including this, there are 5 objects which can be arranged in 5! ways. In each of these A, B, C can be arranged among themselves in 3! ways. Therefore, the total number of required arrangements is $3! \times 5!$

Answer: (D)

10. A total of 6 boys and 5 girls are to be arranged in a row. The number of arrangements such that no two girls stand together is

(A) ${}^{11}\mathrm{P}_5$ (B) ${}^{11}\mathrm{P}_6 + {}^{11}\mathrm{P}_5$
(C) $\dfrac{11!}{6!}$ (D) $\dfrac{7!}{2!} \times 6!$

Solution: Given that each girl should stand in between two boys (there are 5 such places since the number of boys is 6) or before the boys or after the boys. Therefore, there are 7 eligible places for the 5 girls and hence they can be arranged in ${}^{7}\mathrm{P}_5$ ways. But the 6 boys can be arranged among themselves in 6! ways. Therefore the required number of arrangements

$${}^{7}\mathrm{P}_5 \times 6!$$

Answer: (D)

11. A total of 5 mathematics, 3 physics and 4 chemistry books are to be arranged in a shelf such that the books on the same subject are never separated. If one particular mathematics book is to be in the middle of all the mathematics books, then the number of arrangements is

(A) $3!(5! + 4! + 3!)$ (B) $3!(5! \times 4! \times 3!)$
(C) $3!(4! \times 4! \times 3!)$ (D) $3!(4! \times 3! \times 3!)$

Solution: Consider the books on the same subject as a single bundle, so that 3 bundles can be arranged in 3! ways. But mathematics books can be arranged among themselves in 4! ways (since one book is fixed in the middle) the physics books in 3! ways and chemistry books in 4! ways. Therefore, the required number is

$$3!(4! \times 4! \times 3!)$$

Answer: (C)

12. The number of arrangement of the letters of the word BANANA in which two Ns do not appear adjacently is

(A) 40 (B) 60 (C) 80 (D) 100

Solution: Letters other than Ns are BAAA. There are 4 ways of arranging these (B is the first place, B in the second place, etc.). Two Ns are to be arranged in between B, A, A, A. There are five places:

$$\underset{1^{st}}{\uparrow}\ \mathrm{B}\underset{2^{nd}}{\uparrow}\ \mathrm{A}\underset{3^{rd}}{\uparrow}\ \mathrm{A}\underset{4^{th}}{\uparrow}\ \mathrm{A}\underset{5^{th}}{\uparrow}$$

The insertion of Ns can be made in ${}^5P_2 \times (1/2!)$ ways. Therefore the total required number is

$$\frac{5!}{3!2!} \times 4 = 40$$

Answer: (A)

13. A five-letter word is to be formed by using the letters of the word MATHEMATICS such that

(i) odd places of the word are to be filled with unrepeated letters and

(ii) even places are to be filled with repeated letters.

Then the number of words thus formed is

(A) 300 (B) 360 (C) 180 (D) 540

Solution: We have five places:

1	2	3	4	5

There are three odd places and two even places in a five-letter word.

Unrepeated letters: H, E, I, C, S
Repeated letters: AA, MM, TT

The 3 odd places can be filled with the 5 unrepeated letters in

$${}^5P_3 = 60 \quad \text{ways}$$

The 2 even places can be filled with 2 different or 2 alike letters from the repeated letters in

$${}^3P_2 + 3 = 9 \quad \text{ways}$$

Therefore, the number of words thus formed is

$$60 \times 9 = 540$$

Answer: (D)

14. A five-digit number divisible by 3 is to be formed using the numerals 0, 1, 2, 3, 4 and 5 without repetition. The total number of ways this can be done is

(A) 216 (B) 240 (C) 600 (D) 3125

Solution: A number is divisible by 3 if and only if the sum of its digits is divisible by 3. Therefore the numerals to be used are 0, 1, 2, 4, 5 or 1, 2, 3, 4, 5.

In the first case, the number is $5! - 4! = 96$.
In the second case this is $5! = 120$.

Therefore, the required number is $96 + 120 = 216$.

Answer: (A)

15. An n-digit number means a positive integer having n digits. A total of 900 distinct n-digit numbers are to be formed using only the numerals 2, 5 and 7. The smallest value of n for which this is possible is

(A) 6 (B) 7 (C) 8 (D) 9

Solution: Each of the places can be filled in 3 ways (with 3, 5 or 7). The total number of ways

$$3 \times 3 \times \cdots \times 3 (n \text{ times}) = 3^n$$

Now,

$$3^n \geq 900 \Leftrightarrow 3^{n-2} \geq 100 \Leftrightarrow n - 2 \geq 5 \Leftrightarrow n \geq 7$$

Therefore the smallest value of such $n = 7$.

Answer: (B)

16. The letters of the word MOTHER are arranged in all possible ways and the resulting words are written as in a dictionary. Then the rank of the word "Mother" is

(A) 301 (B) 304 (C) 307 (D) 309

Solution:

No. of words beginning with E $= 5! = 120$
No. of words beginning with H $= 5! = 120$
No. of words beginning with ME $= 4! = 24$
No. of words beginning with MH $= 4! = 24$
No. of words beginning with MOE $= 3! = 6$
No. of words beginning with MOH $= 3! = 6$
No. of words beginning with MOR $= 3! = 6$
No. of words beginning with MOTE $= 2! = 2$
No. of words beginning with MOTHER $= 1$

Therefore, the rank of the word "Mother" is

$$120 + 120 + 24 + 24 + 6 + 6 + 6 + 2 + 1 = 309$$

Answer: (D)

17. There are 2 professors each of mathematics, physics and chemistry. The number of ways these 6 professors can be seated in a row so that professors of the same subject be seated together is

(A) 48 (B) 36 (C) 24 (D) 120

Solution: Treat professors of the same subject as a single object. Now 3 objects can be arranged in 3! ways. But, professors of the same subject can be interchanged among themselves in $2! \times 2! \times 2!$ ways. Therefore, the required number is

$$3! \times 2! \times 2! \times 2! = 48$$

Answer: (A)

18. Suppose one has to form 7-digit numbers using the numerals 1, 2, 3, 4, 5, 6 and 7. If the extreme places are occupied by even numerals, then the number of such numbers is

(A) 720 (B) 360 (C) 5040 (D) 120

Solution: There are 3 even numbers (2, 4 and 6) and 2 places (first and last) are to be filled by these. This can be done in ${}^3P_2$ ways. The remaining five places are to be filled by the remaining five numerals. This can be done in 5! ways. Therefore the required number is

$$^{3}P_2 \times 5! = 720$$

Answer: (A)

19. The number of ways of arranging the digits 1, 1, 1, 1, 1, 2, 2, 2, 3, 3, 3, 4, 5, 5, and 6 taken all at a time so that the 3s are separated is

(A) 95315040 (B) 95135040
(C) 95135400 (D) 95153040

Solution: Each 3 can be placed in between the other digits or before them or at the end in

$$\frac{^{13}P_3}{3!} \quad \text{ways}$$

Remaining 12 can be arranged in

$$\frac{12!}{5!\ 3!\ 2!} \quad \text{ways}$$

Therefore the required number is

$$\frac{^{13}P_3}{3!} \times \frac{12!}{5!\ 3!\ 2!} = 95135040$$

Answer: (B)

20. Five letters are to be inserted into five addressed envelopes. The number of ways of inserting the letters so that no letter goes to its corresponding envelope is

(A) 120 (B) 44 (C) 31 (D) 41

Solution: The number of required ways is

$$5!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}\right) = 120\left(\frac{1}{2}-\frac{1}{6}+\frac{1}{24}-\frac{1}{120}\right)$$
$$= 60 - 20 + 5 - 1 = 44$$

(***Note:*** We have made use of the result in "Try it out" after Theorem 6.20.)

Answer: (B)

21. The number of 5-digit numbers by using 1, 1, 1, 2, 2 is

(A) 7 (B) 8 (C) 9 (D) 10

Solution: The required number is

$$\frac{5!}{3!\ 2!} = 10$$

Answer: (D)

22. The number of cyclic permutations of 5 distinct objects is

(A) 24 (B) 120 (C) 60 (D) 42

Solution: The required number is $4! = 24$.

Answer: (A)

23. The total number of numbers that can be formed by using all the digits 1, 2, 3, 4, 3, 2, 1 so that odd digits always occupy odd places is

(A) 24 (B) 20 (C) 18 (D) 28

Solution: Among the given digits, there are four odd digits (1, 3, 3, 1) and there are four odd places (the first, third, fifth, seventh). These four odd places can be filled by 1, 3, 3, 1 in

$$\frac{4!}{2!\ 2!} = 6 \quad \text{ways}$$

The remaining 3 places (second, fourth, sixth) can be filled with 3 digits (2, 4, 2) in

$$\frac{3!}{2!} = 3 \quad \text{ways}$$

The total number of required arrangements is $6 \times 3 = 18$.

Answer: (C)

24. Let there be four lines and four circles in a plane. Let A be the set of all points of intersections of the lines, B the set of all points of intersections of the circles and C the set of all points of intersections of the lines and circles. If $n(S)$ denotes the number of elements in S, then the maximum value of $n(A \cup B \cup C)$ is

(A) 40 (B) 50 (C) 66 (D) 70

Solution: The maximum value of $n(A)$ is $^{4}C_2 = 6$. The maximum value of $n(B)$ is $2 \times {}^{4}C_2 = 12$. Since any line cuts a circle at the most in 2 points, the maximum number of points in which the lines intersect the circles is $8 \times 4 = 32$. Therefore the maximum value of

$$n(A \cup B \cup C) = 6 + 12 + 32 = 50$$

Answer: (B)

25. The total number of mappings from a four element set to a three element set is

(A) 54 (B) 12 (C) 27 (D) 81

Solution: If $A = \{x_1, x_2, x_3, x_4\}$ and $B = \{a_1, a_2, a_3\}$, then any mapping of A into B should take x_1 to any of a_1, a_2, a_3 and likewise x_2, x_3 and x_4. Therefore the number of mapping from A into B is $3 \times 3 \times 3 \times 3 = 81$.

Answer: (D)

26. Let $f(x) \equiv x^2 - 3x + 3$ and x_1, x_2, x_3 and x_4 be solutions of the equation $f(f(x)) = x$. Then the number of arrangements of x_1, x_2, x_3 and x_4, taken all at a time, is

(A) 24 (B) 4 (C) 6 (D) 1

Solution: Note that every solution of $f(x)=x$ is also a solution of $f(f(x))=x$.

$$f(x)=x \Rightarrow x^2-4x+3=0 \Rightarrow x=3 \quad \text{or} \quad 1$$

Therefore, 3 and 1 are roots of $f(x)=x$. Also,

$$f(f(x))=x \Rightarrow (x^2-3x+3)^2-3(x^2-3x+3)+3=x$$
$$\Rightarrow x^4-6x^3+12x^2-10x+3=0$$

Since 3 and 1 are roots of $f(x)=x$, they are roots of $f(f(x))=x$ also and therefore

$$f(f(x))-x=(x-3)(x-1)(x^2-2x+1)=(x-3)(x-1)^3$$

Therefore, 3, 1, 1, 1 are solutions of $f(f(x))=x$. Hence the number of arrangements of the solutions is

$$\frac{4!}{3!}=4$$

Answer: (B)

27. Professors a, b, c and d are conducting an oral examination for a Ph.D. student x on combinatorics. The professors are to sit in chairs in a row. Professors a and b are to sit together. Professor c is the guide of x and he has to sit by the side of Professors a and b. The number of arrangements is

(A) 8 (B) 7 (C) 6 (D) 5

Solution: The arrangements of a, b, c must be abc, bac, cab, cba. In each of these arrangements, d can take his seat at either end. Therefore the number of arrangements is

$$4\times 2=8$$

Answer: (A)

28. There are 2 copies of each of 3 different books. The number of ways they can be arranged in a shelf is

(A) 12 (B) 60 (C) 120 (D) 90

Solution: Totally there are 3 sets of 2 alike books. The total number of books is 6. Therefore the number of arrangements is

$$\frac{6!}{2!\ 2!\ 2!}=\frac{720}{8}=90$$

Answer: (D)

29. The letters of the word COCHIN are permuted and all permutations are arranged in alphabetical order as in a dictionary. The number of words that appear before the word COCHIN is

(A) 360 (B) 192 (C) 96 (D) 48

Solution: The given word is COCHIN.

The no. of words beginning with CC is 4!
The no. of words beginning with CH is 4!
The no. of words beginning with CI is 4!
The no. of words beginning with CN is 4!
The next word is COCHIN

Therefore, the number of words before COCHIN is $4\times 4!=96$.

Answer: (C)

30. The number of 7-digit numbers whose sum of the digits equals 10 and which is formed by using the digits 1, 2 and 3 only is

(A) 55 (B) 66 (C) 77 (D) 88

Solution: In a 7-digit number formed by using 1, 2 and 3, suppose that 1 appears x times, 2 appears y times and 3 appears z times. Then by hypothesis

$$x+2y+3z=10 \quad \text{and} \quad x+y+z=7$$

Solving these equations we get

$$y+2z=3$$

from which we get either

$$y=1, z=1 \quad \text{and} \quad x=5$$

or

$$y=3, z=0 \quad \text{and} \quad x=4$$

Therefore, the total number is

$$\frac{7!}{5!}+\frac{7!}{4!\ 3!}=42+35=77$$

Answer: (C)

31. Eight chairs are numbered 1 to 8. Two women and three men wish to occupy one chair each. First the women choose the chairs from among the chairs numbered 1 to 4 and then men select chairs from among the remaining. The number of possible arrangements is

(A) 120 (B) 1440 (C) 16 (D) 240

Solution: Two women can sit in 4 chairs in ${}^4P_2=12$ ways. After the women, the 3 men can sit in the remaining 6 chairs in ${}^6P_3=120$ ways. Therefore the total number of arrangements

$$12\times 120=1440$$

Answer: (B)

32. Total number of ways in which six "+" signs and four "–" signs can be arranged in a row so that no two "–" signs occur together is

(A) 55 (B) 25 (C) 45 (D) 35

Solution: First arrange the six "+" signs. This can be done in only one way. In between the "+" signs, there are 7 gaps (including the left most and right most places)

The four signs can be arranged in these 7 gaps in

$$\frac{^7P_4}{4!} = \frac{7\cdot 6\cdot 5\cdot 4}{24} = 35$$

Answer: (D)

33. Six different coloured hats of the same size are to be arranged circularly. The number of arrangements is

(A) 60 (B) 50 (C) 40 (D) 45

Solution: The required number of arrangements is

$$\frac{1}{2}(6-1)! = 60$$

Answer: (A)

34. If $^{15}C_{3r} = {^{15}C_{r+3}}$, then the value of r is

(A) 5 (B) 4 (C) 6 (D) 3

Solution: If $^nC_r = {^nC_s}$, then either $r = s$ or $r + s = n$. Therefore

$$^{15}C_{3r} = {^{15}C_{r+3}} \Rightarrow 3r + r + 3 = 15 \text{ (since } r \in \mathbb{Z}^+,\ 3r \neq r+3)$$

$$\Rightarrow r = 3$$

Answer: (D)

35. If $^nC_7 = {^nC_4}$, then $^nC_8$ is equal to

(A) 156 (B) 165 (C) 265 (D) 256

Solution: We have

$$^nC_7 = {^nC_4} \Rightarrow 7+4 = n \text{ (since } 7 \neq 4)$$

Therefore

$$^nC_8 = {^{11}C_8} = {^{11}C_3} = \frac{11\cdot 10\cdot 9}{1\cdot 2\cdot 3} = 165$$

Answer: (B)

36. The value of $^{47}C_4 + \sum_{r=1}^{5} {^{(52-r)}C_3}$ is

(A) $^{51}C_4$ (B) $^{53}C_3$ (C) $^{52}C_4$ (D) $^{53}C_4$

Solution: It is known that $^nC_r + {^nC_{r+1}} = {^{(n+1)}C_{r+1}}$. Therefore

$$^{47}C_4 + \sum_{r=1}^{5} {^{(52-r)}C_3} = (^{51}C_3 + {^{50}C_3} + {^{49}C_3} + {^{48}C_3} + {^{47}C_3}) + {^{47}C_4}$$

$$= {^{51}C_3} + {^{50}C_3} + {^{49}C_3} + {^{48}C_3} + (^{47}C_3 + {^{47}C_4})$$

$$= {^{51}C_3} + {^{50}C_3} + {^{49}C_3} + {^{48}C_3} + {^{48}C_4}$$

$$= {^{51}C_3} + {^{50}C_3} + {^{49}C_3} + {^{49}C_4}$$

$$= {^{51}C_3} + {^{50}C_3} + {^{50}C_4}$$

$$= {^{51}C_3} + {^{51}C_4} = {^{52}C_4}$$

Answer: (C)

37. If $^8C_r - {^7C_3} = {^7C_2}$, then r is equal to

(A) 3 or 5 (B) 5 or 4 (C) 4 or 6 (D) 6 or 5

Solution: We have

$$^8C_r - {^7C_3} = {^7C_2}$$

$$^8C_r = {^7C_2} + {^7C_3} = {^8C_3}$$

This gives

$$r = 3 \quad \text{or} \quad r + 3 = 8 \Rightarrow r = 3 \quad \text{or} \quad 5$$

Answer: (A)

38. If

$$a_n = \sum_{r=0}^{n} \frac{1}{^nC_r}$$

then

$$\sum_{r=0}^{n} \frac{r}{^nC_r}$$

is equal to

(A) $(n-1)a_n$ (B) na_n

(C) $\dfrac{na_n}{2}$ (D) $(n+1)a_n$

Solution: Let

$$s = \sum_{r=0}^{n} \frac{r}{^nC_r} = 0 + \frac{1}{^nC_1} + \frac{2}{^nC_2} + \cdots + \frac{n}{^nC_n}$$

Also

$$s = \frac{n}{^nC_n} + \frac{n-1}{^nC_{n-1}} + \cdots + 0 \quad (\because {^nC_r} = {^nC_{n-r}})$$

Therefore

$$2s = \frac{n}{^nC_n} + \frac{n}{^nC_{n-1}} + \frac{n}{^nC_{n-2}} + \cdots + \frac{n}{^nC_0}$$

$$= n\sum_{r=0}^{n} \frac{1}{^nC_r} = na_n$$

$$s = \frac{na_n}{2}$$

Answer: (C)

39. If $^{28}C_{2r} : {^{24}C_{2r-4}} = 225 : 11$, then r is equal to

(A) 24 (B) 14 (C) 7 (D) 12

Solution: We have

$$\frac{^{28}C_{2r}}{^{24}C_{2r-4}} = \frac{225}{11}$$

Therefore

$$\frac{28!}{(2r)!(28-2r)!} \times \frac{(2r-4)!(28-2r)!}{(24)!} = \frac{225}{11}$$

$$\frac{28\cdot 27\cdot 26\cdot 25}{(2r)(2r-1)(2r-2)(2r-3)} = \frac{225}{11}$$

$$(2r)(2r-1)(2r-2)(2r-3)=\frac{11\cdot 28\cdot 27\cdot 26\cdot 25}{225}$$
$$=11\cdot(14\cdot 2)\cdot(13\cdot 2)\cdot 3$$
$$=11\cdot 12\cdot 13\cdot 14$$

From this we have

$$2r-3=11$$
$$r=7$$

Answer: (C)

40. The inequality ${}^{(n+1)}C_6+{}^{n}C_4>{}^{(n+2)}C_5-{}^{n}C_5$ holds for all n greater than

(A) 8 (B) 9 (C) 7 (D) 1

Solution: Consider

$${}^{n+1}C_6+{}^{n}C_4+{}^{n}C_5={}^{n+1}C_6+{}^{n+1}C_5={}^{n+2}C_6$$

The given inequality holds $\Leftrightarrow$ ${}^{n+2}C_6>{}^{n+2}C_5$

$$\Leftrightarrow \frac{(n+2)!}{6!(n+2-6)!}>\frac{(n+2)!}{5!(n+2-5)!}$$
$$\Leftrightarrow \frac{1}{6!(n-4)!}>\frac{1}{5!(n-3)!}$$
$$\Leftrightarrow \frac{1}{6}>\frac{1}{n-3}$$
$$\Leftrightarrow n-3>6$$
$$\Leftrightarrow n>9$$

Answer: (B)

41. If C_K denotes ${}^4C_K$, then the value of

$$2\sum_{K=1}^{4}\left(\frac{K\cdot C_K}{C_K+C_{4-K}}\right)^2$$

is

(A) 12 (B) 13 (C) 14 (D) 15

Solution: We have

$$\frac{K\cdot C_K}{C_K+C_{4-K}}=\frac{K\cdot C_K}{2\cdot C_K}=\frac{K}{2}$$

Therefore

$$2\sum_{K=1}^{4}\left(\frac{K\cdot C_K}{C_K+C_{4-K}}\right)^2=2\left(\sum_{K=1}^{4}\frac{K}{2}\right)^2=\frac{1}{2}(1^2+2^2+3^2+4^2)=15$$

Answer: (D)

42. If ${}^{n-1}C_r=(K^2-3)\cdot{}^{n}C_{r+1}$ and K is positive, then K belongs to the interval

(A) $(-\sqrt{3},\sqrt{3})$ (B) $(\sqrt{3},2]$
(C) $[0,\sqrt{3}]$ (D) $(\sqrt{3},2)$

Solution: By hypothesis,

$$\frac{{}^{n-1}C_r}{{}^{n}C_{r+1}}=K^2-3$$

Therefore

$$\frac{r+1}{n}=K^2-3$$

Since $n\ge r+1$, we have

$$0<\frac{r+1}{n}\le 1$$

Hence

$$0<K^2-3\le 1$$
$$\Rightarrow 3<K^2\le 4$$
$$\Rightarrow \sqrt{3}<K\le 2$$

Answer: (B)

43. Let T_n denote the number of triangles which can be formed using the vertices of a regular polygon of n sides. If $T_{n+1}-T_n=21$, then n is equal to

(A) 6 (B) 7 (C) 5 (D) 4

Solution: To form a triangle, we need three non-collinear points. Therefore, $T_n={}^nC_3$. Now

$$T_{n+1}-T_n=21$$
$${}^{n+1}C_3-{}^{n}C_3=21$$
$$({}^{n}C_3+{}^{n}C_2)-{}^{n}C_3=21$$
$$\frac{n(n-1)}{2}=21$$
$$n^2-n-42=0$$
$$(n-7)(n+6)=0$$
$$n=7\quad(\text{since } n>0)$$

Answer: (B)

44. The number of selections of 5 distinct letters from the letters of the word INTERNATIONAL is

(A) 140 (B) 56 (C) 21 (D) 66

Solution: The distinct letters are I, N, A, T, E, O, L, R. Therefore the number of selections of 5 letters is

$${}^8C_5=56$$

Answer: (A)

45. The sides AB, BC and CA of ΔABC have 3, 4 and 5 interior points respectively on them (Figure 6.5). The number of triangles that can be formed using these interior points is

(A) 180 (B) 185 (C) 210 (D) 205

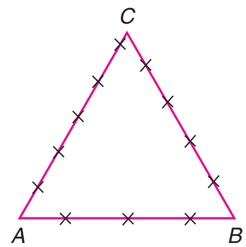


FIGURE 6.5 Single correct choice type question 45.

Solution: The number of ways of selecting 3 points from among $3+4+5\,(=12)$ points is ${}^{12}\mathrm{C}_3$. But from among these, we have to discount collinear sets of points. Therefore the number of triangles is

$${}^{12}\mathrm{C}_3 - {}^{3}\mathrm{C}_3 - {}^{4}\mathrm{C}_3 - {}^{5}\mathrm{C}_3 = \frac{12\cdot 11\cdot 10}{3!} - 1 - 4 - 10 = 205$$

Answer: (D)

46. A box contains two white, three black and four red balls. The number of ways of selecting 3 balls from the box with atleast one black is

(A) 64 (B) 74 (C) 54 (D) 84

Solution: The number of ways is

$${}^{9}\mathrm{C}_3 - {}^{6}\mathrm{C}_3 = 84 - 20 = 64$$

Answer: (A)

47. Five balls of different colours are to be placed in three boxes of different sizes. Each box can hold all the five balls. The number of ways of placing the balls so that no box is empty is

(A) 140 (B) 150 (C) 240 (D) 250

Solution: The number of placings of five different balls in three boxes of different sizes is equal to the number of surjections of a five-element set onto a three-element set which is equal to

$$\sum_{K=0}^{2} (-1)^K\, {}^{3}\mathrm{C}_{3-K} (3-K)^5 = {}^{3}\mathrm{C}_3 \cdot 3^5 - {}^{3}\mathrm{C}_2 \cdot 2^5 + {}^{3}\mathrm{C}_1 \cdot 1^5$$
$$= 243 - 96 + 3 = 150$$

(See Corollary 6.13 or "Try it out" following it.)

Answer: (B)

48. There are 10 points in a plane of which no three are collinear and some four points are concyclic. The maximum number of circles that can be drawn using these is

(A) 116 (B) 120 (C) 117 (D) 110

Solution: We can draw a circle passing through any three given non-collinear points. Therefore, maximum number of circles is

$$({}^{10}\mathrm{C}_3 - {}^{4}\mathrm{C}_3) + 1 = 117$$

Answer: (C)

49. In a polygon (Figure 6.6), no three diagonals are concurrent. If the total number of points of intersections of the diagonals interior to the polygon is 70, then the number of diagonals of the polygon is

(A) 30 (B) 20 (C) 28 (D) 8

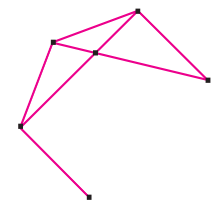


FIGURE 6.6 Single correct choice type question 49.

Solution: To get a point of intersection of two diagonals interior to the polygon, we need 4 vertices of the polygon. It is given that ${}^{n}\mathrm{C}_4 = 70$. Therefore

$$n(n-1)(n-2)(n-3) = 70 \times 4!$$
$$= 8 \times 7 \times 6 \times 5$$
$$n = 8$$

The polygon has 8 vertices and hence 8 sides. Therefore the number of diagonals is

$${}^{8}\mathrm{C}_2 - 8 = 20$$

Answer: (B)

50. A total of 930 greeting cards are exchanged among the residents of flats. If every resident sends a card to every other resident of the same flats, then the number of residents is

(A) 30 (B) 29 (C) 32 (D) 31

Solution: Let n be the number of residents in the flats. Then $2 \times {}^{n}\mathrm{C}_2 = 930$. Therefore

$$n(n-1) = 930$$
$$(n-31)(n+30) = 0$$

This gives $n = 31$ or -30. The second value is not possible and hence, $n = 31$.

Answer: (D)

51. From 5 vowels and 5 consonants, the number of four-letter words (without repetition) having 2 vowels and 2 consonants that can be formed is

(A) 100 (B) 2400 (C) 1600 (D) 24

Solution: Now 2 vowels and 2 consonants can be selected in

$${}^{5}\mathrm{C}_2 \times {}^{5}\mathrm{C}_2 = 10 \times 10 = 100 \quad \text{ways}$$

After selection, the four letters can be permuted in 4! ways. Therefore, the number of words is

$$100 \times 4! = 2400$$

Answer: (B)

52. A total of 6 boys and 6 girls are to sit in a row alternatively and in a circle alternatively. Let m be the number of arrangements in a row and n the number of arrangements in a circle. If $m = Kn$, then the value of K is

(A) 10 (B) 11 (C) 12 (D) 13

Solution: Linear permutations with boy in the first place are of the form B G B G B G B G B G B G and the number of such is $6! \times 6!$. The number of linear permutations with girl in the first place is $6! \times 6!$. Therefore the number of row arrangements is

$$2 \times 6! \times 6!$$

Regarding circular permutation, start with a place which can filled by a boy or a girl and after that the arrangement becomes linear. Placing the boys first and then arranging the girls in 6 gaps, the number of such circular arrangements is $5! \times 6!$. Now, $m = 2 \times 6! \times 6!$ and $n = 5! \times 6!$. Therefore

$$m = Kn$$

$$2 \times 6! \times 6! = K \times 5! \times 6!$$

$$K = 12$$

Answer: (C)

53. From the vertices of a regular polygon of 10 sides, the number of ways of selecting three vertices such that no two vertices are consecutive is

(A) 10 (B) 30 (C) 50 (D) 40

Solution: Let $A_1, A_2, \ldots, A_{10}$ be vertices of a regular polygon of 10 sides.

The number of ways of selecting 3 vertices is ${}^{10}C_3$.

The number of ways of selecting 3 consecutive vertices is (i.e., $A_1\ A_2\ A_3, A_2\ A_3\ A_4, \ldots, A_{10}\ A_1\ A_2) = 10$.

The number of ways of selecting three vertices such that two vertices are consecutive = (First select 2 consecutive vertices, leave their neighboring two vertices and select one more from the remaining 6 vertices) is

$$10 \times {}^6C_1 = 60$$

The total number of required selections is

$${}^{10}C_3 - 10 - 60 = 120 - 70 = 50$$

Answer: (C)

54. The number of proper divisors of 240 is

(A) 18 (B) 20 (C) 19 (D) 24

Solution: Proper divisors of a number are divisors other than unity and itself. We have

$$240 = 2^4 \times 3^1 \times 5^1$$

Any divisor of 240 is of the form $2^a \times 3^b \times 5^c$, 1 where $0 \le a \le 4$, $0 \le b \le 1$ and $0 \le c \le 1$. Therefore the number of proper divisors is

$$5 \times 2 \times 2 - 2 = 20 - 2 = 18$$

Answer: (A)

55. There are 7 distinguishable rings. The number of possible five-ring arrangements on the four fingers (except the thumb) of one hand (the order of the rings on each finger is to be counted and it is not required that each finger has a ring) is

(A) 214110 (B) 211410
(C) 124110 (D) 141120

Solution: There are ${}^7C_5$ ways of selecting the rings to be worn. If a, b, c, d are the numbers of rings on the fingers, we need to find the number of quadruples (a, b, c, d) of non-negative integers such that $a + b + c + d = 5$. The number of such quadruples is

$${}^{(5+4-1)}C_{(4-1)}\ \ (= {}^8C_3)$$

For each set of 5 rings, there are 5! assignments. Therefore, the total number of required arrangements is

$${}^7C_5 \times {}^8C_3 \times 5! = 141120$$

Answer: (D)

56. The number of ordered pairs of positive integers (a, b), such that their least common multiple is the given positive integer $7^2 \times 11^3 \times 19^4$, is

(A) 215 (B) 315 (C) 415 (D) 195

Solution: By Theorem 6.21, the required number of ordered pairs of positive integers (a, b), such that the least common multiple of a and b is the number $7^2 \times 11^3 \times 19^4$, is equal to

$$(2 \times 2 + 1)(2 \times 3 + 1)(2 \times 4 + 1) = 5 \times 7 \times 9 = 315$$

Answer: (B)

Multiple Correct Choice Type Questions

1. Let n and r be positive integers such that $1 \le r \le n$. Which of the following is/are true?

(A) ${}^nP_n = {}^nP_{n-1}$ (B) ${}^nP_n = 2 \times {}^nP_{n-2}$
(C) ${}^nP_r = n \times {}^{n-1}P_{r-1}$ (D) ${}^nP_r = r \times {}^{n-1}P_{r-1}$

Solution: We have

$${}^nP_n = \frac{n!}{(n-n)!} = \frac{n!}{1} = \frac{n!}{[n-(n-1)]!} = {}^nP_{n-1}$$

$$^{n}\mathrm{P}_{n}=\frac{n!}{1}=2\frac{n!}{2!}=2\times{}^{n}\mathrm{P}_{n-2}$$

$$^{n}\mathrm{P}_{r}=\frac{n!}{(n-r)!}=n\cdot\frac{(n-1)!}{[(n-1)-(r-1)]!}=n\cdot{}^{n-1}\mathrm{P}_{r-1}$$

$$r\times{}^{n-1}\mathrm{P}_{r-1}=r\times\frac{(n-1)!}{(n-r)!}\neq{}^{n}P_{r}$$

Answers: (A), (B), (C)

2. Given that $^{n}\mathrm{P}_{r}=n(n-1)(n-2)\ldots(n-r+1)$. Then which of the following are true?

(A) $^{n}\mathrm{P}_{4}=1680 \Rightarrow n=8$ (B) $^{12}\mathrm{P}_{r}=1320 \Rightarrow r=3$

(C) $^{13}\mathrm{P}_{r}=1220 \Rightarrow r=4$ (D) $^{n}\mathrm{P}_{3}=1220 \Rightarrow n=9$

Solution: (A) We have

$$n(n-1)(n-2)(n-3)=1680=8\times7\times6\times5 \Rightarrow n=8$$

(B) We have

$$12(12-1)\cdots(12-r+1)=1320$$
$$11\cdot10\cdots(13-r)=110$$
$$r=3$$

(C) It is not true, since 13 is not a factor of 1220.

(D) For the similar reason as (C), (D) is also not true.

Answers: (A), (B)

3. x is one among n distinct objects and $1\le r\le n$ an integer. Then which of the following are true?

(A) The number of permutations of r objects that involve the object x is $r!\times{}^{n-1}\mathrm{P}_{r-1}$.

(B) The number of permutations of r-objects that do not involve the object x is $^{n-1}\mathrm{P}_{r}$.

(C) $^{n}\mathrm{P}_{r}=(n-1)\mathrm{P}_{r}+r\times{}^{n-1}\mathrm{P}_{r}$.

(D) The number of permutations of r-objects that involve x is $r\times{}^{n-1}\mathrm{P}_{r-1}$.

Solution: A permutation involving x implies that x is in one of the r places. The remaining $r-1$ places are to be filled with $n-1$ objects. This can be done in $^{n-1}\mathrm{P}_{r-1}$ ways. Therefore, the total number of permutations of r objects involving x is $r\times{}^{n-1}\mathrm{P}_{r-1}$. Therefore (D) is correct and (A) is not correct.

The permutations of r-objects not involving x is $^{n-1}\mathrm{P}_{r}$, since r places have to filled with objects of $(n-1)$-element set. Therefore (B) is correct. Also,

$$^{n}\mathrm{P}_{r}={}^{n-1}\mathrm{P}_{r}+(r\times{}^{n-1}\mathrm{P}_{r-1})$$

and hence (C) is correct.

Answers: (B), (C), (D)

4. Consider the word ALLAHABAD. Which of the following statements are true?

(A) The total number of words that can be formed using all the letters of the word is 7560.

(B) The number of words which begin with A and end with A is 1260.

(C) The number of words in which vowels occupy the even places is 60.

(D) The number of words in which all the four vowels occupy adjacent places is 360.

Solution: The word ALLAHABAD consists of 4 As, 2 Ls, 1 B, 1 D and 1 H.

(A) Out of the total 9 objects, 4 are alike of one kind and 2 are alike of another kind. Therefore, the number of words is

$$\frac{9!}{4!\ 2!}=\frac{9\cdot8\cdot7\cdot6\cdot5}{2}=7560$$

(B) Put one A in the first place and another in the last place. In the remaining there are 2 As and 2 Ls. The number of such words is

$$\frac{7!}{2!\ 2!}=1260$$

(C) There are four even places and 4 vowels A, A, A, A. These can be put in even places in only one way. The remaining 5 letters can be arranged in

$$\frac{5!}{2!}=60 \quad \text{ways}$$

(D) Consider all the four as a single letter, so that among six objects, 2 are alike. Therefore, the number of such arrangements is

$$\frac{6!}{2!}=360$$

Answers: (A), (B), (C), (D)

5. Consider the letters of the word INTERMEDIATE. Which of the following is (are) true?

(A) The number of words formed by using all the letters of the given word is (12!)/(3! 2!).

(B) The number of words which begin with I and end with E is (10!)/(2! 2!).

(C) The number of words in which all the vowels come together is $(7!\cdot6!)/(3!\ 2!\ 2!)$.

(D) The number of words in which no two vowels come together is 360×420.

Solution: The given word consists of 12 letters in which there are 3 Es, 2 Is and 2 Ts and the remaining 5 are distinct.

(A) The number of words using all the letters is

$$\frac{(12)!}{3!\ 2!\ 2!}$$

Therefore (A) is false.

(B) Put I in the first place and E in the last place. In the remaining 10 letters, there are 2 Es and 2 Ts. Therefore the number of such words is

$$\frac{(10)!}{2!\ 2!}$$

Therefore (B) is true.

(C) Treat all the vowels as a single object (letter). In the remaining six letters, there are 2 Ts. Now, the 7 letters can be arranged in 7!/2! ways. But the vowels (3 Es, 2 Is and 1 A) can be arranged among themselves in

$$\frac{6!}{3!\ 2!} \text{ ways}$$

Therefore the number of such words is

$$\frac{7!}{2!} \times \frac{6!}{3!\ 2!}$$

Therefore (C) is true.

(D) Among the six consonants, there are seven gaps in which the vowels can be arranged in

$$\frac{{}^7P_6}{3!\ 2!} \text{ ways}$$

The consonants can be arranged in

$$\frac{6!}{2!} \text{ ways}$$

Therefore the number of words in which no two vowels come together is

$$\frac{{}^7P_6}{3!\ 2!} \times \frac{6!}{2!} = 420 \times 360$$

Answers: (B), (C), (D)

6. The letters of the word ARTICLE are arranged in all possible ways. Then which of the following is (are) true?

(A) Number of words formed by using all the letters is 5040.

(B) The number of words with vowels in even places is 144.

(C) The number of words with vowels in odd places is 576.

(D) The number of words with I in the middle is 720.

Solution:

(A) Seven different objects can be arranged in $7! = 5040$ ways.

(B) There are 3 even places and 3 vowels. Therefore the vowels can be arranged in even places in 3!, ways and after that the remaining 4 letters can be arranged in 4! ways. Therefore, the required number is $3! \times 4! = 144$.

(C) As in (B), the required number is ${}^4P_3 \times 4! = 4! \times 4! = 576$.

(D) With I in the middle, the remaining 6 letters can be arranged in $6! = 720$ ways.

Answers: (A), (B), (C) and (D)

7. If ${}^nC_{r-1} = 36$, ${}^nC_r = 84$ and ${}^nC_{r+1} = 126$, then

(A) $n = 8$ (B) $r = 3$ (C) $n = 9$ (D) $r = 4$

Solution: We have

$$\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r} = \frac{84}{36}$$

$$\frac{n-r+1}{r} = \frac{7}{3} \tag{6.1}$$

$$3n - 10r = -3$$

Again

$$\frac{{}^nC_{r+1}}{{}^nC_r} = \frac{n-r}{r+1} = \frac{126}{84} \tag{6.2}$$

$$\frac{3}{2} = \frac{n-r}{r+1}$$

$$2n - 5r = 3$$

Solving Eqs. (6.1) and (6.2), we get $n = 9$ and $r = 3$.

Answers: (B) and (C)

8. If ${}^nP_r = {}^nP_{r+1}$ and ${}^nC_r = {}^nC_{r-1}$, then

(A) $n = 3$ (B) $r = 1$ (C) $r = 2$ (D) $n = 4$

Solution: We have

$${}^nP_r = {}^nP_{r+1}$$

$$\frac{n!}{(n-r)!} = \frac{n!}{(n-r-1)!} \tag{6.3}$$

$$n - r = 1$$

$${}^nC_r = {}^nC_{r-1}$$

$$\frac{n!}{r! \times (n-r)!} = \frac{n!}{(r-1)!(n-r+1)!} \tag{6.4}$$

$$\frac{1}{r} = \frac{1}{n-r+1}$$

$$2r - 1 = n$$

Solving Eqs. (6.3) and (6.4), we get $r = 2$ and $n = 3$.

Answers: (A) and (C)

Matrix-Match Type Questions

1. A total of 6 boys and 6 girls are to be arranged in a row. Certain stipulations on their arrangements are given in Column I and the number of such arrangements is given in Column II. Match the items in Column I with those in Column II.

Column I	Column II
(A) The number of arrangements in which all the girls are together is	(p) $7!\times 6!$
(B) The number of arrangements in which no two girls are together is	(q) $6!\times 6!\times 7$
(C) The number of arrangements in which boys and girls come alternately is	(r) $2\times 6!\times 6!$
(D) The number of arrangements in which the first place is to be occupied by a specified girl and the last place by a specified boy is	(s) $(10)!$

Solution:

(A) Consider all the 6 girls as a single block so that there are $6+1=7$ objects which can be arranged in $7!$ ways. In the block of girls, 6 can be arranged among themselves in $6!$ ways. Therefore the required arrangements are

$$7!\times 6! \quad \text{or} \quad 6!\times 6!\times 7$$

Answer: (A) → (p)

(B) Since no two girls should come together, arrange the 6 girls in 7 gaps (including before the boys and after the boys) which can be done in ${}^7P_6$ ways. After arranging the girls, the 6 boys can be arranged among themselves in $6!$ ways. Therefore, the required arrangements are

$${}^7P_6\times 6! = 7!\times 6! = 6!\times 6!\times 7$$

Answer: (B) → (p), (q)

(C) Since the boys and girls come alternately, the arrangement may begin with a boy or a girl as BG BG BG BG BG BG or GB GB GB GB GB GB. Number of such arrangements is $2\times 6!\times 6!$.

Answer: (C) → (r)

(D) Put the specified girl in the first place and the specified boy in the last place. The remaining 10(5 + 5) can be arranged in (10)! ways.

Answer: (D) → (s)

2. Four-digit numbers, without repetition of digits, are formed using the digits 0, 3, 4, 5. Certain stipulations on arrangements are given in Column I and their numbers are given in Column II. Match these.

Column I	Column II
(A) Total number of four-digit numbers that can be formed is	(p) 8
(B) Total number of even numbers that can be formed is	(q) 10
(C) Total number of odd numbers that can be formed is	(r) 13442
	(s) 18
(D) The sum of all the four-digit numbers is	(t) 13440

Solution:

(A) The total number of four-digit numbers that can be formed is given by
(No. of arrangements of 0, 3, 4 and 5) − (No. of arrangements with 0 in the left end)

$$4!-3!=18$$

Answer: (A) → (s)

(B) A number among these is even if 0 or 4 is in the units place. The number of even numbers

(i) with 0 in the units place $=3!=6$
(ii) with 4 in the units place $=3!-2!=4$

Therefore the total number of even numbers $=6+4=10$.

Answer: (B) → (q)

(C) No. of odd numbers = Total No. − No. of even numbers

$$=18-10=8$$

Answer: (C) → (p)

(D) Contribution of 0 to the sum $=100+10+0=110$

Contribution of 3 to the sum $=3000+300+30+3=3333$
Contribution of 4 to the sum $=4000+400+40+4=4444$
Contribution of 5 to the sum $=5000+500+50+5=5555$

The sum of all the numbers =

$$110+3333+4444+5555=110+1111(3+4+5)$$
$$=110+13332=13442$$

Answer: (D) → (r)

3. In Column I the types of distributions of playing cards and, in Column II, their corresponding number of distributions is given. Match the items in Column I with those in Column II.

Column I	Column II
(A) 52 playing cards are to be equally distributed among four players. The number of possible distributions is	(p) $\dfrac{(52)!}{[(13)!]^4}$
(B) 52 cards are to be divided into four equal groups	(q) $\dfrac{(52)!}{4![(13)!)]^4}$
(C) 52 cards are to be divided into 4 sets, three of them having 17 cards each and the 4th has just one card	(r) $\dfrac{(52)!}{3![(17)!]}$
(D) 52 cards are to be divided equally into two sets	(s) $\dfrac{(52)!}{2!((26)!)^2}$

Solution:

(A) The required number of distributions is

$$^{52}C_{13}\times{}^{39}C_{13}\times{}^{26}C_{13}\times{}^{13}C_{13}=\frac{(52)!}{(13)!(52-13)!}\times\frac{(39)!}{(13)!(39-13)!}\times\frac{(26)!}{(13)!(26-13)!}\times 1=\frac{(52)!}{[(13)!]^4}$$

Answer: (A) → (p)

(B) To be divided into 4 equal groups. Elements of one group can be exchanged with another. This is possible in 4! ways. Therefore, the number of divisions is

$$\frac{(52)!}{4![(13)!]^4}$$

Answer: (B) → (q)

(C) Three of them get 17 each. Again cards can be exchanged among these 3 in 3! ways. Therefore, the number of divisions is

$$\frac{(52)!}{3![(17)!]^3}$$

Answer: (C) → (r)

(D) The number of distributions is

$$\frac{^{52}C_{26}\cdot{}^{26}C_{26}}{2!}=\frac{(52)!}{2![(26)!]^2}$$

Answer: (D) → (s)

4. A committee of 12 members is to be formed from 9 women and 8 men. Match the statements in Column I with the numbers in Column II.

Column I	*Column II*
(A) The number of ways of forming the committee with 6 men and 6 women	(p) 1008
(B) The number of ways of forming the committee with atleast 5 women	(q) 2702
(C) The number of ways of forming the committee with women in majority	(r) 6062
(D) The number of ways of forming the committee with atleast 5 women and with men in majority	(s) 2352

Solution:

(A) The number of ways of forming the committee with 6 men and 6 women from 9 women and 8 men is

$$^9C_6\times{}^8C_6=\frac{9\cdot 8\cdot 7}{3!}\times\frac{8\cdot 7}{2!}=84\times 28=2352$$

Answer: (A) → (s)

(B) The number of ways of forming the committee with atleast 5 women is

$$(^9C_5\times{}^8C_7)+(^9C_6\times{}^8C_6)+(^9C_7\times{}^8C_5)+(^9C_8\times{}^8C_4)+(^9C_9\times{}^8C_3)$$

$$=\left(\frac{9\cdot 8\cdot 7\cdot 6}{4!}\times 8\right)+\left(\frac{9\cdot 8\cdot 7}{3!}\times\frac{8\cdot 7}{2!}\right)+\left(\frac{9\cdot 8}{2!}\times\frac{8\cdot 7\cdot 6}{3!}\right)+\left(9\times\frac{8\cdot 7\cdot 6\cdot 5}{4!}\right)+\left(1\times\frac{8\cdot 7\cdot 6}{3!}\right)$$

$$=1008+2352+2016+630+56=6062$$

Answer: (B) → (r)

(C) The number of ways of forming the committee with women in majority

$$(^9C_7\times{}^8C_5)+(^9C_8\times{}^8C_4)+(^9C_9\times{}^8C_3)$$

$$=\left(\frac{9\cdot 8}{2!}\times\frac{8\cdot 7\cdot 6}{3!}\right)+\left(9\times\frac{8\cdot 7\cdot 6\cdot 5}{4!}\right)+\left(1\times\frac{8\cdot 7\cdot 6}{3!}\right)$$

$$=2016+630+56=2702$$

Answer: (C) → (q)

(D) The number of ways of forming the committee with atleast 5 women and with men in majority

$$^9C_5\times{}^8C_7=\frac{9\cdot 8\cdot 7\cdot 6}{4!}\times 8=1008$$

Answer: (D) → (p)

5. A total of 11 players are to be selected for a cricket match from a cricket squad consisting of 6 specialist batsmen, 3 all rounders, 6 specialist bowlers and 2 wicketkeepers (who can also bat well). Match the items in Column I with those in Column II.

Column I	*Column II*
(A) The number of selections which contain 4 specialist batsmen, 3 all rounders, 3 specialist bowlers and a wicketkeeper	(p) 600
(B) The number of selections which contain 5 specialist batsman, 2 all rounders, 3 specialist bowlers and a wicketkeeper	(q) 720
(C) The number of selections which contain 4 specialist batsman, 1 all rounder, 4 specialist bowlers and 2 wicketkeepers	(r) 675
(D) The number of selections which contain 4 specialist batsmen, 2 all rounders, 3 specialist bowlers and 2 wicketkeepers	(s) 900

Solution: Consider the table on the next page.

Answer: (A) → (p); (B) → (q); (C) → (r); (D) → (s)

Item in column I	*Specialist batsmen (6)*	*All rounders (3)*	*Specialist bowlers (6)*	*Wicketkeepers (2)*	*Number of selections*
(A)	4	3	3	1	${}^6C_4 \times {}^3C_3 \times {}^6C_3 \times {}^2C_1 = 600$ (p)
(B)	5	2	3	1	${}^6C_5 \times {}^3C_2 \times {}^6C_3 \times {}^2C_1 = 720$ (q)
(C)	4	1	4	2	${}^6C_4 \times {}^3C_1 \times {}^6C_4 \times {}^2C_2 = 675$ (r)
(D)	4	2	3	2	${}^6C_4 \times {}^3C_2 \times {}^6C_3 \times {}^2C_2 = 900$ (s)

6. A 17 member hockey squad contains 4 peculiar players A, B, C and D. Players A and B wish to play together or be out of the team together. Players C and D are such that if one plays the other does not want to play. A team of 11 players is to be selected from the squad. Match the items in Column I with those in Column II.

Column I	*Column II*
(A) No. of selections including A and B and one of C, D is	(p) ${}^{13}C_9$
(B) No. of selections including A and B and excluding both C and D is	(q) ${}^{13}C_{11}$
(C) No. of selections excluding A and B and including one of C and D is	(r) ${}^{13}C_8 \times 2$
(D) No. of selections excluding all of A, B, C and D is	(s) $2 \times {}^{13}C_{10}$

Solution:

(A) In addition to A, B and C, 8 more are to be selected from out of 13 (other than A, B, C and D) in ${}^{13}C_8$ ways. Similarly, in addition to A, B and D, another ${}^{13}C_8$ ways. Therefore, the required number is $2 \times {}^{13}C_8$.

Answer: (A) → (r)

(B) In addition to A and B, 9 more are to be selected from among of 13 (other than A, B, C and D) in ${}^{13}C_9$ ways.

Answer: (B) → (p)

(C) 10 players are to be selected, in addition to C, from among 13 (other than A, B, C and D). This can be done in ${}^{13}C_{10}$ ways. Similarly, selections including D can be made in ${}^{13}C_{10}$ ways. Therefore the required no. is $2 \times {}^{13}C_{10}$.

Answer: (C) → (s)

(D) The number of selections of 11 persons from among 13 persons (other than A, B, C and D) is ${}^{13}C_{11}$.

Answer: (D) → (q)

7. Match the statements in Column I with the numbers given in Column II.

Column I	*Column II*
(A) There are 12 points in a plane out of which 5 are collinear and no 3 of the remaining are collinear. Then the number of lines that can be formed by joining pairs of these points is	(p) 1296
(B) The number of triangles that can be formed by using the points mentioned above is	(q) 57
(C) The number of rectangles that can be formed by using the squares in a chess board is	(r) 420
(D) A set of 8 parallel lines are intersected by another set of 6 parallel lines. Then the number of parallelograms thus formed is	(s) 210

Solution:

(A) The five collinear points give us one straight line. Therefore the required number is

$${}^{12}C_2 - {}^5C_2 + 1 = \frac{12 \cdot 11}{2} - \frac{5 \cdot 4}{2} + 1 = 57$$

Answer: (A) → (q)

(B) A triangle is formed with three non-collinear points. Therefore the number of triangle that can be formed is

$${}^{12}C_3 - {}^5C_3 = 210$$

Answer: (B) → (s)

(C) A chess board consists of 9 horizontal and 9 vertical lines. To form a rectangle (it may be a square) we need 2 horizontal and 2 vertical lines. Therefore the number of rectangles is

$${}^9C_2 \times {}^9C_2 = 36 \times 36 = 1296$$

Answer: (C) → (p)

(D) We select 2 from 8-lines set and 2 from 6-lines set. Therefore the number of parallelograms is

$${}^8C_2 \times {}^6C_2 = 28 \times 15 = 420$$

Answer: (D) → (r)

Comprehension-Type Questions

1. Passage: 4 Indians, 3 Americans and 2 Britishers are to be arranged around a round table. Answer the following questions.

(i) The number of ways of arranging them is

(A) $9!$ (B) $\frac{1}{2}9!$ (C) $8!$ (D) $\frac{1}{2}8!$

(ii) The number of ways arranging them so that the two Britishers should never come together is

(A) $7!\times 2!$ (B) $6!\times 2!$ (C) $7!$ (D) $6!\,{}^6P_2$

(iii) The number of ways of arranging them so that the three Americans should sit together is

(A) $7!\times 3!$ (B) $6!\times 3!$ (C) $6!\,{}^6P_3$ (D) $6!\,{}^7P_3$

Solution:

(i) n distinct objects can be arranged around a circular table in $(n-1)!$ ways. Therefore the number of ways of arranging $4+3+2$ people $= 8!$.

Answer: (C)

(ii) First arrange 4 Indians and 3 Americans around a round table in $6!$ ways. Among the six gaps, arrange the two Britishers in ${}^6P_2$ ways. Therefore the total number of arrangements in which Britishers are separated is $6!\times {}^6P_2$.

Answer: (D)

(iii) Treating the 3 Americans as a single object, 7 (= 4 + 1 + 2) objects can be arranged cyclically in $6!$ ways. In each of these, Americans can be arranged among themselves in $3!$ ways. Therefore, the number of required arrangements is $6!\times 3!$.

Answer: (B)

2. Passage: 4 prizes are to be distributed among 6 students. Answer the following three questions.

(i) The number of ways of distributing the prizes, if a student can receive any number of prizes, is

(A) 1296 (B) 16^3 (C) 15 (D) 30

(ii) The number of ways of distributing the prizes, if a student cannot receive all the prizes, is

(A) $16^3 - 16$ (B) 1290 (C) 11 (D) 26

(iii) If a particular student is to receive exactly 2 prizes, then the number of ways of distributing the prizes is

(A) 25 (B) 32 (C) 150 (D) 36

Solution:

(i) Let the prizes be P_1, P_2, P_3 and P_4. P_1 can be given to any one of the 6 students and so are P_2, P_3 and P_4. Therefore the number of distributions is $6^4 = 1296$.

Answer: (A)

(ii) The number of ways in which all the four prizes can be given to any one of the 6 students = 6. Therefore the required number of ways is $6^4 - 6 = 1290$.

Answer: (B)

(iii) Give a set of two prizes to the particular student. Then the remaining 2 can be distributed among 5 students in 5^2 ways. There are ${}^4C_2$ sets, each containing 2 prizes. Therefore the required number of ways of distributing the prizes is

$$5^2 \times {}^4C_2 = 25\times 6 = 150$$

Answer: (C)

3. Passage: A security of 12 persons is to form from a group of 20 persons. Answer the following questions.

(i) The number of times that two particular persons are together on duty is

(A) $\frac{20!}{12!\,8!}$ (B) $\frac{18!}{10!\,8!}$ (C) $\frac{20!}{10!\,8!}$ (D) $\frac{20!}{10!\,10!}$

(ii) The number of times that three particular persons are together on duty is

(A) $\frac{17!}{8!\,9!}$ (B) $\frac{17!}{8!\,8!}$ (C) $\frac{20!}{17!\,3!}$ (D) $\frac{20!}{9!\,8!}$

(iii) The number of ways of selecting 12 guards such that two particular guards are out of duty and three particular guards are together on duty is

(A) $\frac{(20)!}{(15)!\,5!}$ (B) $\frac{(18)!}{9!\,3!}$ (C) $\frac{(15)!}{9!\,6!}$ (D) $\frac{(15)!}{5!\,(10)!}$

Solution:

(i) Let A and B two particular guards who want to be in duty together. We can select 10 more from the remaining 18 persons in ${}^{18}C_{10}$ ways. Therefore the required number is

$$\frac{(18)!}{(10)!\,8!}$$

Answer: (B)

(ii) In addition to the three particular persons who want to be in duty together, we can select 9 more from the remaining 17 persons in ${}^{17}C_9$ ways. Therefore, the required number is

$$\frac{17!}{8!\,9!}$$

Answer: (A)

(iii) Let A and B be out of duty and P, Q, R be particular persons who want to be in duty together. Then, we can choose 9 more from among the remaining 15 persons (excluding A and B) in ${}^{15}C_9$ ways. Therefore, the required number is

$$\frac{(15)!}{9!\,6!}$$

Answer: (C)

4. **Passage:** The letters of the word MULTIPLE are arranged in all possible ways. Answer the following three questions.

(i) The number of arrangements in which the order of the vowels does not change is

(A) 3330 (B) 3320 (C) 3340 (D) 3360

(ii) The number of arrangements in which the vowels' positions are not disturbed is

(A) 60 (B) 260 (C) 160 (D) 320

(iii) The number of arrangements in which the relative order of vowels and consonants is not disturbed is

(A) 460 (B) 420 (C) 360 (D) 440

Solution:

(i) The order of vowels does not change means, first u occurs, then i and then e must occur. The total number of arrangements is

$$\frac{8!}{2!} \quad \text{(since there are two } l\text{)}$$

In each of these arrangements, vowels may occur in 3! ways. Therefore the number of arrangements in which u, i, e occur in this order is

$$\frac{8!}{2!} \times \frac{1}{3!} = 3360$$

Answer: (D)

(ii) Keeping u, i, e in their respective places, the number of arrangements is

$$\frac{5!}{2!} = 60$$

Answer: (A)

(iii) Keeping the relative positions of vowels and consonants means, the vowels can be interchanged among themselves and so can the consonants. Therefore the number of required arrangements is

$$\frac{5!}{2!} \times 3! = 60 \times 6 = 360$$

Answer: (C)

SUMMARY

6.1 ***The symbol n! (Factorial n):*** $0! = 1$. If n is a positive integer, the $n!$ means the number $n(n-1)(n-2)\cdots 2{\cdot}1$. One can note that $n! = n(n-1)!$

Permutations

6.2 ***Permutation:*** Arrangement of objects on a line is called *linear permutation*.

6.3 ***Circular permutation:*** Arrangement of objects in a circular form.

6.4 ***Permutation as a bijection:*** If X is a finite set, then any bijection from X onto X is a permutation. That is arrangement of n distinct objects taken all at a time.

6.5 ***Theorem:*** The number of arrangements of n distinct objects taken all at a time is $n!$.

6.6 ***Theorem:*** The number of permutations of n objects taken r at a time $(0 \le r \le n)$ is

$$n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

6.7 ***Symbol ${}^nP_r$:*** If n is a positive integer and $0 \le r \le n$ is an integer, then ${}^nP_r$ denotes the number of permutations of n distinct objects taken r at a time without repetitions and this

$${}^nP_r = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

6.8 ***Useful formulae:***

(1) ${}^nP_r = \dfrac{n!}{(n-r)!}$

(2) ${}^nP_r = n\cdot{}^{(n-1)}P_{(r-1)}$

(3) ${}^nP_r = {}^{(n-1)}P_r + r\cdot{}^{(n-1)}P_{r-1}$

6.9 ***Permutations with repetitions:***

(1) The number of permutations of n dissimilar things taken r at time, when repetition of objects is allowed any number of times is n^r.

(2) Total number of permutations of n dissimilar objects taken r at a time with atleast one repetition is $n^r - {}^nP_r$.

6.10 ***Circular permutations:*** The number of circular permutations of n dissimilar things is $(n-1)!$. This number includes both anticlockwise and clockwise senses. If the sense is not considered, then the number is

$$\frac{(n-1)!}{2}$$

QUICK LOOK

(1) For live objects the sense will be considered.
(2) For non-live objects the sense will not be considered.

6.11 ***Permutations with alike objects:***

(1) Out of n objects, suppose p objects are alike and the rest are distinct. Then, the number of permutations of the n objects taken all at a time is $n!/p!$.

(2) Suppose $n_1, n_2, \ldots, n_K$ are number of alike objects of different kinds. Then the number of permutations of all these objects is

$$\frac{(n_1+n_2+\cdots+n_K)!}{n_1!n_2!\cdots n_K!}$$

Combinations

6.12 ***Combination:*** Selection of objects.

6.13 ***Symbol $^n\mathrm{C}_r$ or $\binom{n}{r}$:*** The number of combinations of n distinct objects taken $r(0 \le r \le n)$ at a time is denoted by $^n\mathrm{C}_r$.

6.14 ***Value of $^n\mathrm{C}_r$:***

$$^n\mathrm{C}_r = \frac{^n\mathrm{P}_r}{r!} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$$
$$= \frac{n!}{r!(n-r)!}$$
$$^n\mathrm{C}_0 = {^n\mathrm{C}_n} = 1$$

QUICK LOOK

$$^n\mathrm{C}_r = {^n\mathrm{C}_{n-r}}$$

6.15 ***Useful tips:***

(1) $^n\mathrm{C}_r = {^n\mathrm{C}_s} \Rightarrow$ Either $r = s$ or $r+s=n$.

(2) $^n\mathrm{C}_r + {^n\mathrm{C}_{r-1}} = {^{(n+1)}\mathrm{C}_r}$.

(3) $r \times {^n\mathrm{C}_r} = {^{n\times(n-1)}\mathrm{C}_{r-1}}$.

6.16 ***Combinations with alike objects:*** If $p_1, p_2, \ldots, p_K$ are number of alike objects of different kinds, then the number of all selections (with one or more) is $(p_1+1)(p_2+1)\ldots(p_K+1)-1$.

6.17 ***Number of divisons:*** If $1 < n = p_1^{\alpha_1}\cdot p_2^{\alpha_2}\cdots p_K^{\alpha_K}$ is a positive integer where $p_1, p_2, \ldots, p_K$ are distinct prime numbers, then the number of positive divisions of n is $(\alpha_1+1)(\alpha_2+1)\cdots(\alpha_K+1)$.

Note that this number includes both 1 and n.

6.18 Writing a positive integer as a sum of (atleast two) positive integers considering the same set of integers in a different order being different is $2^{n-1}-1$.

6.19 ***Useful results (on integer solutions):***

(1) Let m and n be positive integers such that $m \le n$. Then the number of m-tuples $(x_1, x_2, \ldots, x_m)$ of positive integers satisfying the equation $x_1 + x_2 + \cdots + x_m = n$ is $^{(n-1)}\mathrm{C}_{(m-1)}$.

(2) The number of m-tuples $(x_1, x_2, \ldots, x_m)$ of non-negative integers satisfying the equation $x_1 + x_2 + \cdots + x_m = n$ is $^{(n+m-1)}\mathrm{C}_{(m-1)}$.

6.20 ***Plane divided by lines:*** The maximum number of parts into which a plane is divided by n lines is

$$\frac{n^2+n+2}{2}$$

6.21 ***Number of injections and bijections:*** Let X and Y be non-empty finite sets having m elements and n elements, respectively. Then

(1) The number of functions (mappings) from Y to X is m^n.

(2) The number of injections from Y to X is zero if $m < n$ and $^m\mathrm{P}_n$ if $m \ge n$.

(3) The number of bijections from Y to X is zero if $m < n$ and $m!$ if $m = n$.

6.22 ***Number of surjections:***

(1) ***Recursive formula:*** For any positive integers $m \ge r > 0$, the number $\alpha_m(r)$ of surjections from an m-element set onto an r-element set is given by a recursive formula

$$\alpha_m(r) = r^m - \sum_{s=1}^{r-1} {^r\mathrm{C}_s}\,\alpha_m(s)$$

(2) ***Direct formula:***

$$\alpha_m(r) = \sum_{s=0}^{r-1}(-1)^s\, {^r\mathrm{C}_{(r-s)}}(r-s)^m$$

6.23 ***Derangement:*** Let X be a non-empty set and $f: X \to X$ is a bijection (also called permutation), such that $f(x) \ne x$ for all $x \in x$. Then f is called *derangement* of X.

6.24 ***Number of derangements:***

(1) Let n be a positive integer and $0 \le r \le n$. Let d_r denote the number of Derangements of an r-element set with $d_0 = 1$. Then

$$\sum_{r=0}^{n} {}^{n}C_r\, d_r = n! \quad \text{and} \quad d_n = n! - \sum_{r=0}^{n} {}^{n}C_r d_r$$

(2) ***Direct formula:*** The number d_n of the number of Derangements of an n-element set is given by

$$d_n = (n!)\sum_{K=0}^{n}\frac{(-1)^K}{K!}$$

6.25 ***Set divided into groups:***

(1) If a set contains $m + n$ $(m \ne n)$ elements, then the number of ways the set can be divided into two groups containing m elements and n elements respectably is

$$\frac{(m+n)!}{m!\,n!}$$

(2) If $m = n$, then the number of divisions into two equal groups is

$$\frac{(2n)!}{(n!\,n!)2!}$$

EXERCISES

Single Correct Choice Type Questions

1. The number of different 6 letters words that can be formed by using 5 distinct consonants, 4 vowels and 3 capital letters (distinct) which begin with a capital letter, but consisting of 1 capital, 3 consonants and 2 vowels is

(A) 21500 (B) 21600 (C) 20600 (D) 20500

2. The number of ways of arranging the letters of the word SINGLETON excluding the given word is

(A) 9! (B) 9! – 1

(C) $\frac{1}{2}(9!)-1$ (D) $\frac{1}{2}(9!-1)$

3. Let $A = \{a \mid a$ is a prime number and $a < 31\}$. The number of rational numbers of the form a/b, where a and $b \in A$ and $a \ne b$, is

(A) 180 (B) 92 (C) 91 (D) 90

4. Let A be the set of all four-digit numbers of the form $x_1 x_2 x_3 x_4$ where $x_1 > x_2 > x_3 > x_4$ and each x_i may take the values from 1 to 9. Then $n(A)$ is

(A) 126 (B) 84 (C) 210 (D) 64

5. The numerals $1, 2, 3, \ldots, 9$ are arranged in all possible ways such that the digit in the middle is greater than all its preceding digits and less than all its succeeding digits. The number of such arrangements is

(A) 24 (B) 120 (C) 360 (D) 576

6. There are three coplanar lines. On each of these lines, p number of points are taken. The maximum number of triangles that can be formed with vertices at these points is

(A) $3p^2(p-1)$ (B) $3p^2(p-1)+1$

(C) $p^2(4p-3)$ (D) $3p^2(4p-3)$

7. The number of pairs of words (x, y), x containing 4 letters and y containing 3 letters, from the letters of the word STATICS is

(A) 1260 (B) 396 (C) 829 (D) 796

8. Let $n \ge 3$ and A be a set of n elements. Let $P = (x, y, z)$ where $x, y, z \in A$. Then the number of points P such that atleast two of x, y, z are equal is

(A) $3n^2 - 2n + 1$ (B) n^3

(C) ${}^{n}P_3$ (D) $n^3 - ({}^{n}P_3)$

9. Let $A = \{1, 2, 3, \ldots, n\}$. The number of bijections f from A onto A for which $f(1) \ne 1$ is

(A) $n! - n$ (B) $n! - (n-1)!$

(C) $n^2 - n$ (D) $(n-1)!$

10. The number of triangles whose vertices are the vertices of a polygon of n sides but whose sides are not the sides of the polygon is

(A) $\frac{n}{6}(n-4)(n-5)$ (B) $\frac{n}{6}(n-3)(n-4)$

(C) $\frac{n}{6}(n-1)(n-2)$ (D) $\frac{n}{6}(n-2)(n-3)$

11. Ten lines are given in a plane such that no two are parallel and no three are concurrent. Then the number of regions into which the plane is divided by these lines is

(A) 56 (B) 66 (C) 46 (D) 99

12. The sum of all four-digit numbers (without repetition) that can be formed by using the numerals 2, 3, 4 are 5 is

(A) 93234 (B) 49332

(C) 93324 (D) 94332

13. The number of different nine-digit numbers that can be formed from the number 223355888 such that even digits occupy odd places is

(A) 16 (B) 36 (C) 60 (D) 180

14. A, B are two speakers along with three more to address a public meeting. If B addresses immediately after A, the number of ways of arranging the list is

(A) 24 (B) 36 (C) 48 (D) 30

15. If r, s, t are prime numbers and p, q are two positive integers such that the LCM of p, q is $r^2s^4t^2$, then the number of ordered pairs (p, q) is

(A) 225 (B) 224 (C) 248 (D) 255

16. The number of derangements of a four element set is

(A) 8 (B) 9 (C) 10 (D) 12

17. The number of surjections from a five-element set on to a four-element set is

(A) 340 (B) 220 (C) 320 (D) 240

Multiple Correct Choice Type Questions

1. Consider n points in a plane of which only p points are collinear. Then the number of straight lines that can be drawn by joining these points is

(A) ${}^{(n-p)}C_2 + p(n-p)+1$ (B) ${}^nC_2$

(C) ${}^nC_2 - {}^pC_2 + 1$ (D) ${}^nC_2 - {}^pC_2$

2. Let $f(x) = {}^{(7-x)}P_{(x-3)}$. Then

(A) The domain of f is {3, 4, 5}
(B) Range of f is {2, 3, 24}
(C) The domain of f is {3, 4, 5, 6}
(D) $f(x)$ is one-one

3. Consider the letters of the word TATANAGAR. Which of the following is/are true?

(A) The number of arrangements of all the letters is 7560
(B) The number of words that begin with N is 840
(C) The number of five letter words in which no letter is repeated is 120
(D) The number of words that can be formed using all the letters without changing the position of N is 840

4. Let n be a positive integer and r an integer such that $0 \le r \le n$. Then

(A) ${}^nP_r = n \times {}^{(n-1)}P_{(r-1)}$
(B) ${}^nP_r = r! \times {}^nC_r$
(C) ${}^nP_r = {}^{(n-1)}P_r + r \times {}^{(n-1)}P_{(r-1)}$
(D) The number of permutations of n distinct objects taken r at a time with atleast one repetition is $n^r - {}^nP_r$

5. Certain 5-digit numbers are formed by using the numerals 0, 1, 2, 3, …, 9. Which of the following is/are true?

(A) The total number of numbers without using 0 in the first place from left and using any numeral any number of times is $9 \times (10)^4$
(B) Total number of numbers without repetitions is ${}^{10}P_5 - {}^9P_4$
(C) Total number of numbers with atleast one repeated digit is 62784
(D) If repetitions are allowed, the number of 5-digit numbers not containing 0 is 9^5

6. A total of 5 mathematics books, 4 physics books and 2 chemistry books are to be arranged in a row in a book shelf. Which of the following is/are true?

(A) The number of arrangements that two chemistry books are separated is $9 \times 10!$
(B) The number of arrangements in which four physics books are together is 8! 4!
(C) The number of arrangements in which no two mathematics books are together is (7·6) (6!)
(D) The number of arrangements in which the books on the same subject are all together is 12 (4! 5!)

7. If ${}^{13}C_{r+1} = {}^{13}C_{3r-5}$, then

(A) $r = 4$ (B) $r = 3$ (C) $r = 9$ (D) ${}^rC_2 = 3$

8. Let $x = 2^4 \cdot 3^4$. Which of the following is/are true?

(A) The number of proper divisors of x is 23
(B) The sum of all positive divisors of x is 31×11^2
(C) The sum of all divisors of x is 11^2
(D) $6^4 - 1$ is divisible by 5

9. Consider the word VARANASI. Which of the following is/are true?

(A) The number of words that can be formed using all the letters is 6720
(B) The number of words without disturbing the three A's is 120
(C) The number of words such that all the three A's together is 720
(D) The number of words which begin with A and end with A is 720

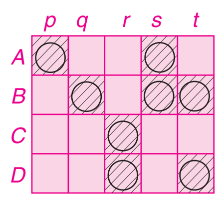
10. Which of the following is (are) correct?

(A) The number of diagonals of a 10-gon is 35
(B) The number of points of intersection of the diagonals of an octagon which lie inside the octagon is 70
(C) If ${}^nP_r = {}^nC_r$, then $r = 1$ or 0
(D) The maximum number of points in which 8 lines intersect 4 circles in the same plane is 64

Matrix-Match Type Questions

In each of the following questions, statements are given in two columns, which have to be matched. The statements in Column I are labeled as **(A), (B), (C)** and **(D),** while those in Column II are labeled as **(p), (q), (r), (s)** and **(t)**. Any given statement in Column I can have correct matching with *one or more* statements in Column II. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following example.

Example: If the correct matches are (A) $\to$ (p),(s); (B) $\to$ (q),(s),(t); (C) $\to$ (r); (D) $\to$(r),(t); that is if the matches are (A) $\to$ (p) and (s); (B) $\to$ (q),(s) and (t); (C) $\to$ (r); and (D) $\to$ (r),(t), then the correct darkening of bubbles will look as follows:



1. In Column I, certain types of arrangements of the letters of the word ORDINATE are given. Column II contains number of arrangements. Match the items in Column I with those in Column II.

Column I	*Column II*
(A) The number of words with I in the fourth place	(p) 576
(B) The number of words with vowels occupying odd places	(q) 676
(C) The number of words with consonants in the odd places	(r) 5040
(D) The number of words beginning with O and ending with E	(s) 720
	(t) 5050

2. Match the items in Column I with those in Column II.

Column I	*Column II*
(A) Total number of arrangements of the letters $a^2b^3c^4$ written in full length is	(p) 1120
(B) Six-digit numbers are to form using the numerals 1, 2, 3, 4. If all the numerals appear atleast once in the same number, then the number of such number is	(q) 120960
	(r) 1260
(C) The number of words that can be formed using all the letters of the word MISSISSIPI which begin with I and end with S is	(s) 1560
(D) The number of words that can be formed from the letters of the word MANESHPURI with vowels together is	(t) 120660

3. Certain requirements of arranging the letters of word ARRANGE are given in Column I and their respective number of arrangements are given in Column II. Match the items in Column I with those in Column II.

Column I	*Column II*
(A) Two Rs are never together	(p) 900
(B) Two As are together, but Rs are separated	(q) 240
(C) Neither two Rs nor two As are together	(r) 660
(D) Rs in the first and last places, but A is in the middle place	(s) 24

4. Match the items in Column I with those in Column II.

Column I	*Column II*
(A) Out of 8 sailors on a boat, 3 can work at row side only and 2 can work at bow side only. The number of arrangements of the sailors, if each side accommodates 4 sailors only, is	(p) $\dfrac{(11)!}{5!6!}(9!)^2$
(B) 18 guests have to be seated, half on each side of a long table. Four particular guests desire to sit on one particular side and three on the other side. The number of seating arrangements is	(q) 4356
(C) ABCD is a parallelogram. Ten lines each are drawn parallel to AB and BC intersecting the sides. The number of parallelograms that are formed is	(r) 15
(D) In a chess tournament, each player should play one game with each of the others. Two players left the tournament on personal reasons having played 3 games each. If the total number of games played is 84, the number of participants in the beginning of the tournament is	(s) 1728

Comprehension-Type Questions

1. Passage: Consider the digits 1, 2, 3, 4, 5 and 6. Answer the following three questions.

(i) The number of four-digit numbers, allowing repetition of digits any number of times, is

(A) 1296 (B) 4096 (C) 3096 (D) 2096

(ii) When repetitions are allowed, the number of four-digit even numbers is

(A) 448 (B) 216 (C) 1296 (D) 648

(iii) When repetitions are allowed, the number of four-digit numbers, that are divisible by 3, is

(A) 632 (B) 532 (C) 432 (D) 332

2. Passage: The letters of the word EAMCET are arranged in all possible ways. Answer the following three questions.

(i) The number of words that can be formed, without disturbing the places of E, is

(A) 120 (B) 24 (C) 48 (D) 720

(ii) The number of words that can be formed without separating the two Es is

(A) 120 (B) 240 (C) 24 (D) 360

(iii) If all possible words are written as in the dictionary, the rank of the word EAMCET is

(A) 134 (B) 135 (C) 132 (D) 133

Assertion–Reasoning Type Questions

Statement I and Statement II are given in each of the questions in this section. Your answers should be as per the following pattern:

(A) If both Statements I and II are correct and II is a correct reason for I
(B) If both Statements I and II are correct and II is not a correct reason for I
(C) If Statement I is correct and Statement II is false
(D) If Statement I is false and Statement II is correct.

1. Statement I: The number of words that can be formed using all the letters of the word ASSASSINATION is 69300.

Statement II: There are m_1 similar objects of one kind, m_2 similar objects of another kind, $\cdots, m_k$ similar objects of different kind. The total number of arrangements of all these objects is

$$\frac{(m_1+m_2+\cdots+m_K)!}{m_1!m_2!\cdots m_K!}$$

2. Statement I: A and B are two speakers to address a public meeting with four more speakers. The number of ways they can address such that A always speaks before B is 360.

Statement II: The number of ways that A can speak before B is equal to the number of ways that B speak before A.

3. Statement I: A number lock has four rings and each ring has 9 digits $1, 2, 3, \ldots, 9$. The number of unsuccessful attempts by a thief who does not know the key code to open the lock is 6560.

Statement II: If repetitions are allowed, the number of permutations of n dissimilar objects taken r at a time is n^r.

4. Statement I: Let $x_1, x_2, \ldots, x_n$ be a permutation of the natural numbers $1, 2, \ldots, n$. If n is odd, then the product $(x_1-1)(x_2-2)\ldots(x_n-n)$ is even.

Statement II: $\sum_{i=1}^{n}(x_i-i)=0$

5. Statement I: If $n \geq 1$ is an integer, then $(n^2)!/(n^2)!^n$ is an integer.

Statement II: mn objects can be divided among n persons in $(mn)!/(m!)^n$ ways.

6. Statement I: Consider n straight lines in a plane of which no two are parallel and no three are concurrent. Then the number of new lines that can be formed by joining the points of intersection of these n lines is

$$\frac{1}{8}(n-3)(n-2)(n-1)n$$

Statement II: Two coplanar non-parallel lines intersect in a point.

7. Statement I: Out of $2n+1$ consecutive positive integers 3 are to be selected such that they are in AP. The number of ways of selecting them is n^2.

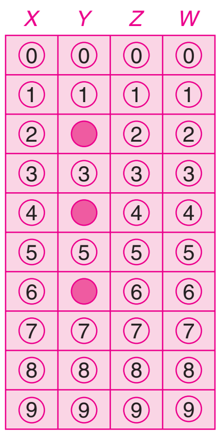
Statement II: Positive integers a, b, c are in AP if and only if either both a and c are even or both a and c are odd.

8. Statement I: In a lake, there are crocodiles each having teeth varying from 1 to 32. The number of crocodiles in the lake is 2^{32}.

Statement II: The number of elements in the power set $\wp(s)$ of a set S containing n elements is 2^n.

Integer Answer Type Questions

The answer to each of the questions in this section is a non-negative integer. The appropriate bubbles below the respective question numbers have to be darkened. For example, as shown in the figure, if the correct answer to the question number Y is 246, then the bubbles under Y labeled as 2, 4, 6 are to be darkened.



1. Out of 10 points in a plane, p points are collinear $(0 < p < 10)$. The number of triangles formed with vertices at these points is 110. Then the value of p is _____.

2. In a panchayat election, the number of candidates contesting for a ward is one more than the maximum number of candidates a voter can vote. If the total number of ways of which a voter can vote is 62, then the number of candidates is _____.

3. From four couples (wife and husband) a four-member team is to be constituted. The number of teams that can be formed which contain no couple is _____.

4. In a test there are n students. 2^{n-k} students gave wrong answers to k questions $(1 \le k \le n)$. If the total number of wrong answers given by the students is 2047, then n is equal to _____.

5. If the number of permutations of n different objects taken $n-1$ at a time is K times the number of permutations of n objects (of which two are identical) taken $n-1$ at a time, then K is equal to _____.

6. There 5 ladies and 10 gentlemen. A committee of 5 members is to be formed with two ladies and three gentlemen. The number of ways of forming the committee, excluding two particular ladies and including two particular gentlemen, is _____.

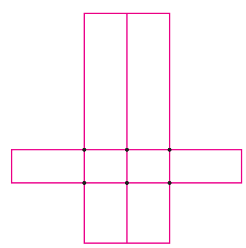
7. The number of arrangements of n distinct object taken all at a time is equal to K times the number of arrangements of n objects which contain two similar objects of one kind and three similar objects of another kind. In such case K is equal to _____.

8. If $A = \{1, 2, 3, 4\}$ and $B = \{a, b\}$, then the number of surjections from A onto B is _____.

9. The number of ordered triplets (x, y, z) of positive integers such that their product is 24 is _____.

10. The least positive integer n such that ${}^{(n-1)}C_5 + {}^{(n-1)}C_6 < {}^{n}C_7$ is _____.

11. Six xs are to be placed in the squares of the given figure (containing 8 squares) with not more than one x in each square and such that each row contains atleast one x. The number of ways that this can be done is _____.



12. Five points on positive x-axis and 10 points on positive y-axis are marked and line segments connecting these points are drawn. Then the maximum number of points of intersection of these 50 line segments in the interior of the first quadrant is _____.

13. The number of triangles whose vertices are at the vertices of an octagon but sides are not the sides of the octagon is _____.

ANSWERS

Single Correct Choice Type Questions

1. (B)
2. (C)
3. (D)
4. (A)
5. (D)
6. (C)
7. (A)
8. (D)
9. (B)
10. (A)
11. (A)
12. (C)
13. (C)
14. (A)
15. (A)
16. (B)
17. (D)

Multiple Correct Choice Type Questions

1. (A), (C)
2. (A), (D)
3. (A), (B), (C), (D)
4. (A), (B), (C), (D)
5. (A), (B), (C), (D)
6. (A), (B), (D)
7. (B), (D)
8. (A), (B), (D)
9. (A), (B), (C), (D)
10. (A), (B), (C), (D)

Matrix-Match Type Questions

1. (A) $\to$ (r), (B) $\to$ (p), (C) $\to$ (p), (D) $\to$ (s)
2. (A) $\to$ (r), (B) $\to$ (s), (C) $\to$ (p), (D) $\to$ (q)
3. (A) $\to$ (p), (B) $\to$ (q), (C) $\to$ (r), (D) $\to$ (s)
4. (A) $\to$ (s), (B) $\to$ (p), (C) $\to$ (q), (D) $\to$ (r)

Comprehension-Type Questions

1. (i) (A); (ii) (D); (iii) (C)
2. (i) (B); (ii) (A); (iii) (D)

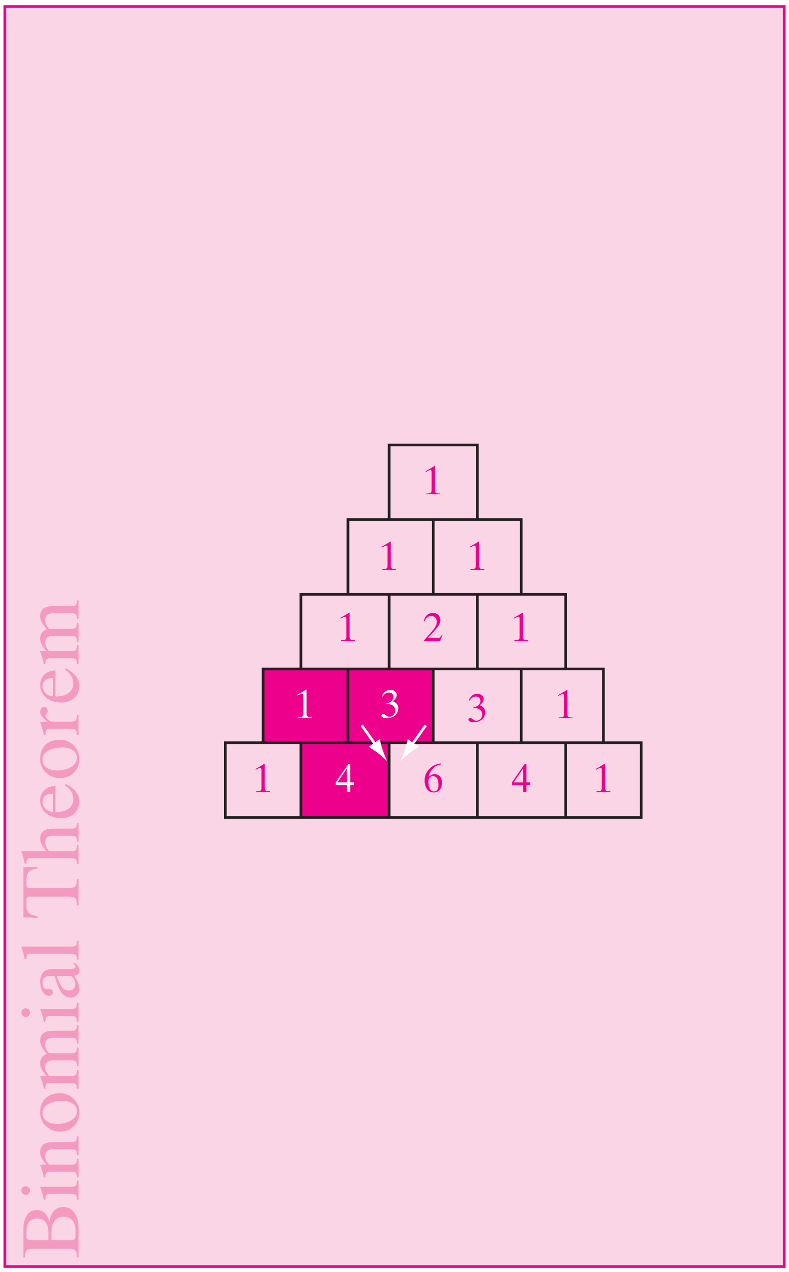
Assertion–Reasoning Type Questions

1. (D)
2. (A)
3. (A)
4. (A)
5. (A)
6. (A)
7. (C)
8. (D)

Integer Answer Type Questions

1. 5
2. 6
3. 16
4. 11
5. 2
6. 24
7. 12
8. 14
9. 30
10. 14
11. 26
12. 450
13. 16

7 Binomial Theorem



Contents

The **binomial theorem** describes the algebraic expansion of powers of a binomial. According to the theorem, it is possible to expand the power $(x + y)^n$ into a sum involving terms of the form ax^by^c. The binomial coefficients appear as the entries of Pascal's triangle.

The theorem which gives expansion of $(a+b)^n$ into the sum of $n+1$ terms, where n is a positive integer, is called the binomial theorem. We have come across formulas like

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

The coefficients involved in these expansions are called *binominal coefficients*. In this chapter, we derive expansion of $(a+b)^n$ for a positive integer n and study the properties of the binominal coefficients in these expansions. This will be further extended to a negative integer n or a rational number.

7.1 | Binomial Theorem for Positive Integral Index

In the expansions of $(a+b)^2$, $(a+b)^3$ and $(a+b)^4$ given above, observe that as we proceed from left to right, the index of a decreases by 1 while the index of b increases by 1. Also, observe that

$$(a+b)^2 = {}^2C_0a^2 + {}^2C_1ab + {}^2C_2b^2$$
$$(a+b)^3 = {}^3C_0a^3 + {}^3C_1a^2b + {}^3C_2ab^2 + {}^3C_3b^3$$
$$(a+b)^4 = {}^4C_0a^4 + {}^4C_1a^3b + {}^4C_2a^2b^2 + {}^4C_3ab^3 + {}^4C_4b^4$$

Keeping these in mind, we derive a formula for $(x+a)^n$ in the following. The idea behind writing x as one of the summands in $(x+a)^n$ is just to look at it as a polynomial of degree n, so that we can apply various results of addition and multiplication of polynomials in the study of the binominal coefficients,

THEOREM 7.1 (BINOMIAL THEOREM FOR POSITIVE INTEGRAL INDEX) For any positive integer n and any real or complex number a,

$$(x+a)^n = {}^nC_0x^n + {}^nC_1x^{n-1}a + {}^nC_2x^{n-2}a^2 + \cdots + {}^nC_na^n$$
$$= \sum_{r=0}^{n} {}^nC_rx^{n-r}a^r$$

PROOF We use induction on n. For $n = 1$, this is trivial, since ${}^1C_0 = 1 = {}^1C_1$.

Let $n > 1$ and assume the theorem for $n-1$; that is

$$(x+a)^{n-1} = \sum_{r=0}^{n-1} {}^{n-1}C_rx^{n-1-r}a^r$$

Then, we have

$$(x+a)^n = (x+a)^{n-1}(x+a)$$
$$= \left(\sum_{r=0}^{n-1} {}^{n-1}C_rx^{n-1-r}a^r\right)(x+a)$$
$$= \sum_{r=0}^{n-1} {}^{n-1}C_rx^{n-r}a^r + \sum_{r=0}^{n-1} {}^{n-1}C_rx^{n-1-r}a^{r+1}$$
$$= {}^{n-1}C_0x^{n-0}a^0 + \sum_{r=1}^{n-1} {}^{n-1}C_rx^{n-r}a^r + \sum_{r=0}^{n-2} {}^{n-1}C_rx^{n-1-r}a^{r+1} + {}^{n-1}C_{n-1}x^0a^{(n-1)+1}$$
$$= x^n + \sum_{r=1}^{n-1} {}^{n-1}C_rx^{n-r}a^r + \sum_{r=1}^{n-1} {}^{n-1}C_{r-1}x^{n-r}a^r + a^n$$
$$= {}^nC_0x^na^0 + \sum_{r=1}^{n-1}({}^{n-1}C_r + {}^{n-1}C_{r-1})x^{n-r}a^r + {}^nC_nx^0a^n$$

$$= {}^nC_0 x^n a^0 + \sum_{r=1}^{n-1} {}^nC_r x^{n-r} a^r + {}^nC_n x^0 a^n$$

$$= \sum_{r=0}^{n} {}^nC_r x^{n-r} a^r$$

Thus, the theorem is valid for all positive integers n. ■

DEFINITION 7.1 Note that there are $n+1$ terms in the above expansion of $(x+a)^n$. The $(r+1)$th term is called the *general term* and is denoted by T_{r+1}. It is given by

$$T_{r+1} = {}^nC_r x^{n-r} a^r \quad (0 \le r \le n)$$

COROLLARY 7.1 For any positive integer n and for any real number a,

$$\begin{aligned}(x-a)^n &= \sum_{r=0}^{n} {}^nC_r x^{n-r} (-a)^r \\ &= x^n - {}^nC_1 x^{n-1} a + {}^nC_2 x^{n-2} a^2 - {}^nC_3 x^{n-3} a^3 + \cdots \\ &\quad + (-1)^r\, {}^nC_r x^{n-r} a^r + \cdots + (-1)^{n-1}\, {}^nC_{n-1} x a^{n-1} + (-1)^n a^n\end{aligned}$$

The general term in the above expansion is $(-1)^r\ {}^nC_r x^{n-r} a^r$.

Examples

(1) The fourth term in the expansion of $(2x+5a)^8$ is

$$\begin{aligned}{}^8C_{4-1}(2x)^{8-(4-1)}(5a)^{4-1} &= {}^8C_3 (2)^5 x^5 (5)^3 a^3 \\ &= 224000 \cdot x^5 a^3\end{aligned}$$

(2) The ninth term in the expansion of $(2x-3a)^{17}$ is

$${}^{17}C_8 (2x)^{17-8}(-3a)^8 = {}^{17}C_8 (2)^9 (3)^8 x^9 a^8$$

COROLLARY 7.2 For any positive integer n and any real numbers a, b and c,

$$(a+b+c)^n = \sum_{\substack{0 \le r,\, s,\, t \le n \\ r+s+t=n}} \frac{n!}{r!\,s!\,t!} a^r b^s c^t$$

PROOF First, treat $b+c$ as single real number and use Theorem 7.1 to get

$$(a+b+c)^n = \sum_{r=0}^{n} {}^nC_r a^{n-r} (b+c)^r$$

and hence expand $(b+c)^r$ to get $(r+1)$ terms. Therefore, the expansion of $(a+b+c)^n$ contains

$$\sum_{r=0}^{n} (r+1) = \sum_{s=1}^{n+1} s = \frac{(n+1)(n+2)}{2} = {}^{(n+2)}C_2$$

number of terms and we can see that

$$(a+b+c)^n = \sum_{\substack{0 \le r,\, s,\, t \le n \\ r+s+t=n}} \frac{n!}{r!\,s!\,t!} a^r b^s c^t$$

Here the summation is taken over all ordered triples (r, s, t) of non-negative integers such that $r+s+t=n$.

Note: In general, for any positive integers n and m,

$$(x_1 + x_2 + \cdots + x_m)^n = \sum_{r_1+r_2+\cdots+r_m=n} \frac{n!}{r_1!r_2!\cdots r_m!} x_1^{r_1} x_2^{r_2} \cdots x_m^{r_m}$$

The summation is taken over all ordered m-tuples $(r_1, r_2, \ldots, r_m)$ of non-negative integers such that $r_1 + r_2 + \cdots + r_m = n$.

By Theorem 6.16, the number of terms in the above expansion is ${}^{(n+m-1)}C_{(m-1)}$. In particular, the number of terms in $(a+b+c)^n$ is

$${}^{(n+2)}C_2 = \frac{(n+2)(n+1)}{2}$$

Also, the number of terms in the expansion of $(a+b+c+d)^n$ is

$${}^{(n+4-1)}C_{4-1} = {}^{(n+3)}C_3 = \frac{(n+1)(n+2)(n+3)}{6}$$

DEFINITION 7.2 The middle term(s) in the expansion of $(x+a)^n$ is defined to be the term $T_{(n/2)+1}$ if n is even and the terms $T_{(n+1)/2}$ and $T_{(n+3)/2}$ if n is odd.

Note that, if n is even then the expansion of $(x+a)^n$ contains $n+1$ terms and there are equal number of terms before and after the term $T_{(n/2)+1}$. If n is odd, then the expansion of $(x+a)^n$ has even $(n+1)$ number of terms and there are exactly $(n-1)/2$ terms each before $T_{(n+1)/2}$ and after $T_{(n+3)/2}$. The total number of terms is

$$\frac{n-1}{2} + 1 + 1 + \frac{n-1}{2} = n+1$$

Examples

1. The middle term in the expansion of $(3x+4a)^{16}$ is $T_{(16/2)+1} = T_9$ which is given by

$${}^{16}C_8 (3x)^{16-8} (4a)^8 = {}^{16}C_8 \cdot 3^8 \cdot 4^8 \cdot a^8 \cdot x^8$$

2. There are two middle terms in the expansions of $(2x-3a)^{11}$. These are

$$T_{(11+1)/2} = T_6 = {}^{11}C_5 (2x)^{11-5} (-3a)^5 = -{}^{11}C_5 2^6 \cdot 3^5 \cdot a^5 \cdot x^6$$

and $$T_{(11+3)/2} = T_7 = {}^{11}C_6 (2x)^{11-6} (-3a)^6 = {}^{11}C_6 2^5 \cdot 3^6 \cdot a^6 \cdot x^5$$

DEFINITION 7.3 The binomial expansion of $(1+x)^n$ is

$${}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \cdots + {}^nC_{n-1} x^{n-1} + {}^nC_n x^n$$

This is called the *standard binomial* expansion and the coefficients in this are called the *binomial coefficients*. That is, ${}^nC_0, {}^nC_1, {}^nC_2, \ldots, {}^nC_r, \ldots, {}^nC_{n-1}, {}^nC_n$ are called the binomial coefficients. These are simply denoted by $C_0, C_1, \ldots, C_r, \ldots, C_{n-1}, C_n$.

Note that C_r alone has no meaning, unless we specify n also. If we say that $C_0, C_1, \ldots, C_n$ are the binomial coefficients means that these are ${}^nC_0, {}^nC_1, \ldots, {}^nC_n$, respectively. In the following we prove certain important properties of the binomial coefficients.

THEOREM 7.2 Let $C_0, C_1, C_2, \ldots, C_n$ be the binomial efficients. Then the following hold good.

1. $C_r = C_{n-r}$ for all $0 \le r \le n$
2. $C_0 + C_1 + C_2 + \cdots + C_n = 2^n$
3. $C_0 + C_2 + C_4 + \cdots = C_1 + C_3 + C_5 + \cdots = 2^{n-1}$. That is,

$$\sum_{r\,\text{even}} C_r = \sum_{r\,\text{odd}} C_r = 2^{n-1}$$

4. $C_0 + 2 \cdot C_1 + 3 \cdot C_2 + \cdots + (n+1) \cdot C_n = (n+2)2^{n-1}$

PROOF Note that

$$C_r = {}^nC_r = \frac{n!}{(n-r)!r!} \quad \text{for each } 0 \le r \le n$$

1. It is trivial.

2. We have the standard binomial expansion

$$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \cdots + {}^nC_{n-1} x^{n-1} + {}^nC_n x^n$$

By substituting 1 for x, we get that

$$\begin{aligned} 2^n &= {}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_{n-1} + {}^nC_n \\ &= C_0 + C_1 + C_2 + \cdots + C_{n-1} + C_n \end{aligned}$$

3. By substituting –1 for x in the standard binomial expansion, we get that

$$0 = C_0 - C_1 + C_2 - \cdots + (-1)^r \cdot C_r + \cdots + (-1)^n \cdot C_1$$

Therefore

$$\begin{aligned} &C_0 - C_1 + C_2 - C_3 + C_4 - \cdots = 0 \\ &C_0 + C_2 + C_4 + \cdots = C_1 + C_3 + C_5 + \cdots \end{aligned}$$

and by part (1), each of these is $2^n/2 = 2^{n-1}$.

4. Consider

$$S = C_0 + 2 \cdot C_1 + 3 \cdot C_2 + \cdots + (n+1) \cdot C_n \tag{7.1}$$

Writing the terms in the reverse order, we get that

$$S = (n+1) \cdot C_n + n \cdot C_{n-1} + \cdots + 3 \cdot C_2 + 2 \cdot C_1 + C_2$$

Since $C_r = C_{n-r}$ for each $0 \le r \le n$, we get

$$S = (n+1) \cdot C_0 + n \cdot C_1 + \cdots + 3 \cdot C_{n-2} + 2 \cdot C_{n-1} + C_n$$

Adding this to Eq. (7.1), we get that

$$\begin{aligned} 2S &= (n+2) \cdot C_0 + (n+2) \cdot C_1 + \cdots + (n+2) \cdot C_n \\ &= (n+2)(C_0 + C_1 + \cdots + C_n) \\ &= (n+2)2^n \end{aligned}$$

Therefore $S = (n+2)2^{n-1}$. That is,

$$C_0 + 2C_1 + 3C_2 + \cdots + (n+1)C_n = (n+2)2^{n-1}$$

■

THEOREM 7.3 Let $C_0, C_1, C_2, \ldots, C_n$ be the binomial coefficients. Then the following hold good.

1. For any real numbers a and d,

$$a \cdot C_0 + (a+d) \cdot C_1 + (a+2d) \cdot C_2 + \cdots + (a+nd) \cdot C_n = (2a+nd) \cdot 2^{n-1}$$

2. $\sum_{r=1}^{n} r \cdot C_r = n \cdot 2^{n-1}$

3. $\sum_{r=1}^{n} r(r-1)C_r = n(n-1) \cdot 2^{n-2}$

4. $\sum_{r=1}^{n} r^2 \cdot C_r = n(n+1) \cdot 2^{n-2}$

PROOF

1. Put $S = a\cdot C_0 + (a+d)\cdot C_1 + (a+2d)\cdot C_2 + \cdots + (a+nd)\cdot C_n$. Writing the terms in the reverse order and using $C_r = C_{n-r}$, we have

$$S = (a+nd)\cdot C_0 + [a+(n-1)d]\cdot C_1 + \cdots + (a+d)\cdot C_{n-1} + a\cdot C_n$$

Adding two equations, we get

$$2S = (2a+nd)\cdot C_0 + (2a+nd)\cdot C_1 + \cdots + (2a+nd)\cdot C_n$$

Therefore

$$S = \frac{1}{2}(2a+nd)(C_0 + C_1 + \cdots + C_n) = \frac{1}{2}(2a+nd)2^n$$

Thus $S = (2a+nd)2^{n-1}$.

2. Substituting 0 for a and 1 for d in part (1), we get

$$0\cdot C_0 + 1\cdot C_1 + 2\cdot C_2 + \cdots + n\cdot C_n = (0 + n\cdot 1)2^{n-1}$$

Therefore

$$\sum_{r=1}^{n} r\cdot C_r = n\cdot 2^{n-1}$$

In the following solution, differentiation is used which we discuss in Vol. III.

3. Consider $(1+x)^n = C_0 + C_1\cdot x + C_2\cdot x^2 + \cdots + C_{n-1}x^{n-1} + C_n x^n$. On differentiating both sides with respect to x, we get

$$n(1+x)^{n-1} = C_1 + C_2\cdot 2x + C_3\cdot 3x^2 + \cdots + C_n\cdot nx^{n-1}$$

Again on differentiating we get that

$$n(n-1)(1+x)^{n-1} = 2\cdot C_2 + 3\cdot 2\cdot C_3 x + 4\cdot 3\cdot C_4 + \cdots + n(n-1)\cdot C_n x^{n-1}$$

Substituting 1 for x in the above, we get that

$$n(n-1)2^{n-2} = 2\cdot 1\cdot C_2 + 3\cdot 2\cdot C_3 + \cdots + n(n-1)C_n$$

Thus

$$\sum_{r=1}^{n} r(r-1)C_r = n(n-1)2^{n-2}$$

4. We have

$$\begin{aligned}\sum_{r=1}^{n} r^2\cdot C_r &= \sum_{r=1}^{n} (r(r-1)+r)\cdot C_r \\ &= \sum_{r=1}^{n} r(r-1)\cdot C_r + \sum_{r=1}^{n} r\cdot C_r \\ &= \sum_{r=2}^{n} r(r-1)\cdot C_r + \sum_{r=1}^{n} r\cdot C_r \\ &= n(n-1)\cdot 2^{n-2} + n\cdot 2^{n-1} \\ &= (n(n-1)+2n)2^{n-2} \\ &= n(n+1)2^{n-2}\end{aligned}$$

■

Next, we will discuss about numerically greatest term among the $(n+1)$ terms in the expansion of $(1+x)^n$. Before this, let us recall that, for any real number x, $[x]$ denotes the largest integer less than or equal to x and that $[x]$ is called the *integral part* of x. Also, $x-[x]$ is called the *fractional part* of x and is denoted by $\{x\}$. Note that $x=[x]+\{x\}$, $[x]\in\mathbb{Z}$ and $0\le\{x\}<1$.

DEFINITION 7.4 In the binomial expansion of $(1+x)^n$, a term T_r is called *numerically greatest* if $|T_i|\le|T_r|$ for all $1\le i\le n+1$.

THEOREM 7.4 Let x be a non-zero real number, n a positive integer and m the integral part of $(n+1)|x|/(1+|x|)$.

1. If $m<(n+1)|x|/(1+|x|)<m+1$, then T_{m+1} is the numerically greatest term in the binomial expansion of $(1+x)^n$.
2. If $m=[(n+1)|x|]/(1+|x|)$, then T_m and T_{m+1} are the numerically greatest terms in the binomial expansion of $(1+x)^n$.

PROOF Let $T_1, T_2, \ldots, T_{n+1}$ be all the terms in the binomial expansion of $(1+x)^n$. Then

$$T_{r+1}={}^n\mathrm{C}_r x^r \quad \text{for all } 0\le r\le n$$

Since $x\neq 0$, T_{r+1} is a non-zero real number for each r. Now, consider

$$\frac{T_{r+1}}{T_r}=\frac{{}^n\mathrm{C}_r x^r}{{}^n\mathrm{C}_{r-1}x^{r-1}}=\frac{n!x^r}{(n-r)!r!}\times\frac{(n-r+1)!(r-1)!}{n!x^{r-1}}$$

$$=\frac{(n-r+1)}{r}\cdot x$$

Therefore

$$\left|\frac{T_{r+1}}{T_r}\right|=\frac{(n-r+1)}{r}|x| \tag{7.2}$$

Now

$$|T_{r+1}|\ge|T_r|\Leftrightarrow\frac{(n-r+1)}{r}|x|\ge 1$$

$$\Leftrightarrow\frac{n+1}{r}-1\ge\frac{1}{|x|}$$

$$\Leftrightarrow\frac{n+1}{r}\ge\frac{1+|x|}{|x|}$$

$$\Leftrightarrow\frac{1}{r}\ge\frac{1+|x|}{(n+1)|x|}$$

$$\Leftrightarrow r\le\frac{(n+1)|x|}{1+|x|}$$

Therefore, for any integer r with $1\le r\le n+1$, we have

$$|T_{r+1}|\ge|T_r|\Leftrightarrow r\le\left[\frac{(n+1)|x|}{1+|x|}\right]=m \tag{7.3}$$

Also, by Eq. (7.1) again

$$|T_{r+1}| = |T_r| \quad \Leftrightarrow \quad r = \frac{(n+1)|x|}{1+|x|} \tag{7.4}$$

and

$$|T_{r+1}| \le |T_r| \quad \Leftrightarrow \quad r \ge \frac{(n+1)|x|}{1+|x|} \tag{7.5}$$

Now, we shall distinguish two cases. First note that since m is the integral part of $(n+1)|x|/(1+|x|)$, we have

$$m \le \frac{(n+1)|x|}{1+|x|} < m+1$$

1. Suppose that $m < [(n+1)|x|]/(1+|x|) < m+1$. From Eqs. (7.3)–(7.5), we have

$$|T_1| < |T_2| < \cdots < |T_m| < |T_{m+1}| \quad \text{and} \quad |T_{m+1}| > |T_{m+2}| > \cdots > |T_{n+1}|$$

   Thus

$$|T_{m+1}| > |T_i| \quad \text{for all } i \ne m+1$$

   and therefore T_{m+1} is the numerically greatest term.

2. Suppose that $m = [(n+1)|x|]/(1+|x|)$. From Eq. (7.4), we have

$$|T_{m+1}| = |T_m|$$

   Again, by Eqs. (7.3) and (7.5), we get

$$|T_1| < |T_2| < \cdots < |T_m| = |T_{m+1}| > |T_{m+2}| > \cdots > |T_{n+1}|$$

   Thus, T_m and T_{m+1} are the numerically greatest terms in the binomial expansion of $(1+x)^n$. ■

COROLLARY 7.3 For any non-zero real numbers a and x, if T_m is numerically greatest term in $[1+(x/a)]^n$, then $a^n T_m$ is numerically greatest term in $(a+x)^n$.

QUICK LOOK 1

1. If $[(n+1)|x|]/(1+|x|)$ is not an integer, and m is the integral part of $[(n+1)|x|]/(1+|x|)$, then T_{m+1} is the numerically greatest term in $(1+x)^n$.
2. If $[(n+1)|x|]/(1+|x|)$ is an integer and is equal to m, then T_m and T_{m+1} are both numerically greatest terms in $(1+x)^n$. In this case note that $|T_m| = |T_{m+1}|$ and that T_m may not be equal to T_{m+1} and we can only infer that $T_m = \pm T_{m+1}$.

Example 7.1

Find the numerically greatest term(s) in the binomial expansion of $(1-2x)^{12}$ for $x = 1/5$.

Solution: Put $X = -2/5$ and consider $(1+X)^{12}$. We have

$$\frac{(12+1)|X|}{1+|X|} = \frac{13\cdot(2/5)}{1+(2/5)} = \frac{26}{7}$$

and the integral part of this is 3. Therefore, by part (1) of Theorem 7.4, T_4 is the numerically greatest term in $(1+X)^{12}$ and

$$T_4 = {}^{12}C_3 X^{12-3} = \frac{12\cdot 11\cdot 10}{1\cdot 2\cdot 3}\left(\frac{-2}{5}\right)^9 = -220\left(\frac{2}{5}\right)^9$$

Example 7.2

Find the numerically greatest term in the binomial expansion of $(2x-3y)^{19}$ when $x = 1/4$ and $y = 1/6$.

Solution: We have

$$(2x-3y)^{19} = (2x)^{19}\left(1-\frac{3y}{2x}\right)^{19}$$

Put $X = -3y/2x$ and consider

$$\frac{(19+1)|X|}{1+|X|} = \frac{20|-1|}{1+|-1|} = 10$$

which is an integer. By part (2) of Theorem 7.4, T_{10} and T_{11} are the numerically greatest terms in the binomial expansion of $(1+X)^{19}$ and of $(2x-3y)^{19}$. These terms are given by

$$T_{10} = {}^{19}C_9(2x)^{10}(-3y)^9 = -{}^{19}C_9\left(2\cdot\frac{1}{4}\right)^{10}\left(3\cdot\frac{1}{6}\right)^9 = -{}^{19}C_9\cdot\frac{1}{2^{19}}$$

and $$T_{11} = {}^{19}C_{10}(2x)^9(-3y)^{10} = {}^{19}C_{10}\left(2\cdot\frac{1}{4}\right)^9\left(3\cdot\frac{1}{6}\right)^{10}$$

$$= {}^{19}C_{10}\cdot\frac{1}{2^{19}}$$

Note that, in this case $T_{10} = -T_{11}$.

Theorem 7.5

1. If n is even, then ${}^nC_{n/2}$ is the greatest among the binomial coefficients ${}^nC_0, {}^nC_1, {}^nC_2, \ldots, {}^nC_n$.
2. If n is odd, then ${}^nC_{(n-1)/2} = {}^nC_{(n+1)/2}$ are the greatest among ${}^nC_0, {}^nC_1, {}^nC_2, \ldots, {}^nC_n$.

Proof This follows from Theorem 7.4 by taking $x = 1$ and from the part that $T_m = {}^nC_{m-1}$. ■

7.2 | Binomial Theorem for Rational Index

In the earlier section, we have proved that, for any positive integer n and for any real number x,

$$(1+x)^n = {}^nC_0 + {}^nC_1x^1 + {}^nC_2x^2 + \cdots + {}^nC_nx^n$$

This also can be expressed as

$$(1+x)^n = 1+\sum_{r=1}^{\infty}\frac{n(n-1)(n-2)\cdots(n-r+1)}{1\cdot 2\cdot 3\cdots r}x^r$$

For $r > n$, the coefficient of x^r becomes zero and the above is an expression of $n+1$ terms only. However, for a negative integer n or for a fraction (rational number) n, we have a similar formula consisting of infinitely many terms, provided $|x| < 1$. The proof of this is beyond the scope of this book and we state the following without proof and derive certain useful consequences.

Theorem 7.6 Let x be a real number such that $-1 < x < 1$. Then for any rational number m,

$$(1+x)^m = 1+\sum_{r=1}^{\infty}\frac{m(m-1)\cdots(m-r+1)}{1\cdot 2\cdot 3\cdots r}x^r$$

Corollary 7.4 Let n be a positive integer and x a real number such that $-1 < x < 1$. Then

1. $(1+x)^{-n} = \sum_{r=0}^{\infty}(-1)^r\,{}^{(n+r-1)}C_rx^r$
2. $(1-x)^{-n} = \sum_{r=0}^{\infty}{}^{n+r-1}C_rx^r$

PROOF From Theorem 7.6, we get that

$$(1+x)^{-n} = 1 + \sum_{r=1}^{\infty} \frac{(-n)(-n-1)(-n-2)\cdots(-n-r+1)}{1\cdot 2\cdot 3\cdots r} x^r$$

$$= 1 - \frac{n}{1}x + \frac{n(n+1)}{1\cdot 2}x^2 - \frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}x^3 + \cdots + (-1)^r \frac{n(n+1)\cdots(n+r-1)}{1\cdot 2\cdot 3\cdots r} + \cdots$$

$$= 1 + \sum_{r=1}^{\infty} (-1)^r \frac{(n+r-1)(n+r-2)\cdots(n+1)n}{1\cdot 2\cdot 3\cdots r} x^r$$

$$= \sum_{r=0}^{\infty} (-1)^r\, {}^{(n+r-1)}C_r x^r$$

Also, by replacing x with $-x$ in the above, we have

$$(1-x)^{-n} = \sum_{r=0}^{\infty} (-1)^r \cdot {}^{(n+r-1)}C_r (-x)^r$$

$$= \sum_{r=0}^{\infty} {}^{(n+r-1)}C_r x^r$$

■

COROLLARY 7.5 Let m and n be positive integers and x a number such that $-1 < x < 1$. Then we have the following.

1. $(1+x)^{m/n} = 1 + \frac{m}{1}\frac{x}{n} + \frac{m(m-n)}{1\cdot 2}\left(\frac{x}{n}\right)^2 + \cdots + \frac{m(m-n)\cdots[m-(r-1)n]}{1\cdot 2\cdots r}\left(\frac{x}{n}\right)^r + \cdots$
2. $(1-x)^{m/n} = 1 - \frac{m}{1}\cdot\frac{x}{n} + \frac{m(m-n)}{1\cdot 2}\left(\frac{x}{n}\right)^2 + \cdots + (-1)^r \frac{m(m-n)\cdots[m-(r-1)n]}{1\cdot 2\cdots r}\left(\frac{x}{n}\right)^r + \cdots$
3. $(1+x)^{-m/n} = 1 - \frac{m}{1}\cdot\frac{x}{n} + \frac{m(m+n)}{1\cdot 2}\left(\frac{x}{n}\right)^2 + \cdots + (-1)^r \frac{m(m+n)\cdots[m+(r-1)n]}{1\cdot 2\cdots r}\left(\frac{x}{n}\right)^r + \cdots$
4. $(1-x)^{-m/n} = 1 + \frac{m}{1}\cdot\frac{x}{n} + \frac{m(m+n)}{1\cdot 2}\left(\frac{x}{n}\right)^2 + \cdots + \frac{m(m+n)\cdots[m+(r-1)n]}{1\cdot 2\cdots r}\left(\frac{x}{n}\right)^r + \cdots$

PROOF Here we will prove part (1) only. Parts (2), (3) and (4) can be similarly proved. From Theorem 7.6,

$$(1+x)^{m/n} = 1 + \sum_{r=1}^{\infty} \frac{(m/n)[(m/n)-1]\cdots[(m/n)-r+1]}{1\cdot 2\cdot 3\cdots r} x^r$$

$$= 1 + \frac{(m/n)}{1}x + \frac{(m/n)[(m/n)-1]}{1\cdot 2}x^2 + \frac{(m/n)[(m/n)-1][(m/n)-2]}{1\cdot 2\cdot 3}x^3 + \cdots$$

$$+ \frac{(m/n)[(m/n)-1]\cdots[(m/n)-(r-1)]}{1\cdot 2\cdot 3\cdots r}x^r + \cdots$$

$$= 1 + \frac{m}{1}\cdot\frac{x}{n} + \frac{m(m-n)}{1\cdot 2}\left(\frac{x}{n}\right)^2 + \frac{m(m-n)(m-2n)}{1\cdot 2\cdot 3}\left(\frac{x}{n}\right)^3 + \cdots$$

$$+ \frac{m(m-n)\cdots(m-(r-1)n)}{1\cdot 2\cdots n}\left(\frac{x}{n}\right)^r + \cdots$$

■

Try it out Solve parts (2)–(4) of Corollary 7.4.

QUICK LOOK 2

For any real number x with $-1 < x < 1$, we have the following:

1. $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots + (-1)^r x^r + \cdots$

2. $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots + x^r + \cdots$

3. $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots + (-1)^r \cdot (r+1)x^r + \cdots$

4. $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots + (r+1)x^r + \cdots$

5. $(1+x)^{-3} = 1 - 3x + \dfrac{3\cdot 4}{1\cdot 2}x^2 - \dfrac{3\cdot 4\cdot 5}{1\cdot 2\cdot 3}x^3 + \cdots + (-1)^r \dfrac{3\cdot 4\cdot 5\cdots r(r+1)(r+2)}{1\cdot 2\cdot 3\cdots r}x^r + \cdots$

$$= 1 - 3x + \frac{3\cdot 4}{1\cdot 2}x^2 - \frac{4\cdot 5}{1\cdot 2}x^3 + \cdots + (-1)^r \cdot \frac{(r+1)(r+2)}{1\cdot 2}x^r + \cdots$$

6. $(1-x)^{-3} = 1 + 3x + \dfrac{3\cdot 4}{1\cdot 2}x^2 + \dfrac{4\cdot 5}{1\cdot 2}x^3 + \cdots + \dfrac{(r+1)(r+2)}{1\cdot 2}x^r + \cdots$

In the following examples we will use the fact that the general term in the expansion of $(1+x)^{-m}$ is given by

$$T_{r+1} = (-1)^r \frac{m(m+1)\cdots(m+r-1)}{1\cdot 2\cdot 3\cdots r}x^r$$

This is the $(r+1)$th term in the expansion of $(1+x)^{-m}$ for $r > 0$ and first term is always 1.

Example 7.3

Obtain the fifth term of $[1+(x/3)]^{-8}$.

Solution: The fifth term is given by

$$T_5 = T_{4+1} = (-1)^4 \frac{8(8+1)\cdots[8+(4-1)]}{1\cdot 2\cdot 3\cdot 4}\left(\frac{x}{3}\right)^4$$

$$= \frac{8\cdot 9\cdot 10\cdot 11}{1\cdot 2\cdot 3\cdot 4}\frac{x^4}{3^4} = \frac{110}{27}x^4$$

Example 7.4

Obtain the sixth term in the expansion of $[1-(x^2/4)]^{-4}$.

Solution: The sixth term is given by

$$T_6 = T_{5+1} = \frac{4\cdot 5\cdot 6\cdot 7\cdots[4+(5-1)]}{1\cdot 2\cdot 3\cdot 4\cdot 5}\left(\frac{x^2}{4}\right)^5$$

$$= \frac{4\cdot 5\cdot 6\cdot 7}{1\cdot 2\cdot 3\cdot 4\cdot 5}\frac{x^{10}}{4^5} = \frac{7}{4^5}x^{10}$$

Example 7.5

Obtain the fifth term in the expansion of $[6+(5y/11)]^{6/5}$.

Solution: The given expression can be written as

$$\left(6+\frac{5y}{11}\right)^{6/5} = 6^{6/5}\left(1+\frac{5y}{6\cdot 11}\right)^{6/5}$$

The fifth term in the expansion is given by

$$T_5 = T_{4+1} = 6^{6/5}\left[\frac{6(6-5)(6-2\cdot 5)(6-3\cdot 5)}{1\cdot 2\cdot 3\cdot 4}\left(\frac{5y}{66}\right)^4\right]$$

$$= 6^{6/5}\,\frac{6\cdot 1\cdot(-4)(-9)}{1\cdot 2\cdot 3\cdot 4}\left(\frac{5}{66}\right)^4 y^4$$

Example 7.6

Obtain the 10th term in the expansion of $(3-4x)^{2/3}$.

Solution: We can write the given expression as

$$(3-4x)^{2/3} = 3^{2/3}\left(1-\frac{4}{3}x\right)^{2/3}$$

The 10th term in the expansion is given by

$$T_{10} = T_{9+1} = (-1)^9 \frac{2(2-3)(2-2\cdot 3)\cdots(2-8\cdot 3)}{1\cdot 2\cdots 9}\left(\frac{4x}{3}\right)^9 \cdot 3^{2/3}$$

$$= -\frac{2\cdot 1\cdot 4\cdot 7\cdot 10\cdot 13\cdot 16\cdot 19\cdot 22}{1\cdot 2\cdot 3\cdots 9}\left(\frac{4}{3}\right)^9 \cdot 3^{2/3}\cdot x^9$$

Example 7.7

Obtain the values of x for which the binomial expansion of $(3-4x)^{-7}$ is valid.

Solution: The given expression can be written as

$$(3-4x)^{-7} = 3^{-7}\left(1-\frac{4x}{3}\right)^{-7}$$

Therefore, the binomial expansion of $(3-4x)^{-7}$ is valid if and only if that of $[1-(4x/3)]^{-7}$ is valid and this is valid if $-1 < 4x/3 < 1$. That is

$$\frac{-3}{4} < x < \frac{3}{4} \quad \text{or} \quad |x| < \frac{3}{4}$$

Example 7.8

Find the sum of the infinite series

$$1+\frac{1}{3}+\frac{1\cdot 3}{3\cdot 6}+\frac{1\cdot 3\cdot 5}{3\cdot 6\cdot 9}+\cdots$$

Solution: This sum can be written as

$$S = 1+\frac{1}{1}\cdot\frac{1}{3}+\frac{1\cdot 3}{1\cdot 2}\left(\frac{1}{3}\right)^2+\frac{1\cdot 3\cdot 5}{1\cdot 2\cdot 3}\left(\frac{1}{3}\right)^3+\cdots$$

$$= 1+\sum_{r=1}^{\infty}\left[\frac{m(m+n)\cdots(m+(r-1)n)}{1\cdot 2\cdots r}\cdot\left(\frac{x}{n}\right)^r\right]$$

where $m=1, n=2, x=2/3$. Substituting these values we get

$$S = \left(1-\frac{2}{3}\right)^{-1/2} \quad \text{[by part (4) of Corollary 7.4]}$$

$$= \sqrt{3}$$

Example 7.9

Find the coefficient of x^4 in $(1-4x)^{-3/5}$.

Solution: The general term in the expansion of $(1-4x)^{-3/5}$ is

$$T_{r+1} = \frac{3(3+5)(3+2\cdot 5)+\cdots+(3+(r-1)5)}{1\cdot 2\cdot 3\cdots r}\left(\frac{4x}{5}\right)^r$$

Therefore x^4 appears in T_5 only and hence

$$T_5 = \frac{3\cdot 8\cdot 13\cdot 18}{1\cdot 2\cdot 3\cdot 4}\left(\frac{4x}{5}\right)^4 = \frac{13\cdot 18\cdot 4^4}{5^4}x^4$$

The coefficient of x^4 in the expansion of $(1-4x)^{-3}$ is $(13\cdot 18\cdot 256)/625$.

Example 7.10

Using the binomial theorem for rational index, find the approximate value of $(242)^{1/5}$ correct to 4 decimals.

Solution: Consider

$$(242)^{1/5} = (243-1)^{1/5}$$

$$= (243)^{1/5}\left(1-\frac{1}{243}\right)^{1/5}$$

$$= 3\left[1-\frac{1}{5}\cdot\frac{1}{243}+\frac{(1/5)[(1/5)-1]}{1\cdot 2}\left(\frac{1}{243}\right)^2\cdots\right]$$

$$= 3\left[1-\frac{1}{5}(0.00243)-\frac{2}{25}(0.00243)^2\cdots\right]$$

$$\left[\text{since } \frac{1}{243} = \left(\frac{1}{3}\right)^5 \simeq (0.3)^5\right]$$

$$= 3 - \frac{3}{5}(0.00243) - \frac{6}{25}(0.00243)^2 \cdots$$

$$= 3 - 0.001458 - 0.000001417176$$
(by neglecting other terms)

$$= 2.998541$$

$$\simeq 2.9985 \quad \text{(corrected to 4 decimals)}$$

Note that when we are required to find an approximation of an expression correct to K decimal places, we choose r, the number of terms to be taken in consideration, such that the magnitude of the rth term is less than $1/10^{K+2}$ so that its decimal representation has atleast $K + 2$ zeros immediately after the decimal.

Example 7.11

Find the approximate value of $\sqrt{4+3x}/(3-2x)^2$, when $|x|$ is so small that x^2 and higher powers of x can be neglected.

Solution: The given expression can be written as

$$\frac{\sqrt{4+3x}}{(3-2x)^2} = \frac{2[1+(3x/4)]^{1/2}}{9[1-(2x/3)]^2}$$

$$= \frac{2}{9}\left(1+\frac{3x}{4}\right)^{1/2} \cdot \left(1-\frac{2x}{3}\right)^{-2}$$

$$= \frac{2}{9}\left(1+\frac{1}{2}\cdot\frac{3}{4}x\right)\left[1+(-2)\left(\frac{-2x}{3}\right)\right]$$

(by neglecting x^2 and the higher powers of x)

$$= \frac{2}{9}\left(1+\frac{3}{8}x\right)\left(1+\frac{4}{3}x\right)$$

$$= \frac{2}{9}\left(1+\frac{3}{8}x+\frac{4}{3}x\right) \text{ (by neglecting } x^2 \text{ too)}$$

$$= \frac{2}{9}\left(1+\frac{41}{24}x\right)$$

$$= \frac{2}{9} + \frac{41}{108}x = \frac{24+41x}{108}$$

WORKED-OUT PROBLEMS

Single Correct Choice Type Questions

1. If the sum of the fifth and sixth terms is zero in the binomial expansion of $(a - b)^n$, $n \le 5$, then the value of a/b is

(A) $(n-4)/5$ (B) $5/(n-4)$ (C) 5 (D) 1/5

Solution: If $n \le 5$, then the fifth and sixth terms exist in the binomial expansion of $(a-b)^n$ only when $n = 5$, then in $(a-b)^5$,

$$\text{Fifth term} = {}^5C_4 a^{5-4}(-b)^4 = 5ab^4$$
$$\text{Sixth term} = {}^5C_5 a^{5-5}(-b)^5 = -b^5$$

If the sum of fifth and sixth terms is zero, then

$$5ab^4 - b^5 = 0$$

$$5ab^4 = b^5$$

$$\frac{a}{b} = \frac{1}{5}$$

Answer: (D)

2. If a and b are the coefficients of x^n in the expansions of $(1+x)^{2n}$ and $(1+x)^{2n-1}$, respectively, then

(A) $a = 2b$ (B) $b = 2a$
(C) $a = 3b$ (D) $b = 3a$

Solution: x^n occurs in the $(n+1)$th term of $(1+x)^{2n}$ or $(1+x)^{2n-1}$. Therefore

$$a = {}^{2n}C_n = \frac{(2n)!}{n!n!} \quad \text{and} \quad b = {}^{(2n-1)}C_n = \frac{(2n-1)!}{(n-1)!n!}$$

Hence

$$\frac{2n}{n}b = a \quad \text{or} \quad a = 2b$$

Answer: (A)

3. If in the expansion of $(1+x)^m(1-x)^n$, the coefficients of x and x^2 are 3 and –6, respectively, then m and n are respectively

(A) 12, 9 (B) 13, 9 (C) 9, 13 (D) 9, 12

Solution: We have

$$(1+x)^m(1-x)^n = (1 + mC_1x + mC_2x^2 + \cdots)$$
$$\times (1 - nC_1x + nC_2x^2 + \cdots)$$

Therefore coefficient of x is given by

$$^{m}C_1 - {}^{n}C_1 = m - n = 3 \qquad (7.6)$$

Now coefficient of x^2 is

$$mC_2 + nC_2 - mC_1 \cdot nC_1 = -6$$
$$m(m-1) + n(n-1) - 2mn = -12$$
$$(m-n)^2 - (m+n) = -12 \qquad (7.7)$$
$$9 - (m+n) = -12$$
$$m + n = 21$$

Solving Eqs. (7.6) and (7.7), we get $m = 12,\ n = 9$.

Answer: (A)

4. In the expansion of $(1 + x)^{15}$, if the coefficients of $(r-1)$th and $(2r+3)$rd terms are equal, then r is equal to

(A) 4 (B) 6 (C) 5 (D) 7

Solution: The coefficients of $(r-1)$th and $(2r+3)$rd terms are

$$T_{r-1} = {}^{15}C_{r-2}x^{r-2} \quad \text{and} \quad T_{2r+3} = {}^{15}C_{2r+2}x^{2r+2}$$

Since it is given that they are equal, we have

$$^{15}C_{r-2} = {}^{15}C_{2r+2}$$
$$r - 2 \neq 2r + 2$$
$$\Rightarrow (r-2) + (2r+2) = 15$$
$$\Rightarrow r = 5$$

Answer: (C)

5. Let a be the coefficient of x^{10} in the expansion of $(1-x^2)^{10}$ and b the term independent of x in the expansion $[x-(2/x)]^{10}$. Then $a : b$ is equal to

(A) 1 : 12 (B) 1 : 22 (C) 6 : 1 (D) 1 : 32

Solution: We have $x^{10} = (x^2)^5$. The coefficient of x^{10} is $^{10}C_5$. Independent term in the expansion $[x-(2/x)]^{10}$ is $-2^5 \cdot {}^{10}C_5$. Therefore

$$a : b = 1 : 2^5 = 1 : 32$$

Answer: (D)

6. The value of the term independent of x in the expansion of

$$\left(\frac{x+1}{\sqrt[3]{x^2} - \sqrt[3]{x} + 1} - \frac{x-1}{x - \sqrt{x}}\right)^{10}$$

is

(A) 110 (B) 90 (C) 210 (D) 200

Solution: Put $\sqrt[3]{x} = y$ and $\sqrt{x} = z$ so that

$$\frac{x+1}{\sqrt[3]{x^2} - \sqrt[3]{x} + 1} = y + 1$$

and
$$\frac{x-1}{x-\sqrt{x}} = 1 + \frac{1}{z}$$

The given expansion is

$$\left(y + 1 - 1 - \frac{1}{z}\right)^{10} = \left(\sqrt[3]{x} - \frac{1}{\sqrt{x}}\right)^{10}$$

Therefore

$$T_{r+1} = {}^{10}C_r \cdot x^{(10-r)/3} \cdot (-1)^r \cdot x^{-r/2}$$

T_{r+1} is independent of x implies

$$\frac{10-r}{3} - \frac{r}{2} = 0$$

Hence

$$20 - 5r = 0 \quad \text{or} \quad r = 4$$

Therefore independent term value is

$$^{10}C_4(-1)^4 = 210$$

Answer: (C)

7. The coefficient of x^{50} in the expression $(1+x)^{1000} + 2x(1+x)^{999} + 3x^2(1+x)^{998} + \cdots + (1001)x^{1000}$ is

(A) $^{(1000)}C_{53}$ (B) $^{(1002)}C_{52}$ (C) $^{(1002)}C_{50}$ (D) $^{(1002)}C_{51}$

Solution: Let

$$s = (1+x)^{1000} + 2x(1+x)^{999} + 3x^2(1+x)^{998} + \cdots + (1001)x^{1000}$$

The terms of the given sum follow arithmetic geometric progression with first factors of the terms as 1, 2, 3, …, and second factors are in GP with common ratio $r = x/(1+x)$. Therefore

$$s = \frac{(1+x)^{1001}\{1 - [x/(1+x)]^{1001}\}}{1 - [x/(1+x)]} - (1001)x^{1001}$$
$$= (1+x)^{1002}\left[1 - \left(\frac{x}{1+x}\right)^{1001}\right] - (1001)x^{1001}$$
$$= (1+x)^{1002} - (1+x)x^{1001} - (1001)x^{1001}$$
$$= (1+x)^{1002} - (1002)x^{1001} - x^{1002}$$

Therefore the coefficient of x^{50} in the expression = coefficient of x^{50} in the expansion of $(1+x)^{1002}$ which is equal to $^{(1002)}C_{50}$.

Answer: (C)

8. Let n be a positive integer and

$$(1+x^2)^2(1+x)^n = \sum_{K=0}^{n+4} a_K x^K$$

If a_1, a_2 and a_3 are in AP, then the number of values of n is

(A) 2 (B) 3 (C) 4 (D) 5

Solution: We have

$$\sum_{K=0}^{n+4} a_K x^K = (1+2x^2+x^4)[1+n\mathrm{C}_1 x + n\mathrm{C}_2 x^2 + \cdots + x^n]$$

On the LHS, a_1, a_2 and a_3 are respectively the coefficients of x, x^2 and x^3. Therefore equating the coefficients of x, x^2 and x^3 on both the sides we get

$$a_1 = {}^n\mathrm{C}_1;\quad a_2 = 2 + {}^n\mathrm{C}_2;\quad a_3 = 2\cdot{}^n\mathrm{C}_1 + {}^n\mathrm{C}_3$$

From these we have

$$a_1 + a_3 = 2a_2$$
$${}^n\mathrm{C}_1 + 2\cdot{}^n\mathrm{C}_1 + {}^n\mathrm{C}_3 = 2(2 + {}^n\mathrm{C}_2)$$
$$3n + \frac{n(n-1)(n-2)}{6} = 4 + n(n-1)$$
$$n^3 - 9n^2 + 26n - 24 = 0$$
$$(n-2)(n-3)(n-4) = 0$$
$$n = 2, 3, 4$$

Answer: (B)

9. Let n be a positive integer. If the coefficients of second, third and fourth terms in the expansion of $(1+x)^n$ are in AP, then the value of n is

(A) 2 (B) 5 (C) 6 (D) 7

Solution: By hypothesis ${}^n\mathrm{C}_1, {}^n\mathrm{C}_2$ and ${}^n\mathrm{C}_3$ are in AP. Therefore

$${}^n\mathrm{C}_1 + {}^n\mathrm{C}_3 = 2\times{}^n\mathrm{C}_2$$
$$n + \frac{n(n-1)(n-2)}{6} = n(n-1)$$
$$6 + n^2 - 3n + 2 = 6n - 6$$
$$n^2 - 9n + 14 = 0$$
$$(n-2)(n-7) = 0$$

Since, there are more than three terms in the expansion, the value of n must be 7.

Answer: (D)

10. For $x > 1$, if the third term in the expansion of $[(1/x) + x^{\log_{10} x}]^5$ is 1000, then the value of x is

(A) 10 (B) 100 (C) $5\sqrt{2}$ (D) 50

Solution: Put $\log_{10} x = y$. Therefore

$$\left(\frac{1}{x} + x^{\log_{10} x}\right)^5 = (10^{-y} + 10^{y^2})^5$$

The third term is given by

$${}^5\mathrm{C}_2 10^{-3y}\cdot 10^{2y^2} = 1000$$
$$10^{-3y}\cdot 10^{2y^2} = 100 = 10^2$$
$$2y^2 - 3y - 2 = 0$$
$$(y-2)(2y+1) = 0$$
$$y = 2 \quad\text{or}\quad -1$$

For these values of y we have

$$x = 10^2 \quad\text{or}\quad x = 10^{-1}$$

Since it is given that $x > 1$, $x = 10^2$.

Answer: (B)

11. The coefficient of x^4 in the expansion of $[(x/2) - (3/x^2)]^{10}$ is

(A) $\frac{405}{256}$ (B) $\frac{504}{259}$

(C) $\frac{450}{263}$ (D) $\frac{400}{263}$

Solution: The $(r+1)$th term is given by

$$T_{r+1} = {}^{10}\mathrm{C}_r\left(\frac{x}{2}\right)^{10-r}\left(\frac{-3}{x^2}\right)^r$$
$$= {}^{10}\mathrm{C}_r \times \frac{1}{2^{10-r}}(-3)^r x^{10-3r}$$

Therefore

$$10 - 3r = 4 \Rightarrow r = 2$$

Hence coefficient of x^4 is

$${}^{10}\mathrm{C}_2 \times \frac{3^2}{2^8} = \frac{9\cdot 10}{2}\times\frac{9}{256} = \frac{405}{256}$$

Answer: (A)

12. The expression $(x+\sqrt{x^3-1})^5 + (x-\sqrt{x^3-1})^5$ is a polynomial of degree:

(A) 5 (B) 6

(C) 7 (D) 8

Solution: We know that

$$(a+b)^5 + (a-b)^5 = 2[{}^5\mathrm{C}_0 a^5 + {}^5\mathrm{C}_2 a^3 b^2 + {}^5\mathrm{C}_4 ab^4]$$
$$= 2(a^5 + 10a^3b^2 + 5ab^4)$$
$$= 2[x^5 + 10x^3(x^3-1) + 5x(x^3-1)^2]$$

where $x = a$ and $b = \sqrt{x^3-1}$. Therefore the given expression is a polynomial of degree 7.

Answer: (C)

13. The coefficient of t^{24} in $(1+t^2)^{12}(1+t^{12})(1+t^{24})$ is

(A) ${}^{12}\mathrm{C}_6 + 3$ (B) ${}^{12}\mathrm{C}_6 + 1$

(C) ${}^{12}\mathrm{C}_6$ (D) ${}^{12}\mathrm{C}_6 + 2$

Solution: We have

$$(1+t^2)^{12}(1+t^{12})(1+t^{24}) = (1+t^2)^{12}[1+t^{12}+t^{24}+t^{36}]$$
$$= [1+{}^{12}\text{C}_1 t^2 + {}^{12}\text{C}_2(t^2)^2 + \cdots + t^{24}]$$
$$\times (1+t^{12}+t^{24}+t^{36})$$

Therefore the coefficient of t^{24} is $1+{}^{12}\text{C}_6+1 = {}^{12}\text{C}_6+2$

Answer: (D)

14. The sum of the rational terms in the expansion of $(\sqrt{2}+3^{1/5})^{10}$ is

(A) 31 (B) 41 (C) 51 (D) 61

Solution: The general term is given by

$$T_{r+1} = {}^{10}\text{C}_r 2^{(10-r)/2} \cdot 3^{r/5}$$

This is rational, if $10-r$ is even and r is a multiple of 5.

$$r=5 \Rightarrow \frac{10-r}{2} \text{ is not an integer}$$

$$r=10 \Rightarrow 10-r=0 \quad \text{and} \quad \frac{r}{5}=2$$

$$r=0 \Rightarrow \frac{10-r}{2}=5$$

Therefore the sum of the rational terms is

$${}^{10}\text{C}_0 \cdot 2^5 + {}^{10}\text{C}_{10} \cdot 3^2 = 32+9=41$$

Answer: (B)

15. The sum of the coefficients of the polynomial $(1+x-3x^2)^{2163}$ is

(A) 0 (B) 2^{2163} (C) 1 (D) –1

Solution: If

$$f(x) \equiv a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n$$

then the sum of the coefficients is

$$a_0 + a_1 + a_2 + \cdots + a_n = f(1)$$

Now let $f(x) \equiv (1+x-3x^2)^{2163}$. Therefore the sum of the coefficients is

$$f(1) = (-1)^{2163} = -1$$

Answer: (D)

16. The coefficient of x^{99} in the expansion of $(x+1)(x+2)(x+3)\cdots(x+99)(x+100)$ is

(A) 5050 (B) 5500 (C) 5005 (D) 5000

Solution: We have

$$(x+\alpha)(x+\beta) = x^2 + (\alpha+\beta)x + \alpha\beta$$

$$(x+\alpha)(x+\beta)(x+\gamma) = x^3 + (\alpha+\beta+\gamma)x^2$$
$$+ (\alpha\beta+\beta\gamma+\gamma\alpha)x + \alpha\beta\gamma$$

By mathematical induction, we can show that the coefficient of x^{n-1} in the expansion of $(x+\alpha_1)(x+\alpha_2)(x+\alpha_3)\cdots(x+\alpha_n)$ is $\alpha_1+\alpha_2+\cdots+\alpha_n$. Therefore the coefficient of x^{99} in the given expansion is

$$1+2+3+\cdots+100 = \frac{100\times 101}{2} = 5050$$

Answer: (A)

17. Let T_r denote the rth term in the expansion of $[2^x+(1/4^x)]^n$. If the ratio $T_3 : T_2 = 7:1$ and sum of the coefficients of second and third terms is 36, then x value is

(A) $\dfrac{1}{2}$ (B) $\dfrac{-1}{2}$ (C) $\dfrac{1}{3}$ (D) $\dfrac{-1}{3}$

Solution: It is given that

$$\frac{T_3}{T_2} = 7$$

This implies

$$\left(\frac{n-2+1}{2}\right)\left(\frac{2^{-2x}}{2^x}\right) = 7$$

$$\frac{n-1}{2} \cdot 2^{-3x} = 7 \qquad (7.8)$$

Also

$${}^n\text{C}_1 + {}^n\text{C}_2 = 36$$
$$n^2+n-72=0$$
$$(n+9)(n-8)=0$$
$$n=8$$

Putting the value $n=8$ in Eq. (7.8), we have

$$\frac{7}{2} \cdot 2^{-3x} = 7$$
$$2^{-3x}=2$$

Therefore

$$x = \frac{-1}{3}$$

Answer: (D)

18. If the sixth term in the expansion of

$$\left[2^{\log_2\sqrt{9^{x-1}+7}} + \frac{1}{2^{(1/5)\log_2(3^{x-1}+1)}}\right]^7$$

is 84, then the sum of the values of x is

(A) 3 (B) 4 (C) 9 (D) 16

Solution: By hypothesis

$${}^7\text{C}_5(\sqrt{9^{x-1}+7})^2\left(\frac{1}{3^{x-1}+1}\right) = 84$$

Therefore

$$\frac{3^{2(x-1)}+7}{3^{x-1}+1}=4$$

Substituting $y=3^{x-1}$ we get

$$y^2+7=4y+4$$

$$y^2-4y+3=0$$

This gives $y=1,3$ which implies that $3^{x-1}=1$ or $3^{x-1}=3$. Therefore $x=1,2$. The sum of the values $=1+2=3$.

Answer: (A)

19. The integral part of $(\sqrt{2}+1)^6$ is

(A) 298 (B) 297 (C) 198 (D) 197

Solution: We have

$$(\sqrt{2}+1)^6+(\sqrt{2}-1)^6=2[(\sqrt{2})^6+15(\sqrt{2})^4+15(\sqrt{2})^2+1]$$
$$=198$$

Now $0<\sqrt{2}-1<1$ implies $197<(\sqrt{2}+1)^6<198$. Therefore the integral part of $(\sqrt{2}+1)^6=197$.

Answer: (D)

20. The last term in $(2^{1/3}+2^{-1/2})^n$ is $(3^{-5/3})^{\log_3 8}$. Then, the value of the fifth term is

(A) 110 (B) 210 (C) 310 (D) 220

Solution: We are given that

$$2^{-n/2}=3^{-5/3\cdot(3\cdot\log_3 2)}=2^{-5}$$

Therefore, $n=10$. The fifth term is

$$^{10}C_4\cdot(2^{1/3})^6\cdot(2^{-1/2})^4=\frac{7\cdot 8\cdot 9\cdot 10}{24}=210$$

Answer: (B)

21. Let $(1-x+x^2)^n=a_0+a_1x+a_2x^2+\cdots+a_{2n}x^{2n}$. If $a_0, a_1, a_2, \ldots, a_{2n}$ are in AP, then a_n equals

(A) $2n+1$ (B) $\frac{1}{2n+1}$ (C) $2n-1$ (D) $\frac{1}{2n-1}$

Solution: We have

$$1=a_0+a_1+a_2+\cdots+a_{2n}=\frac{2n+1}{2}(a_0+a_{2n})$$

Therefore

$$a_0+a_{2n}=\frac{2}{2n+1}$$

$$a_0+(a_0+2nd)=\frac{2}{2n+1}$$

(where d is the common difference)

$$a_0+nd=\frac{1}{2n+1}$$

Hence

$$a_n=(n+1)\text{th term}=a_0+nd=\frac{1}{2n+1}$$

Answer: (B)

22. If a_1, a_2, a_3, and a_4 are the coefficients of any four consecutive terms in the expansion of $(1+x)^n$, then

$$\frac{a_1}{a_1+a_2},\ \frac{a_2}{a_2+a_3},\ \frac{a_3}{a_3+a_4}$$

are in

(A) AP (B) GP (C) HP (D) AGP

Solution: Let a_1, a_2, a_3, a_4 be the coefficients of rth, $(r+1)$th, $(r+2)$th and $(r+3)$th terms, respectively. Then

$$a_1={}^nC_{r-1},\ a_2={}^nC_r,\ a_3={}^nC_{r+1},\ a_4={}^nC_{r+3}$$

We know that

$$\frac{^nC_K}{^nC_{K-1}}=\frac{n-K+1}{K}$$

Therefore

$$\frac{a_2}{a_1}=\frac{n-r+1}{r}\Rightarrow 1+\frac{a_2}{a_1}=\frac{n+1}{r}$$

$$\frac{a_3}{a_2}=\frac{n-r}{r+1}\Rightarrow 1+\frac{a_3}{a_2}=\frac{n+r}{r+1}$$

$$\frac{a_4}{a_3}=\frac{n-r-1}{r+2}\Rightarrow 1+\frac{a_4}{a_3}=\frac{n+1}{r+2}$$

and hence

$$\frac{a_1}{a_1+a_2}+\frac{a_3}{a_3+a_4}=\frac{r}{n+1}+\frac{r+2}{n+1}=2\left(\frac{r+1}{n+1}\right)=2\left(\frac{a_2}{a_2+a_3}\right)$$

This gives that the following are in AP:

$$\frac{a_1}{a_1+a_2},\ \frac{a_2}{a_2+a_3},\ \frac{a_3}{a_3+a_4}$$

Answer: (A)

23. If the middle term in the expansion of $(1+x)^{2n}$ is $K(2^n/n!)x^n$, then K is equal to

(A) $(2n)!$ (B) $1\cdot 3\cdot 5\cdots(2n-1)$

(C) $(2n-1)!$ (D) $\frac{1}{2}(2n-1)!$

Solution: Since there are $2n+1$ terms in the expansion, the $(n+1)$th term will be the middle term. Therefore the middle term is given by

$$^{(2n)}C_n x^n=\frac{(2n)!}{n!n!}x^n$$
$$=\frac{1\cdot 2\cdot 3\cdots 2n}{n!n!}x^n$$

$$= \frac{[1\cdot 3\cdot 5\cdots(2n-1)][2\cdot 4\cdot 6\cdots(2n)]x^n}{n!n!}$$

$$= \frac{[1\cdot 3\cdot 5\cdots(2n-1)]2^n(n!)x^n}{n!n!}$$

Therefore $K = 1\cdot 3\cdot 5\cdots(2n-1)$.

Answer: (B)

24. The coefficient of x^{53} in the expansion of $\sum_{K=0}^{100} {}^{100}C_K (x-3)^{10-K}\cdot 2^K$.

(A) $-{}^{100}C_{53}$ (B) ${}^{100}C_{53}$ (C) $-{}^{100}C_{52}$ (D) ${}^{100}C_{52}$

Solution: We have

$$\sum_{K=0}^{100} {}^{100}C_K (x-3)^{100-K}\cdot 2^K = (x-3+2)^{100} = (1-x)^{100}$$

Therefore coefficient of $x^{53} = -{}^{100}C_{53}$.

Answer: (A)

25. The coefficient of x^r in the expansion of $(x+3)^{n-1} + (x+3)^{n-2}(x+2) + (x+3)^{n-3}(x+2)^2 + \cdots + (x+2)^{n-1}$ is

(A) ${}^nC_r(3^{n-r} - 2^{n-r})$ (B) ${}^nC_{r-1}(3^{n-r+1} - 3^{n-r+1})$

(C) ${}^nC_{r+1}(3^r - 2^r)$ (D) ${}^nC_r(3^{n-r} + 2^{n-r})$

Solution: The terms of the given sum follow GP with first term $(x+3)^{n-1}$ and common ratio $(x+2)/(x+3)$. Therefore the given sum is

$$(x+3)^{n-1}\frac{\{1-[(x+2)/(x+3)]^n\}}{1-[(x+2)/(x+3)]} = (x+3)^n - (x+2)^n$$

Therefore the coefficient of x^r of the given sum is

$${}^nC_{n-r}3^{n-r} - {}^nC_{n-r}2^{n-r} = {}^nC_r(3^{n-r} - 2^{n-r}) \quad (\because\ {}^nC_r = {}^nC_{n-r})$$

Answer: (A)

26. The coefficient of x^8 in the expansion of $(1+x+x^2+x^3)^4$ is

(A) 30 (B) 31 (C) 32 (D) 36

Solution: We have

$$\begin{aligned}(1+x+x^2+x^3)^4 &= (1+x)^4(1+x^2)^4 \\ &= (1+4x+6x^2+4x^3+x^4) \\ &\quad (1+4x^2+6x^4+4x^6+x^8)\end{aligned}$$

Therefore the coefficient of $x^8 = 6\times 4 + 1\times 6 + 1 = 31$.

Answer: (B)

27. If $(1+x+2x^2)^{20} \equiv a_0 + a_1x + a_2x^2 + \cdots + a_{40}x^{40}$, then the value of $a_0 + a_2 + a_4 + \cdots + a_{38}$ is

(A) $2^{20}(2^{20}-1)$ (B) $2^{20}(2^{20}+1)$

(C) $2^{39} - 2^{19}$ (D) $2^{39} + 2^{19}$

Solution: In the given identity, substituting $x = 1$ and $x = -1$ both sides and adding

$$2(a_0 + a_2 + a_4 + \cdots + a_{40}) = 2^{40} + 2^{20}$$

Therefore

$$a_0 + a_2 + a_4 + \cdots + a_{40} = 2^{39} + 2^{19} \tag{7.9}$$

But a_{40} is the coefficient of x^{40} which is 2^{20}. Therefore from Eq. (7.9), we get

$$a_0 + a_2 + a_4 + \cdots + a_{38} = 2^{39} + 2^{19} - 2^{20} = 2^{39} - 2^{19}$$

Answer: (C)

28. The sum

$$\sum_{r=0}^{n} (-1)^r\, {}^nC_r\left[\frac{1}{2^r} + \frac{3}{2^{2r}} + \frac{5}{2^{3r}} + \frac{7}{2^{4r}} + \cdots \text{upto } m \text{ terms}\right]$$

is equal to

(A) $\dfrac{2^{mn}+1}{2^{mn}(2^n-1)}$ (B) $\dfrac{2^{mn}-1}{2^m(2^n-1)}$

(C) $\dfrac{2^{2mn}-1}{2^{mn}(2^m-1)}$ (D) $\dfrac{2^{mn}-1}{2^{mn}(2^n-1)}$

Solution: We have

$$\sum_{r=0}^{n} (-1)^r\, {}^nC_r\left(\frac{1}{2}\right)^r = \left(1-\frac{1}{2}\right)^n = \left(\frac{1}{2}\right)^n$$

$$\sum_{r=0}^{n} (-1)^r\, {}^nC_r\left(\frac{3}{2^{2r}}\right) = \sum_{r=0}^{n} (-1)^r\, {}^nC_r\left(\frac{3}{4}\right)^r = \left(1-\frac{3}{4}\right)^n = \left(\frac{1}{4}\right)^n$$

and so on. Therefore the given sum is

$$\left(\frac{1}{2}\right)^n + \left(\frac{1}{4}\right)^n + \left(\frac{1}{8}\right)^n + \left(\frac{1}{16}\right)^n + \cdots \text{upto } m \text{ terms}$$

$$= \left(\frac{1}{2}\right)^n\left[1 + \left(\frac{1}{2}\right)^n + \left(\frac{1}{2}\right)^{2n} + \left(\frac{1}{2}\right)^{3n} + \cdots \text{upto } m \text{ terms}\right]$$

$$= \frac{1}{2^n}\,\frac{1-(1/2^n)m}{1-(1/2^n)} = \frac{2^{mn}-1}{2^{mn}(2^n-1)}$$

Answer: (D)

29. Let p and q be positive integers. Let

$${}^pC_q = \begin{cases} \dfrac{p!}{p!\cdot(p-q)!} & \text{when } p \geq q \\ 0 & \text{when } p < q \end{cases}$$

Then the sum $\sum_{r=0}^{m}({}^{10}C_r \times {}^{20}C_{m-r})$ is maximum when m is

(A) 5 (B) 10 (C) 15 (D) 20

Solution: We have

$$(1+x)^{10} = C_0 + C_1x + C_2x^2 + \cdots + C_{10}x^{10} \text{ (where } C_r = {}^{10}C_r)$$

$$(1+x)^{20} = C_0 + C_1x + C_2x^2 + \cdots + C_{20}x^{20} \text{ (where } C_r = {}^{20}C_r)$$

Therefore

$$\sum_{r=0}^{m}({}^{10}C_r \times {}^{20}C_{m-r}) = \text{coefficient of } x^m \text{ in the expansion of}$$

$$(1+x)^{10}(1+x)^{20} = \text{coefficient of } x^m \text{ in } (1+x)^{30} = {}^{30}C_m$$

It is known that ${}^nC_r$ is maximum if $r = n/2$ when n is even. Therefore the given sum is maximum, if

$$m = \frac{30}{2} = 15$$

Answer: (C)

Multiple Correct Choice Type Questions

1. If the third term in the expansion of $(x + x^{\log_{10} x})^5$ is 10,00,000, then the value(s) of x may be

(A) 10 (B) 10^2 (C) $10^{5/2}$ (D) $10^{-5/2}$

Solution: Put $\log_{10} x = y$. Therefore

$$ {}^5C_2 x^3 (x^y)^2 = 10^6$$
$$x^{3+2y} = 10^5$$
$$(3+2y)\log_{10} x = 5$$
$$(3+2y)y = 5$$
$$2y^2 + 3y - 5 = 0$$
$$(y-1)(2y+5) = 0$$

This gives

$$y = 1 \quad \text{or} \quad y = \frac{-5}{2}$$

Therefore $x = 10$ or $10^{-5/2}$.

Answers: (A), (D)

2. If $(1+ax)^n = 1 + 8x + 24x^2 + \cdots$, then

(A) $a = 3$ (B) $n = 4$ (C) $a = 2$ (D) $n = 5$

Solution: We have

$$(1+ax)^n = 1 + {}^nC_1(ax) + {}^nC_2(ax)^2 + \cdots$$

Therefore

$$({}^nC_1)a = 8 \Rightarrow an = 8$$
$$({}^nC_2)a^2 = 24 \Rightarrow n(n-1)a^2 = 48$$

Now, $48 = n(n-1)a^2 = (an)(an-a) = 8(8-a)$. Therefore

$$8 - a = 6$$
$$a = 2$$
$$a = 2 \Rightarrow n = 4$$

Hence the answer is $a = 2$, $n = 4$.

Answers: (B) and (C)

3. If the ninth term in the expansion of $[3^{\log_3 \sqrt{(25)^x+7}} + 3^{-(1/8)\log_3(5^{x-1}+1)}]^{10}$ is equal to 180 where $x > 1$, then x value is

(A) $\log_5 3 + 1$ (B) $\log_5 15$ (C) $\log_5 3 + 2$ (D) $\log_{10} 15$

Solution: Put $a = \sqrt{25^{x-1} + 7} = \sqrt{5^{2(x-1)} + 7}$ and $b = (5^{x-1} + 1)^{-1/8}$. Therefore the ninth term is

$${}^{10}C_8 a^2 b^8 = 45(5^{2(x-1)} + 7)(5^{x-1} + 1)^{-1} = 180$$

Substituting $y = 5^{x-1}$ in this we get

$$y^2 + 7 = 4(y+1)$$
$$y^2 - 4y + 3 = 0$$
$$y = 1 \quad \text{or} \quad 3$$

Now

$$y = 1 \Rightarrow 5^{x-1} = 1 \Rightarrow x = 1 \text{ (reject as } x > 1)$$
$$y = 3 \Rightarrow 5^{x-1} = 3 \Rightarrow x = \log_5 15$$

Answers: (A) and (B)

4. Which of the following statements are true?

(A) The digit at unit place in the number $17^{1995} + 11^{1995} - 7^{1995}$ is 1.

(B) $(106)^{85} - (85)^{106}$ is divisible by 7.

(C) The positive integer which is just greater than $(1 + 0.0001)^{1000}$ is 2.

(D) If $(1 + 2x - 3x^2)^{2010} = a_0 + a_1x + a_2x^2 + \cdots + a_{4020}x^{4020}$, then $a_0 + a_2 + a_4 + a_6 + \cdots + a_{4020}$ is an even integer.

Solution:

(A) $(17)^{1995} + 11^{1995} - 7^{1995} = (10+7)^{1995} + (10+7)^{1995} - 7^{1995} = 1 + \text{(a multiple of 10)}$ as 7^{1995} and -7^{1995} cancelled with each other. Therefore the digit at the unit place is 1. Therefore (A) is true.

(B) $(106)^{85} - (85)^{106} = (1+105)^{85} - (1+84)^{106}$. When binomially expanded 1, –1 will be cancelled and in the remaining terms, 105 and 84 occur and are divisible by 7. Therefore (B) is true.

(C) It is easy to see that

$$1 < (1+0.0001)^{1000} = \left(1+\frac{1}{10^4}\right)^{1000} < 2$$

Therefore the integer just greater than $[1+(1/10^4)]^{1000}$ is 2. Hence (C) is true.

(D) Put $x = 1$ and -1 and add.

$$0 + (-4)^{2010} = a_0 + a_2 + a_4 + a_6 + \cdots + a_{4020}$$

Therefore (D) is true.

Answers: (A), (B), (C), (D)

5. If $(7+4\sqrt{3})^n = I + f$, where I is an integer and $0 < f < 1$, then

(A) I is an even integer (B) I is an odd integer

(C) $(I+f)(1-f) = 1$ (D) $f^2 - f + 1 = 0$

Solution: By hypothesis $(7 + 4\sqrt{3})^n = I + f$. Let $(7 - 4\sqrt{3})^n = G$, where $0 < G < 1$. Therefore

$$(I+f)+G = 2[7^n + nC_2 7^{n-2}(\sqrt{3})^2 + nC_4 7^{n-4}(\sqrt{3})^4 + \cdots]$$

When $0 < f + G < 2$ and $I + f + G$ is an integer $\Rightarrow f + G = 1$. Therefore

$$I = 2\text{ (some integer)} - 1 \text{ is an odd number}$$

Hence (B) is true. Again

$$(I+f)(1-f) = (I+f)G = (7+4\sqrt{3})^n(7-4\sqrt{3})^n$$
$$= (49-48)^n = 1$$

Therefore (C) is true.

Answers: (B) and (C)

6. If $(6\sqrt{6}+14)^{2n+1} = P$, then

(A) $[P]$ is an even integer (B) $PF = 2^{2n+1}$

(C) $PF = 20^{2n+1}$ (D) $[P]$ is an odd integer

Note: $[\cdot]$ denotes integral part and F is the fractional part of P.

Solution: Let

$$(6\sqrt{6}+14)^{2n+1} = P = I + F$$

where $I = [P]$ and $F = P - [P]$. Let

$$(6\sqrt{6}-14)^{2n+1} = G$$

so that $0 < G < 1$. Now

$$P - G = 2[(2n+1)C_1(6\sqrt{6})^{2n}(14)$$
$$+ (2n+1)C_3(6\sqrt{6})^{2n-2} \times (14)^2 + \cdots]$$
$$= 2\text{ (some positive integer)}$$

Therefore $I + F - G$ is an integer and $0 < F < 1, 0 < G < 1$ implies that $F = G$.

Also $[P] = I$ is an even integer and

$$PF = (I+F)F = (I+F)G$$
$$= (6\sqrt{6}+14)^{2n+1}(6\sqrt{6}-14)^{2n+1} = (20)^{2n+1}$$

Answers: (A) and (C)

7. If $(1+x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n$, where C_r denotes ${}^nC_r$, then

(A) $\dfrac{C_0}{x} - \dfrac{C_1}{x+1} + \dfrac{C_2}{x+2} - \cdots + (-1)^n\dfrac{C_n}{x+n}$

$$= \frac{n!}{x(x+1)(x+2)\cdots(x+n)}$$

for all $x \neq -1, -2, -3, \ldots$

(B) $\dfrac{C_0}{1} - \dfrac{C_1}{2} + \dfrac{C_2}{3} - \dfrac{C_3}{4} + \cdots + (-1)^n\dfrac{C_n}{n+1} = \dfrac{1}{n+1}$

(C) $\dfrac{C_0}{2} - \dfrac{C_1}{3} + \dfrac{C_2}{4} - \dfrac{C_3}{5} + \cdots + (-1)^n\dfrac{C_n}{n+2}$

$$= \frac{1}{(n+1)(n+2)}$$

(D) $C_0 - C_1 + C_2 - C_3 + \cdots + (-1)^nC_n = 0$

Solution: We prove (A) by mathematical induction. For $n = 1$,

$$\frac{C_0}{x} - \frac{C_1}{x+1} = \frac{1}{x} - \frac{1}{x+1} = \frac{1}{x(x+1)} = \frac{1!}{x(x+1)}$$

Assume for n, that is

$$\frac{C_0}{x} - \frac{C_1}{x+1} + \frac{C_2}{x+2} - \frac{C_3}{x+3} + \cdots + (-1)^n\frac{C_n}{x+n}$$
$$= \frac{n!}{x(x+1)(x+2)\cdots(x+n)}$$

Change x to $x+1$ on both sides

$$\frac{C_0}{x+1} - \frac{C_1}{x+2} + \frac{C_2}{x+3} - \cdots + (-1)^n\frac{C_n}{x+n+1}$$
$$= \frac{n!}{(x+1)(x+2)\cdots(x+n+1)}$$

On subtraction and using $nC_r + nC_{r-1} = (n+1)C_r$, we have

$$\frac{(n+1)C_0}{x} - \frac{(n+1)C_1}{x+1} + \frac{(n+1)C_2}{x+2} - \frac{(n+1)C_3}{x+3} + \cdots$$
$$+ (-1)^n\frac{(n+1)C_{n+1}}{x+n+1}$$

$$= \frac{n![(x+n+1)-x]}{x(x+1)(x+2)\cdots(x+n+1)}$$

$$= \frac{(n+1)!}{x(x+1)(x+2)\cdots(x+n+1)}$$

The result is also true for $n+1$. Hence (A) is true.
In (A), by substituting $x = 1, 2$ we see that (B) and (C) are true.
In $(1+x)^n$ expansion, putting $x = -1$ we get (D) is true.

Answers: (A), (B), (C), (D)

8. If $(1+x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n$, where $C_r = {}^nC_r$, then

(A) $a \cdot C_0 + (a+d) \cdot C_1 + (a+2d) \cdot C_2 + \cdots + (a+nd) \cdot C_n$
$= 2^{n-1}(2a+nd)$

(B) $1 \cdot C_0 + 2 \cdot C_1 + 3 \cdot C_2 + \cdots + (n+1) \cdot C_n = 2^n(n+1)$

(C) $1 \cdot C_0 + 2 \cdot C_1 + 3 \cdot C_2 + \cdots + (n+1) \cdot C_n = 2^{n-1}(2n+1)$

(D) $\dfrac{C_1}{C_0} + 2 \cdot \dfrac{C_2}{C_1} + 3 \cdot \dfrac{C_3}{C_2} + \cdots + n\dfrac{C_n}{C_{n-1}} = \dfrac{n(n+1)}{2}$

Solution:

(A) Let

$$s = a \cdot C_0 + (a+d) \cdot C_1 + (a+2d) \cdot C_2 + \cdots + (a+nd) \cdot C_n$$

Since $C_r = C_{n-r}$ we have

$$s = (a+nd) \cdot C_0 + [a+(n-1)d] \cdot C_1 + \cdots + a \cdot C_n$$

By adding both the above equations we get

$$2s = (2a+nd)[C_0 + C_1 + \cdots + C_n] = (2a+nd)2^n$$

Therefore

$$s = 2^{n-1}(2a+nd)$$

Therefore (A) is true.
Putting $a = 1 = d$, we get

$$s = 2^{n-1}(n+2)$$

Therefore (B) and (C) are not true.

(D) Let

$$u_r = r \cdot \frac{C_r}{C_{r-1}} = r \cdot \frac{n!}{(n-r)!r!} \cdot \frac{(n-r+1)!(r-1)!}{n!}$$

Therefore

$$u_r = r \cdot \frac{n-r+1}{r} = n-r+1$$

$$\sum_{r=1}^{n} r \cdot \frac{C_r}{C_{r-1}} = \sum_{r=1}^{n} u_r = \sum_{r=1}^{n}(n-r+1) = \frac{n(n+1)}{2}$$

Therefore (D) is true.

Answers: (A) and (D)

9. Let C_r be the binomial coefficient in the expansion of $(1+x)^n$. Then

(A) $\dfrac{C_0}{1} + \dfrac{C_2}{3} + \dfrac{C_4}{5} + \dfrac{C_6}{7} + \cdots = \dfrac{2^n}{n+1}$

(B) $\dfrac{C_1}{2} + \dfrac{C_3}{4} + \dfrac{C_5}{6} + \cdots = \dfrac{2^n - 1}{n+1}$

(C) $C_0C_1 + C_1C_2 + C_2C_3 + \cdots + C_{n-1}C_n = {}^{2n}C_{n+1}$

(D) If $s_n = C_0C_1 + C_1C_2 + C_2C_3 + \cdots + C_{n-1}C_n$ and if $s_{n+1}/s_n = 15/4$, then $n = 2$ or 4.

Solution:

(A) We have

$$\frac{C_{2r}}{2r+1} = \frac{n!}{(n-2r)!(2r)!(2r+1)} = {}^{(n+1)}C_{(2r+1)} \cdot \frac{1}{(n+1)}$$

Therefore

$$\sum \frac{C_{2r}}{2r+1} = \sum {}^{(n+1)}C_{(2r+1)} \cdot \frac{1}{(n+1)}$$

$$= \frac{1}{n+1}[{}^{(n+1)}C_1 + {}^{(n+1)}C_3 + {}^{(n+1)}C_5 + \cdots]$$

$$= \frac{1}{n+1}2^{(n+1)}$$

$$= \frac{2^n}{n+1} \quad \text{[see Theorem 7.2 part (3)]}$$

Therefore (A) is true.

(B) $\dfrac{C_{2r-1}}{2r} = \dfrac{1}{n+1} \cdot {}^{(n+1)}C_{2r}$

Therefore

$$\sum \frac{C_{2r-1}}{2r} = \frac{1}{n+1}[{}^{(n+1)}C_2 + {}^{(n+1)}C_4 + \cdots] = \frac{1}{n+1}(2^n - 1)$$

Therefore (B) is true.

(C) $(1+x)^{2n} = (1+x)^n(1+x)^n$

$$= (C_0 + C_1x + \cdots + C_nx^n)$$
$$\times (C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \cdots + C_n)$$

Equating the coefficients of x^{n+1} (or x^{n-1}) both sides, we get

$$C_0C_1 + C_1C_2 + C_2C_3 + \cdots + C_{n-1}C_n = {}^{2n}C_{n+1}$$

Therefore (C) is true.

(D) In (C) above, replace n with $n+1$. Then

$$\frac{15}{4} = \frac{s_{n+1}}{s_n} = \frac{(2n+2)(2n+1)}{n(n+2)}$$

Solving we get

$$n^2 - 6n + 8 = 0$$
$$(n-2)(n-4) = 0$$
$$n = 2 \quad \text{or} \quad 4$$

Hence (D) is true.

Answers: (A), (B), (C), (D)

10. For any positive integers m, n (with $m \le n$). Let

$$\binom{n}{m} = {}^nC_m = \frac{n!}{m!(n-m)!}$$

Then

(A) $\binom{n}{m} + \binom{n-1}{m} + \binom{n-2}{m} + \cdots + \binom{m}{m} = \binom{n+1}{m+1}$

(B) $\binom{n}{m} + 2\binom{n-1}{m} + 3\cdot\binom{n-2}{m} + \cdots + (n-m+1)\cdot\binom{m}{m} = \binom{n+2}{m+2}$

(C) $\binom{n}{m} + 2\binom{n-1}{m} + 3\cdot\binom{n-2}{m} + \cdots + (n-m+1)\cdot\binom{m}{m} = \binom{n+1}{m+2}$

(D) $\binom{10}{2} + \binom{9}{2} + \binom{8}{2} + \cdots + \binom{2}{2} = \binom{11}{3}$

Solution:

(A) We have

$$\binom{n}{m} + \binom{n-1}{m} + \binom{n-2}{m} + \cdots + \binom{m}{m}$$

= Coefficient of x^m in the expression

$$(1+x)^n + (1+x)^{n-1} + (1+x)^{n-2} + \cdots + (1+x)^m$$
$$= {}^nC_m + {}^{(n-1)}C_m + {}^{(n-2)}C_m + \cdots + {}^mC_m$$
$$= [{}^mC_m + {}^{(m+1)}C_m] + [{}^{(m+2)}C_m + {}^{(m+3)}C_m + \cdots + {}^nC_m]$$
$$= [{}^{(m+1)}C_{m+1} + {}^{(m+1)}C_m] + [{}^{(m+2)}C_m + {}^{(m+3)}C_m + \cdots + {}^nC_m]$$
$$= [{}^{(m+2)}C_{m+1} + {}^{(m+2)}C_m] + [{}^{(m+3)}C_m + {}^{(m+4)}C_m + \cdots + {}^nC_m]$$
$$= [{}^{(m+3)}C_{m+1} + {}^{(m+3)}C_m] + [{}^{(m+4)}C_m + \cdots + {}^nC_m]$$

Finally ${}^nC_{m+1} + {}^nC_m = {}^{(n+1)}C_{m+1}$.

Therefore (A) is true.

(B) Let

$$s = \binom{n}{m} + 2\binom{n-1}{m} + 3\cdot\binom{n-2}{m} + \cdots + (n-m+1)\cdot\binom{m}{m}$$

Therefore

$$s = \left[\binom{n}{m} + \binom{n-1}{m} + \binom{n-2}{m} + \cdots + \binom{m}{m}\right]$$
$$+ \left[\binom{n-1}{m} + \binom{n-2}{m} + \cdots + \binom{m}{m}\right]$$
$$+ \left[\binom{n-2}{m} + \binom{n-3}{m} + \cdots + \binom{m}{m}\right]$$

....

....

First row sum $= \binom{n+1}{m+1}$

Second row sum $= \binom{n}{m+1}$

Third row sum $= \binom{n-1}{m+1}$, etc.

Hence

$$s = \binom{n+1}{m+1} + \binom{n}{m+1} + \binom{n-1}{m+1} + \cdots + \binom{m+1}{m+1} = \binom{n+2}{m+2}$$

Therefore (B) is true.

(D) In (A), take $n = 10$ and $m = 2$. So, (D) is true.

Answers: (A), (B), (D)

11. Let $(1+x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n$ where C_r means ${}^nC_r$. Then

(A) $C_0^2 + C_1^2 + C_2^2 + \cdots + C_n^2 = \dfrac{(2n)!}{(n!)^2}$

(B) $a\cdot C_0^2 + (a+d)\cdot C_1^2 + (a+2d)\cdot C_2^2 + \cdots + (a+nd)\cdot C_n^2 = \dfrac{(2n-1)!}{n!(n-1)!}$

(C) Sum of the products of $C_0, C_1, C_2, \ldots, C_n$ taken two at a time is equal to $2^{2n-1} - (2n-1)!/[n!(n-1)!]$

(D) $C_2 + 2\cdot C_3 + 3\cdot C_4 + \cdots + (n-1)\cdot C_n = 1 + (n-2)2^{n-1}$

Solution:

(A) $(1+x)^{2n} = (1+x)^n(1+x)^n$

$$= (C_0 + C_1x + C_2x^2 + \cdots + C_nx^n)$$
$$\times (C_0x^n + C_1x^{n-1} + \cdots + C_n) \quad (\because C_r = C_{n-r})$$

Equating coefficient of x^n on both sides

$${}^{2n}C_n = C_0^2 + C_1^2 + C_2^2 + \cdots + C_n^2$$

Therefore (A) is true.

(B) Let

$$s = a\cdot C_0^2 + (a+d)\cdot C_1^2 + (a+2d)\cdot C_2^2 + \cdots + (a+nd)\cdot C_n^2$$

and $$s = (a+nd)\cdot C_0^2 + [a+(n-1)d]\cdot C_1^2 + [a+(n-2)d]\cdot C_2^2 + \cdots + a\cdot C_n^2$$

Therefore

$$2s = (2a+nd)(C_0^2 + C_1^2 + C_2^2 + \cdots + C_n^2)$$

$$= (2a+nd)\frac{(2n)!}{(n!)(n!)}$$

$$s = (2a+nd)\frac{(2n-1)!}{(n!)(n-1)!}$$

Hence (B) is true.

(C) We have

$$2\sum_{0\le i<j\le n} C_iC_j = (C_0 + C_1 + C_2 + \cdots + C_n)^2 - (C_0^2 + C_1^2 + C_2^2 + \cdots + C_n^2)$$

$$= 2^{2n} - \frac{(2n)!}{(n!)(n!)}$$

So (C) is true.

(D) We have

$$s = C_2 + 2\cdot C_3 + 3\cdot C_4 + \cdots + (n-1)\cdot C_n$$

$$= [(-1)\cdot C_0 + 0\cdot C_1 + 1\cdot C_2 + 2\cdot C_3 + \cdots + (n-1)\cdot C_n] + 1 \quad (\because C_0 = 1)$$

$$= [-1 + (n-1)]\cdot 2^{n-1} + 1 \quad \text{[see Q38 part (A)]}$$

$$= 1 + (n-2)\cdot 2^{n-1}$$

Therefore (D) is true.

Answers: (A), (B), (C), (D)

Matrix-Match Type Questions

1. Match the items of Column I with those of Column II.

Column I	*Column II*
(A) The term independent of x in the expansion of $(1-x)^2[x+(1/x)]^{10}$ is	(p) ${}^{17}C_8 + {}^{17}C_7$
(B) The coefficient of a^6 in the expansion of $(1+a)^6 + (1+a)^7 + \cdots + (1+a)^{15}$ is	(q) ${}^{16}C_7$
(C) The coefficient of x^8y^{10} in the expansion of $(x+y)^{18}$ is	(r) ${}^{11}C_5$
(D) $\frac{1}{1!10!} + \frac{1}{3!8!} + \frac{1}{5!6!} + \cdots + \frac{1}{11!}$ equals	(s) $\frac{2^{10}}{11!}$

Solution:

(A) We have

$$(1-x)^2\left(x+\frac{1}{x}\right)^{10} = (1-2x+x^2)\left[x^{10} + {}^{10}C_1x^9\left(\frac{1}{x}\right) + {}^{10}C_2x^8\left(\frac{1}{x}\right)^2 + \cdots + \frac{1}{x^{10}}\right]$$

Therefore the term independent of x is

$${}^{10}C_5 + (-2)(0) + {}^{10}C_6 = {}^{11}C_6 = {}^{11}C_5$$

Answer: (A) → (r)

(B) The coefficient of a^6 is given by

$$(1+a)^6 + (1+a)^7 + \cdots + (1+a)^{15} = (1+a)^6[1 + (1+a) + (1+a)^2 + \cdots + (1+a)^9]$$

$$= (1+a)^6\frac{[(1+a)^{10} - 1]}{(1+a) - 1}$$

$$= \frac{(1+a)^{16} - (1+a)^6}{a}$$

Therefore coefficient of $a^6 = {}^{16}C_7$.

Answer: (B) → (q)

(C) We have $T_{r+1} = {}^{18}C_r x^{18-r} y^r$. Therefore the coefficient of x^8y^{10} will be obtained when $r = 10$ and the coefficient is ${}^{18}C_{10} = {}^{18}C_8 = {}^{17}C_8 + {}^{17}C_7$.

Answer: (C) → (p)

(D) The given sum is

$$\frac{1}{11!}\left[\frac{11!}{1!10!} + \frac{11!}{3!8!} + \frac{11!}{5!6!} + \cdots + \frac{11!}{11!}\right]$$

$$= \frac{1}{11!}\left[{}^{11}C_1 + {}^{11}C_3 + {}^{11}C_5 + \cdots + {}^{11}C_{11}\right]$$

$$= \frac{2^{10}}{11!}$$

Answer: (D) → (s)

2. Match the items of Column I with the items of Column II.

Column I	*Column II*
(A) If the sum of the coefficients in the expansion of $(a^2x^2 - 2ax + 1)^{50}$ is zero, then the value of $a + 2a + 3a + \cdots + 10a$ is	(p) 1
(B) If $x = (3+2\sqrt{2})^6 = [x] + f$ where $[x]$ is the integral part of x and $f = x - [x]$, then $x(1-f)$ is	(q) 15
(C) If the sum of the coefficients of x^5 and x^{10} in the expansion of $[x^3 - (1/x^2)]^n$ is zero, then n equals	(r) 129
(D) Number of integral terms in the expansion of $(\sqrt{5} + \sqrt[8]{7})^{1024}$ is	(s) 55

Solution:

(A) Sum of the coefficients in the expansion of $(ax^2 - 2ax + 1)^{50} = (a-1)^{50} = 0 \Rightarrow a = 1$. Therefore

$$a + 2a + 3a + \cdots + 10a = 1 + 2 + 3 + \cdots + 10 = \frac{10 \cdot 11}{2} = 55$$

Answer: (A) → (s)

(B) Let $x = (3+2\sqrt{2})^6 = [x] + f$ and $g = (3-2\sqrt{2})^6$. Then

$$[x] + f + g = x + g = 2 \text{ (some integer)}$$

Now $0 < f, g < 1$ and $[x] + f + g$ is an integer. This implies $f + g = 1$. Therefore

$$x(1-f) = xg = (3+2\sqrt{2})^6(3-2\sqrt{6})^6 = 1$$

Answer: (B) → (p)

(C) We have

$$T_{r+1} = {}^nC_r(x^3)^{n-r} \cdot \left(\frac{-1}{x^2}\right)^r = (-1)^r \cdot {}^nC_r x^{3n-5r}$$

Now

$$3n - 5r = 5 \Rightarrow r = \frac{3n-5}{5} = p \quad \text{(say)} \tag{7.10}$$

$$3n - 5r = 10 \Rightarrow \frac{3n-10}{5} = q \quad \text{(say)} \tag{7.11}$$

Therefore from Eqs. (7.10) and (7.11),

$$p - q = 1 \tag{7.12}$$

Hence

$${}^nC_p(-1)^p + {}^nC_q(-1)^q = 0$$

$${}^nC_p(-1)^p + {}^nC_{p-1}(-1)^{p-1} = 0$$

$${}^nC_p = {}^nC_{p-1}$$

$$2p - 1 = n$$

$$2\left(\frac{3n-5}{5}\right) = n + 1$$

$$n = 15$$

Answer: (C) → (q)

(D) We have $T_{r+1} = (1024)C_r 5^{(1024-r)/2} \cdot 7^{r/8}$ is an integer. Therefore r is a multiple of 8. Since $0 \le r \le 1024$, the number of multiples of 8 which lie between 0 and 1024 (both inclusive) is 129.

Answer: (D) → (r)

3. Match the items of Column I with the items of Column II.

Column I	*Column II*
(A) Coefficient of x^n in the expansion of $(1+x)^{2n}$ is	(p) ${}^{(2n)}C_n$
(B) Coefficient of x^n in the expansion of $(x^2 + 2x)^n$ is	(q) 2^n
(C) Coefficient of x^n in the expansion of $n(x^2+2x)^{n-1}$ is	(r) ${}^nC_1 \times {}^{(n-1)}C_1 \times 2^{n-2}$
(D) Coefficient of x^n in the expansion of ${}^nC_2(x^2+2x)^{n-2}$ is	(s) ${}^nC_2 \times {}^{(n-2)}C_2 \times 2^{n-4}$

Solution:

(A) $(1+x)^{2n} = {}^{2n}C_0 + {}^{2n}C_1 x + {}^{2n}C_2 x^2 + \cdots + {}^{2n}C_{2n} x^{2n}$

Therefore coefficient of $x^n = {}^{2n}C_n$.

Answer: (A) → (p)

(B) $T_{r+1} = (r+1)$th term in the expansion of $(x^2+2x)^n$

$$= {}^nC_r(x^2)^{n-r} \cdot (2x)^r = {}^nC_r x^{2n-r} \cdot 2^r$$

$$2n - r = n \Rightarrow r = n$$

Therefore coefficient of $x^n = nC_n \cdot 2^n = 2^n$.

Answer: (B) → (q)

(C) $T_{r+1} = [(n-1)C_r(x^2)^{n-1-r} \cdot (2x)^r]n$

$$= (n-1)C_r x^{2n-2-r} \cdot 2^r \cdot n$$

$$2n - 2 - r = n \Rightarrow r = n - 2$$

Therefore coefficient of x^n in $n(x^2+2x)^{n-1}$ is

$${}^{(n-1)}C_{n-2} \cdot n \cdot 2^{n-2} = {}^nC_1 \times {}^{(n-1)}C_1 \cdot 2^{n-2}$$

Answer: (C) → (r)

(D) Similarly, the coefficient of x^n in ${}^nC_2(x^2+2x)^{n-2}$ is

$${}^nC_2 \times {}^{(n-2)}C_2 \times 2^{n-4}$$

Answer: (D) → (s)

4. Match the items of Column I to the items of Column II, if $(1+x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n$ where C_r is ${}^nC_r$.

Column I	*Column II*
(A) $\frac{C_0}{1} - \frac{C_1}{2} + \frac{C_2}{3} - \frac{C_3}{4} + \cdots + (-1)^n \cdot \frac{C_n}{n+1}$ is	(p) $\frac{2^n}{1\cdot 3\cdot 5\cdots(2n+1)}$
(B) $\frac{C_0}{2} - \frac{C_1}{3} + \frac{C_2}{4} - \frac{C_3}{5} + \cdots + (-1)^n \cdot \frac{C_n}{n+2}$ is	(q) $\frac{2^n}{1\cdot 3\cdot 5\cdots(2n-1)}$
(C) $\frac{1}{n!}\left(\frac{C_0}{1} - \frac{C_1}{3} + \frac{C_2}{5} - \frac{C_3}{7} + \cdots + (-1)^n \cdot \frac{C_n}{2n+1}\right)$ is	(r) $\frac{1}{n+1}$
(D) $\frac{-1}{n!}\left[-C_0 - \frac{C_1}{1} + \frac{C_2}{3} - \frac{C_3}{5} + \frac{C_4}{7} + \cdots + (-1)^n \cdot \frac{C_n}{2n-1}\right]$ is	(s) $\frac{1}{(n+1)(n+2)}$

Solution: It is known that (from Q7 in Multiple Correct Choice Type Questions)

$$\frac{C_0}{x} - \frac{C_1}{x+1} + \frac{C_2}{x+2} - \frac{C_3}{x+3} + \cdots + (-1)^n \cdot \frac{C_n}{x+n} = \frac{n!}{x(x+1)(x+2)\cdots(x+n)}$$

for all $x \neq 0, -1, -2, -3, \ldots$

Put $x = 1, 2, 1/2$ and $-1/2$ to get the result. This is a simple exercise left to the students.

Answer: (A) → (r), B → (s), (C) → (p), (D) → (q)

Comprehension-Type Questions

1. Passage: In the expansion of $(x+a)^n$, the general term is ${}^nC_r x^{n-r} \cdot a^r$ and the number of terms in the expansion is $n+1$. Answer the following three questions:

(i) If the fourth term in the expansion of $[px + (1/x)]^n$ is 5/2, then np is equal to

(A) 4 (B) 3 (C) 9/2 (D) 10

(ii) The number of terms in the expansion of $(x+y+z)^n$ is

(A) $\frac{n(n+1)}{2}$ (B) $(n+1)(n+2)$

(C) $\frac{(n+1)(n+2)}{2}$ (D) $(n+2)(n+3)$

(iii) The coefficient of x^5 in the expansion of $[3x^2 - (1/3x^3)]^{10}$ is

(A) –9520 (B) 9520 (C) 9720 (D) –9720

Solution:

(i) The fourth term in the expansion of $[px + (1/x)]^n$ is 5/2, that is

$${}^nC_3 (px)^{n-3} \cdot \left(\frac{1}{x^3}\right) = \frac{5}{2}$$

Therefore

$$\frac{n(n-1)(n-2)}{6} \cdot p^{n-3} \cdot x^{n-6} = \frac{5}{2}$$

Now $n = 6$ (since the term is independent of x), hence

$$\frac{6\cdot 5\cdot 4}{6} \cdot p^3 = \frac{5}{2}$$

$$p^3 = \left(\frac{1}{2}\right)^3 \Rightarrow p = \frac{1}{2}$$

Therefore $np = 3$.

Answer: (B)

(ii) $(x+y+z)^n = x^n + {}^nC_1 x^{n-1}(y+z) + {}^nC_2 x^{n-2}(y+z)^2 + \cdots + (y+z)^2$

It can be observed that second, third, fourth, …, $(n+1)$th terms contain 2, 3, 4, …, $(n+1)$ terms in their respective expansions. Therefore the number of terms in the given expansion is

$$1 + 2 + 3 + \cdots + (n+1) = \frac{(n+1)(n+2)}{2}$$

Answer: (C)

(iii) We have

$$T_{r+1} = {}^{10}C_r(3x^2)^{10-r}\left(-\frac{1}{3x^3}\right)^r$$

$$= {}^{10}C_r 3^{10-r}(-1)^r x^{20-5r}\cdot\frac{1}{3^r}$$

$$x^5 = x^{20-5r} \Rightarrow r = 3$$

Therefore the coefficient of x^5 is

$$-({}^{10}C_3\cdot 3^4) = -\left(\frac{8\cdot 9\cdot 10}{6}\right)\cdot 81 = -(120\times 81) = -9720$$

Answer: (D)

2. Passage: Let $(1 + x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n$ where $C_r = {}^nC_r$. For a given value of x, let $p = (n+1)|x|/(|x|+1)$. If p is an integer, then the numerical values of pth and $(p + 1)$th terms are equal and they are the numerically greatest terms in the expansion of $(1 + x)^n$. If p is not an integer and $[p]$ denotes the integral part of p, then $([p]+1)$th term is numerically greatest term. Answer the following questions:

(i) The value of numerically greatest term in the expansion of $\sqrt{2}[1+(1/\sqrt{2})]^{16}$ is

(A) $\frac{1}{16}\cdot{}^{16}C_7$ (B) $\frac{1}{16}\cdot{}^{16}C_8$

(C) $\frac{1}{18}\cdot{}^{16}C_8$ (D) $\frac{1}{8}\cdot{}^{16}C_7$

(ii) Numerically greatest term in the expansion of $(3-5x)^{15}$ when $x = 1/5$ is (are)

(A) fourth and fifth term (B) sixth term

(C) seventh term (D) eighth term

(iii) The greatest value of ${}^{20}C_r (0 \le r \le 20)$ is

(A) ${}^{20}C_8$ (B) ${}^{20}C_9$ (C) ${}^{20}C_{10}$ (D) ${}^{20}C_7$

Solution:

(i) We have

$$p = \frac{(16+1)(1/\sqrt{2})}{(1/\sqrt{2})+1} = 17(\sqrt{2}-1)$$

Therefore $[p] = 7$. The eighth term is numerically greatest and its value is

$$\sqrt{2}\cdot{}^{16}C_7\cdot\left(\frac{1}{\sqrt{2}}\right)^7 = \frac{1}{8}\cdot{}^{16}C_7$$

Answer: (D)

(ii) We have

$$(3-5x)^{15} = 3^{15}\left(1-\frac{5x}{3}\right)^{15} = 3^{15}\left(1-\frac{1}{3}\right)^{15} \quad \left(\because x = \frac{1}{5}\right)$$

So

$$p = \frac{(15+1)(1/3)}{1+(1/3)} = \frac{16}{4} = 4$$

is an integer. Therefore numerically greatest terms are fourth and fifth terms.

Answer: (A)

(iii) Consider $(1+1)^{20}$. Then

$$p = \frac{(20+1)1}{1+1}$$

Therefore $[p] = 10$ and hence 11th term is greatest and its value is ${}^{20}C_{10}$.

Answer: (C)

3. Passage: If n is a positive integer, x and a are real (complex), then $(r + 1)$th term in the expansion of $(x + a)^n$ is ${}^nC_r x^{n-r}\cdot a^r$. Answer the following questions:

(i) If the coefficient of x^7 in $[ax^2+(1/bx)]^{11}$ is equal to the coefficient of x^{-7} in $[ax-(1/bx^2)]^{11}$, then

(A) $ab = 1$ (B) $ab = -1$ (C) $ab = 2$ (D) $ab = -2$

(ii) If the coefficients $(2r+4)$th and $(r-2)$th terms in the expansion of $(1+x)^{18}$ are equal, thus r is equal to

(A) 5 (B) 4 (C) 6 (D) 7

(iii) Coefficient of x^{50} in the expansion of $(1+x)^{41}(1-x+x^2)^{40}$ is

(A) –1 (B) 1 (C) 40 (D) 0

Solution:

(i) We have that the $(r+1)$th term is

$${}^{11}C_r(ax^2)^{11-r}\left(\frac{1}{bx}\right)^r = {}^{11}C_r a^{11-r}\cdot\frac{1}{b^r}\cdot x^{22-3r}$$

$$22 - 3r = 7$$

$$r = 5$$

Therefore the coefficient of x^7 in the first expansion is ${}^{11}C_5(a^6/b^5)$.

Similarly the coefficient of x^{-7} in the second expansion is ${}^{11}C_6(a^5/b^6)$. Therefore

$${}^{11}C_5\left(\frac{a^6}{b^5}\right) = {}^{11}C_6\left(\frac{a^5}{b^6}\right) \Rightarrow ab = 1$$

Answer: (A)

(ii) By hypothesis,

$${}^{18}C_{2r+3} = {}^{18}C_{r-3}$$

$$2r + 3 \ne r - 3$$

$$(2r+3)+(r-3) = 18$$

$$r = 6$$

Answer: (C)

(iii) We have

$$(1+x)^{41}(1-x+x^2)^{40} = (1+x)(1-x^3)^{40}$$

The power of x in this expansion is of the form either $3r$ or $3r+1$ but 50 is of the form $3k+2$. Hence, coefficient of x^{50} is zero.

Answer: (D)

4. Passage: If n is a positive integer, then

$$(1+x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n$$
$$= C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \cdots + C_n$$

where C_r means ${}^nC_r$. Answer the following questions:

(i) $C_0 \cdot C_2 + C_1 \cdot C_3 + C_2 \cdot C_4 + \cdots + C_{n-2} \cdot C_n =$

(A) ${}^{2n}C_n$ (B) ${}^{2n}C_{n-2}$ (C) ${}^{2n}C_{n+1}$ (D) 2^{2n}

(ii) $C_0 - 2 \cdot C_1 + 3 \cdot C_2 - 4 \cdot C_3 + \cdots + (-1)^n \cdot (n+1) \cdot C_n =$

(A) 0 (B) $2^{2n}-1$

(C) $(2n)C_{n-1}$ (D) $2^{n-1}(n+1)$

(iii) $(C_1)^2 + 2 \cdot (C_2)^2 + 3 \cdot (C_3)^2 + \cdots + n \cdot (C_n)^2 =$

(A) $(2n)!/n!n!$

(B) $(2n)!/n!(n-1)!$

(C) coefficient of x^{n-1} in $n(1+x)^{2n-1}$

(D) $(n+1) \cdot 2^n$

Solution:

(i) $(1+x)^{2n} = (1+x)^n(x+1)^n$

$$= (C_0 + C_1x + C_2x^2 + \cdots + C_nx^n)$$
$$(C_0x^n + C_1x^{n-1} + \cdots + C_n)$$

Equating the coefficients x^{n-2}(or x^{n+2}) on both sides we get

$${}^{2n}C_{n-2} = C_0 \cdot C_2 + C_1 \cdot C_3 + C_2 \cdot C_4 + \cdots + C_{n-2} \cdot C_n$$

Answer: (B)

(ii) Let

$$T_{r+1} = (-1)^r \cdot (r+1) \cdot C_r$$
$$= (-1)^r \cdot r \cdot C_r + (-1)^r \cdot C_r$$
$$= (-1)^r \cdot n \cdot (n-1)C_{r-1} + (-1)^r \cdot C_r \text{ for } r = 1, 2, 3, \ldots, n$$

Therefore the given sum is

$$C_0 + n\sum_{r=1}^{n}(-1)^r \cdot (n-1)C_{r-1} + \sum_{r=1}^{n}(-1)^r \cdot C_r$$
$$= C_0 - (1-1)^{n-1} + [(1-1)^n - C_0] = 0$$

Answer: (A)

Aliter: $(1+x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n$

Therefore

$$x(1+x)^n = C_0x + C_1x^2 + C_2x^3 + \cdots + C_nx^{n+1}$$

Differentiating both sides with respect to x and then substituting $x = -1$ on both sides we get

$$0 = C_0 - 2 \cdot C_1 + 3 \cdot C_2 - 4 \cdot C_3 + \cdots + (-1)^n \cdot (n+1) \cdot C_n$$

(iii) Let $s = 0 \cdot C_0^2 + 1 \cdot C_1^2 + 2 \cdot C_2^2 + \cdots + n \cdot C_n^2$

Therefore

$$s = n \cdot C_0^2 + (n-1) \cdot C_1^2 + (n-2) \cdot C_2^2 + \cdots + 0 \cdot C_n^2$$
$$(\because C_r = C_{n-r})$$

This gives

$$2s = n[C_0^2 + C_1^2 + C_2^2 + \cdots + C_n^2] = n({}^{2n}C_n)$$

Hence

$$s = \frac{n}{2} \cdot {}^{2n}C_n = \frac{n}{2}\left[\frac{(2n)!}{n!n!}\right] = n\left[\frac{(2n-1)!}{(n-1)!n!}\right]$$
$$= \text{ Coefficient of } x^{n-1} \text{ in } n(1+x)^{2n-1}$$

Answer: (C)

Assertion–Reasoning Type Questions

In the following set of questions, a Statement I is given and a corresponding Statement II is given just below it. Mark the correct answer as:

(A) Both I and II are true and II is a correct reason for I

(B) Both I and II are true and II is not a correct reason for I

(C) I is true, but II is false

(D) I is false, but II is true

1. Statement I: If C_r is the binomial coefficient in the expansion of $(1+x)^n$, then

$$C_r + 2 \cdot C_{r-1} + C_{r-2} = {}^{(n+2)}C_r \quad \text{for } 2 \le r \le n$$

Statement II: ${}^nC_{r-1} + {}^nC_r = {}^{(n+1)}C_r$

Solution: We have

$${}^nC_{r-1} + {}^nC_r = \frac{n!}{(n-r+1)!(r-1)!} + \frac{n!}{(n-r)!(r)!}$$
$$= \frac{n!(r+n-r+1)}{r!(n-r+1)!}$$
$$= \frac{(n+1)!}{r!(n+1-r)!}$$
$$= {}^{(n+1)}C_r$$

Therefore Statement II is true.

$$\begin{aligned}C_r + 2\cdot C_{r-1} + C_{r-2} &= (C_r + C_{r-1}) + (C_{r-1} + C_{r-2})\\&= ({}^nC_r + {}^nC_{r-1}) + ({}^nC_{r-1} + {}^nC_{r-2})\\&= {}^{(n+1)}C_r + {}^{(n+1)}C_{r-1}\\&= {}^{(n+2)}C_r\end{aligned}$$

Hence Statement I is also true and Statement II is a correct explanation of Statement I.

Answer: (A)

2. Statement I: If

$$\sum_{r=0}^{2n} a_r (x-2)^r = \sum_{r=0}^{2n} b_r (x-3)^r$$

and $a_k = 1$ for all $k \ge n$, then $b_n = {}^{(2n+1)}C_{n+1}$.

Statement II: Two polynomials of same degree are equal, if the corresponding coefficients are equal.

Solution: Statement II is true according to the definition of equality of polynomials. Put $x - 3 = y$. Therefore

$$\sum_{r=0}^{2n} a_r (y+1)^r = \sum_{r=0}^{2n} b_r y^r$$

Equating coefficient of y^n on both sides, we get

$$a_n + {}^{(n+1)}C_1\cdot a_{n+1} + {}^{(n+2)}C_2\cdot a_{n+2} + \cdots + {}^{(2n)}C_n\cdot a_{2n} = b_n$$

$${}^{(n+1)}C_0 + {}^{(n+1)}C_1 + {}^{(n+2)}C_2 + \cdots + {}^{(2n)}C_n = b_n \quad (\because a_k = 1 \text{ for } k \ge n)$$

Using ${}^nC_r + {}^nC_{r-1} = {}^{(n+1)}C_r$, we have ${}^{(2n+1)}C_n = b_n$

Hence Statement I is also true and Statement II is a correct explanation of Statement I.

Answer: (A)

3. Statement I: If $(1+x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n$, then

$$\begin{aligned}&C_0 - 2^2\cdot C_1 + 3^2\cdot C_2 - 4^2\cdot C_3 + \cdots\\&+ (-1)^n\cdot(n+1)^2\cdot C_n = 0 \quad \text{for } n \ge 2\end{aligned}$$

Statement II: Any polynomial function in x is differentiable for all real values of x.

Solution: Statement II is true is clear.

$$(1+x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n$$

Therefore

$$x(1+x)^n = C_0x + C_1x^2 + C_2x^3 + \cdots + C_nx^{n+1}$$

Differentiating both w.r.t. x we get

$$\begin{aligned}(1+x)^n + nx(1+x)^{n-1} &= C_0 + 2\cdot C_1x + 3\cdot C_2x^2 + \cdots\\&+ (n+1)\cdot C_n\end{aligned}$$

Again multiplying both sides with x and differentiating w.r.t. x we get

$$\begin{aligned}&(1+x)^n + 3nx(1+x)^{n-1} + n(n-1)x^2(1+x)^{n-2}\\&= C_0 + 2^2\cdot C_1x + 3^2\cdot C_2x^2 + \cdots + (n+1)^2\cdot C_nx^n\end{aligned}$$

Substituting $x = -1$ on both sides, we have

$$0 = C_0 - 2^2\cdot C_1 + 3^2\cdot C_2 - 4^2\cdot C_3 + \cdots + (-1)^n\cdot(n+1)^2\cdot C_n$$

Hence Statement I is true and Statement II is true. Also Statement II is a correct explanation of Statement I.

Answer: (A)

4. Statement I: If $(1+x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n$, then

$$1^2\cdot C_1 + 2^2\cdot C_2 + 3^2\cdot C_3 + \cdots + n^2\cdot C_n = n(n+1)2^{n-2} \quad \text{for } n \ge 1$$

Statement II: Any polynomial function in x is differentiable for all real values of x.

Solution: Clearly Statement II is true:

$$(1+x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n$$

Differentiating both sides w.r.t. x we get

$$n(1+x)^{n-1} = C_1 + 2\cdot C_2x + 3\cdot C_3x^2 + \cdots + n\cdot C_nx^{n-1}$$

Now, multiplying both sides with x, differentiating both sides w.r.t. x, and then substituting $x = 1$ on both sides we get

$$1^2\cdot C_1 + 2^2\cdot C_2 + 3^2\cdot C_3 + \cdots + n^2\cdot C_n = n(n+1)2^{n-2}$$

Hence both statements are true and Statement II is a correct explanation of Statement I.

Answer: (A)

5. Statement I: If n is an even positive integer and $K = 3n/2$ then

$$\sum_{r=1}^{K} (-3)^{r-1}\cdot{}^{(3n)}C_{2r-1} = 0$$

Statement II: If m is a positive integer, and θ is real then $(\cos\theta + i\sin\theta)^m = \cos m\theta + i\sin m\theta$.

Solution: We have

$$1 + i\sqrt{3} = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) \quad (\text{where } i = \sqrt{-1})$$

By De Moivre's Theorem we have

$$(1+i\sqrt{3})^n = 2^n\left[\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3}\right] \tag{7.13}$$

Let $n = 2m$ where m is a positive integer. Replacing n with $6m$ in Eq. (7.13), we have

$$2^{6m}[\cos(2m\pi) + i\sin(2m\pi)] = (1 + i\sqrt{3})^{6m}$$

Therefore

$$2^{6m} = (1+i\sqrt{3})^{6m} = 1 + {}^{6m}C_1(i\sqrt{3}) + {}^{6m}C_2(i\sqrt{3})^2 + {}^{6m}C_3(i\sqrt{3})^3 + \cdots + (i\sqrt{3})^{6m}$$

Equating imaginary parts we get

$$\sqrt{3}[{}^{6m}C_1 - {}^{6m}C_3(3) + {}^{6m}C_5 3^2 - {}^{6m}C_7 3^3 + \cdots (-1)^{3m-1}\,{}^{6m}C_{6m-1}3^{3m-1}] = i$$

$$\sum_{r=1}^{K}(-3)^{r-1}\,{}^{(3n)}C_{2r-1} = 0 \quad \text{where } K = \frac{3n}{2}$$

Hence both statements are true and Statement II is a correct explanation of Statement I.

Answer: (A)

6. Statement I: If

$$\sum_{K=1}^{n} K^3\left(\frac{C_K}{C_{K-1}}\right)^2 = 196 \text{ where } C_r = {}^nC_r$$

then sum of the coefficients in the expansion of $(x - 3x^2 + x^3)^n$ is 1.

Statement II: $\dfrac{C_r}{C_{r-1}} = \dfrac{n-r+1}{r+1}$

Solution: We have

$$\frac{C_r}{C_{r-1}} = \frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n!}{r!(n-r)!} \times \frac{(r-1)!(n-r+1)!}{n!} = \frac{n-r+1}{r}$$

Therefore Statement II is not true. Now

$$\sum_{K=1}^{n} K^3\left(\frac{C_K}{C_{K-1}}\right)^2 = \sum_{K=1}^{n} K(n-K+1)^2$$

$$= \sum_{K=1}^{n} K[(n+1)^2 - 2K(n+1) + K^2]$$

$$= (n+1)^2\sum_{K=1}^{n} K - 2(n+1)\sum_{K=1}^{n} K^2 + \sum_{K=1}^{n} K^3$$

$$= \frac{(n+1)^3 n}{2} - \frac{2(n+1)n(n+1)(2n+1)}{6} + \frac{n^2(n+1)^2}{4}$$

$$= \frac{n(n+1)^2}{12}[6(n+1) - 4(2n+1) + 3n]$$

$$= \frac{n(n+1)^2(n+2)}{12}$$

$$= 14^2 = 2^2\cdot 7^2 = \frac{6\cdot 7^2\cdot 8}{12}$$

This gives $n = 6$. Therefore sum of the coefficients in the expansion of $(x - 3x^2 + x^3)^6 = (1 - 3 + 1)^6 = 1$. Hence Statement I is true and Statement II is not true.

Answer: (C)

7. Statement I: If ${}^{(2n+1)}C_0 + {}^{(2n+1)}C_3 + {}^{(2n+1)}C_6 + \cdots = 170$, then $n = 4$.

Statement II: If w in non-real cube root of unity, $w^3 = 1$ and $1 + w + w^2 = 0$.

Solution: We have

$$(1+x)^{2n+1} = {}^{(2n+1)}C_0 + {}^{(2n+1)}C_1 x + {}^{(2n+1)}C_2 x^2 + \cdots + {}^{(2n+1)}C_{2n+1}x^{2n+1}$$

Put $x = 1$, w and w^2 and add. We now have

$$2^{2n+1} + (1+w)^{2n+1} + (1+w^2)^{2n+1} = 3[{}^{(2n+1)}C_0 + {}^{(2n+1)}C_3 + {}^{(2n+1)}C_6 + \cdots] \qquad (7.14)$$

LHS of Eq. (7.14)

$$= 2^{2n+1} - w^{4n+2} - w^{2n+2} = 2^{2n+1} - w^{4n+2} - w^{2n+1}$$

From Eq. (7.14), we have

$$2^{2n+1} - w^{n+2} - w^{2n+1} = 3 \times 170 = 510$$

Therefore

$$2^{2n+1} - w^3(w^{n-1} + w^{2(n-1)}) = 510$$

$$2^{2n+1} - (w^{n-1} + w^{2(n-1)}) = 510 \qquad (7.15)$$

It is known that $1 + w^n + w^{2n} = 3$ or 0 according as n is a multiple of 3 or not. Therefore $w^{n-1} + w^{2(n-1)} = 2$ or -1 according as $n - 1$ is a multiple of 3 or not.

If $w^{n-1} + w^{2(n-1)} = -1$, then from Eq. (7.15)

$$2^{2n+1} = 509$$

which is not possible. Therefore

$$w^{n-1} + w^{2(n-1)} = 2$$

From Eq. (7.15),

$$2^{2n+1} = 512 = 2^9$$

which implies $n = 4$.

Answer: (A)

Integer Answer Type Questions

1. Let C_r denote ${}^nC_r$. If

$$\frac{C_r + 4\cdot C_{r+1} + 6\cdot C_{r+2} + 4\cdot C_{r+3} + C_{r+4}}{C_r + 3\cdot C_{r+1} + 3\cdot C_{r+2} + C_{r+3}} = \frac{n+k}{r+k}$$

then the value of K is ______.

Solution: The numerator is

$$(C_r + C_{r+1}) + 3(C_{r+1} + C_{r+2}) + 3(C_{r+2} + C_{r+3}) + (C_{r+3} + C_{r+4})$$
$$= {}^{(n+1)}C_{r+1} + 3\cdot{}^{(n+1)}C_{r+2} + 3\cdot{}^{(n+1)}C_{r+3} + {}^{(n+1)}C_{r+4}$$
$$= [{}^{(n+1)}C_{r+1} + {}^{(n+1)}C_{r+2}] + 2\cdot[{}^{(n+1)}C_{r+2} + {}^{(n+1)}C_{r+3}]$$
$$+ [{}^{(n+1)}C_{r+3} + {}^{(n+1)}C_{r+4}]$$
$$= {}^{(n+2)}C_{r+2} + 2\cdot{}^{(n+2)}C_{r+3} + {}^{(n+2)}C_{r+4}$$
$$= [{}^{(n+2)}C_{r+2} + {}^{(n+2)}C_{r+3}] + [{}^{(n+2)}C_{r+3} + {}^{(n+2)}C_{r+4}]$$
$$= {}^{(n+3)}C_{r+3} + {}^{(n+3)}C_{r+4}$$
$$= {}^{(n+4)}C_{r+4}$$

Similarly, the denominator $= {}^{(n+3)}C_{r+3}$. Therefore

$$\frac{{}^{(n+4)}C_{r+4}}{{}^{(n+3)}C_{r+3}} = \frac{n+K}{r+K} \Rightarrow \frac{n+4}{r+4} = \frac{n+K}{r+K} \Rightarrow K = 4$$

Answer: 4

2. If $(1 + 2x + 3x^2)^{10} = a_0 + a_1x + a_2x^2 + \cdots + a_{20}x_{20}$, then $a_1 + a_2$ is equal to ______.

Solution: $(1 + 2x + 3x^2)^{10} = [1 + x(2+3x)]^{10}$

$$= 1 + {}^{10}C_1 x(2+3x) + {}^{10}C_2 x^2(2+3x)^2 + \cdots$$

Therefore

$$a_1 = {}^{10}C_1 \times 2 = 20$$

and $\quad a_2 = {}^{10}C_2 \times 2^2 + 3 \times {}^{10}C_1 = 180 + 30 = 210$

Adding the two we get

$$a_1 + a_2 = 230$$

Answer: 230

3. The number of distinct terms in the expansion of

$$\left(x^3 + \frac{1}{x^3} + 1\right)^{20}$$

when x is real and $x \neq \pm 1$ is ______.

Solution: We have

$$\left(x^3 + \frac{1}{x^3} + 1\right)^{20} = \left[1 + \left(x^3 + \frac{1}{x^3}\right)\right]^{20} = 1 + {}^{20}C_1\left(x^3 + \frac{1}{x^3}\right)$$
$$+ {}^{20}C_2\left(x^3 + \frac{1}{x^3}\right)^2 + \cdots + {}^{20}C_{20}\left(x^3 + \frac{1}{x^3}\right)^{20}$$

Therefore

$$1, x^3, (x^3)^2, (x^3)^3, \ldots, (x^3)^{20}, \frac{1}{x^3}, \left(\frac{1}{x^3}\right)^2, \left(\frac{1}{x^3}\right)^3, \ldots, \left(\frac{1}{x^3}\right)^{20}$$

are all distinct whose number is $1 + 20 + 20 = 41$.

Answer: 41

4. The greatest value of the term independent of x in the expansion of $[x\sin\alpha + (\cos\alpha/x)]^{20}$ as α is real is ${}^{20}C_{10}2^{-\beta}$. Then β value is ______.

Solution: We have

$$T_{r+1} = {}^{20}C_r(x\sin\alpha)^{20-r}\left(\frac{\cos\alpha}{x}\right)^r$$
$$= {}^{20}C_r(\sin\alpha)^{20-r}(\cos\alpha)^r x^{20-2r}$$

This is independent of x. Since $20 - 2r = 0$ so $r = 10$. Therefore

$$T_{11} = {}^{20}C_{10}(\sin\alpha\cos\alpha)^{10} = {}^{20}C_{10}\frac{1}{2^{10}}(\sin 2\alpha)^{10} \leq {}^{20}C_{10}2^{-10}$$

and equality holds when $2\alpha = (4n \pm 1)(\pi/2)$. Therefore $\beta = 10$.

Answer: 10

5. The number of rational terms in the expansion of $(5^{2/3} + 10^{-1/4})^{20}$ is ______.

Solution: We have

$$T_{r+1} = {}^{20}C_r\cdot(5^{2/3})^{20-r}(10^{-1/4})^r = {}^{20}C_r 5^{(160-11r)/12}\cdot 2^{-r/4}$$

which is rational when $r = 8$ and 20.

Answer: 2

6. The number of non-zero terms in the expansion of $(\sqrt{11}+1)^{75} - (\sqrt{11}-1)^{75}$ is ______.

Solution: We have

$$(\sqrt{11}+1)^{75} - (\sqrt{11}-1)^{75} = 2[C_1(\sqrt{11})^{74} + C_3(\sqrt{11})^{72}$$
$$+ C_5(\sqrt{11})^{70} + \cdots]$$

Therefore number of non-zero terms is

$$\frac{75+1}{2} = 38$$

Answer: 38

7. If $\dfrac{1}{1!10!} + \dfrac{1}{3!8!} + \dfrac{1}{5!6!} + \cdots + \dfrac{1}{11!1!} = \dfrac{2^K}{11!}$ then K value is ______.

Solution: We have

$$11!\left[\frac{1}{1!10!} + \frac{1}{3!8!} + \frac{1}{5!6!} + \cdots + \frac{1}{11!1!}\right]$$

$$= \frac{11}{1!10!} + \frac{11}{3!8!} + \frac{11}{5!6!} + \cdots + \frac{11}{11!}$$

$$= {}^{11}C_1 + {}^{11}C_3 + {}^{11}C_5 + {}^{11}C_7 + {}^{11}C_9 + {}^{11}C_{11}$$

= sum of the even coefficient in the expansion of $(1+x)^{11}$

$$= 2^{10}$$

Therefore $K = 10$.

Answer: 10

8. ${}^{11}C_2 + 2[{}^{10}C_2 + {}^{9}C_2 + {}^{8}C_2 + \cdots + {}^{2}C_2]$ is equal to ______.

Solution: In general we prove that for any positive integer $n \geq 2$,

$$^{(n+1)}C_2 + 2[{}^{n}C_2 + {}^{(n-1)}C_2 + {}^{(n-2)}C_2 + \cdots + {}^{2}C_2] = 1^2 + 2^2 + 3^2 + \cdots + n^2$$

We know that

$$^{K}C_2 = \frac{K(K-1)}{2}$$

Therefore

$$2 \cdot {}^{K}C_2 = K^2 - K$$

Put $K = n, n-1, n-2, \ldots, 2$ and add. We get

$$2[{}^{n}C_2 + {}^{(n-1)}C_2 + {}^{(n-2)}C_2 + \cdots + {}^{2}C_2]$$

$$= (1^2 + 2^2 + 3^2 + \cdots + n^2) - (1 + 2 + 3 + \cdots + n)$$

$$= (1^2 + 2^2 + 3^2 + \cdots + n^2) - \frac{n(n+1)}{2}$$

$$= (1^2 + 2^2 + 3^2 + \cdots + n^2) - {}^{(n+1)}C_2$$

Therefore

$$^{(n+1)}C_2 + 2[{}^{n}C_2 + {}^{(n-1)}C_2 + {}^{(n-2)}C_2 + \cdots + {}^{2}C_2]$$

$$= 1^2 + 2^2 + 3^2 + \cdots + n^2$$

Now substituting $n = 10$, we have

$$^{11}C_2 + 2[{}^{10}C_2 + {}^{9}C_2 + {}^{8}C_2 + \cdots + {}^{2}C_2] = 1^2 + 2^2 + 3^2 + \cdots + 10^2$$

$$= \frac{10 \cdot 11 \cdot 21}{6} = 385$$

Answer: 385

9. If $\dfrac{{}^{15}C_1}{2} + \dfrac{{}^{15}C_3}{4} + \dfrac{{}^{15}C_5}{6} + \cdots + \dfrac{{}^{15}C_{15}}{16} = \dfrac{2^K - 1}{16}$ then K is equal to ______.

Solution: Since

$$\frac{C_r}{r+1} \cdot x^r = \frac{{}^{(n+1)}C_{r+1}}{(n+1)x} \cdot x^{r+1}$$

for $r = 0, 1, 2, \ldots, n$, we have

$$C_0 + \frac{C_1}{2}x + \frac{C_2}{3}x^2 + \cdots + \frac{C_n}{n+1}x^n = \frac{(1+x)^{n+1} - 1}{x(n+1)}$$

Substituting $x = 1, -1$ we have

$$C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \cdots + \frac{C_n}{n+1} = \frac{2^{n+1} - 1}{n+1}$$

$$C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \frac{C_3}{4} + \cdots + \frac{(-1)^n \cdot C_n}{n+1} = \frac{1}{n+1}$$

On subtraction we get

$$2\left[\frac{C_1}{2} + \frac{C_3}{4} + \frac{C_5}{6} + \cdots\right] = \frac{2^{n+1} - 2}{n+1}$$

Therefore

$$\frac{C_1}{2} + \frac{C_3}{4} + \frac{C_5}{6} + \cdots = \frac{2^n - 1}{n+1}$$

Putting $n = 15$, we get

$$\frac{C_1}{2} + \frac{C_3}{4} + \frac{C_5}{6} + \cdots + \frac{C_{15}}{16} = \frac{2^{15} - 1}{16}$$

Hence $K = 15$.

Answer: 15

SUMMARY

7.1 ***Binomial theorem:*** If n is a positive integer and a is any real or complex number, then

$$(x+a)^n = {}^nC_0x^n + {}^nC_1x^{n-1}a + {}^nC_2x^{n-2}a^2 + \cdots + {}^nC_na^n$$

7.2 ***General term:*** The $(r+1)$th term ${}^nC_rx^{n-r}a^r$ is called the general term in the expansion of $(x+a)^n$.

7.3 ***Number of terms:***

(1) The number of terms in the expansion of $(x+a)^n$ is $n+1$.

(2) The number of terms in the expansion of $(x+y+z)^n$ is $(n+1)(n+2)/2$ which is ${}^{(n+2)}C_2$.

(3) The number of terms in the expansion of $(x_1 + x_2 + \cdots + x_K)^n$ is ${}^{(n+K-1)}C_{(K-1)}$.

7.4 ***Middle term(s):***

(1) If n is even, then the $[(n/2)+1]$th term is the middle term in the expansion of $(x+a)^n$.

(2) If n is odd, then $[(n+1)/2]$th and $[(n+3)/2]$th are the middle terms.

7.5 ***Binomial coefficients:*** If n is a positive integer, then $(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \cdots + {}^nC_nx^n$. The coefficients $x^r (r = 0, 1, 2, \ldots n)$ viz.

${}^nC_0, {}^nC_1, {}^nC_2, \ldots, {}^nC_n$ are called binomial coefficients and they will be denoted by $C_0, C_1, C_2, \ldots, C_n$.

7.6 ***Properties of binomial coefficients:*** Let C_r be the binomial coefficient in the expansion of $(1+x)^n$ for $r = 0, 1, 2, \ldots, n$. Then

(1) $C_r = C_{n-r}$

(2) $C_0 + C_1 + C_2 + \cdots + C_n = 2^n$.

(3) $C_0 + C_2 + C_4 + \cdots = C_1 + C_3 + C_5 + \cdots = 2^{n-1}$

(4) $1\cdot C_0 + 2\cdot C_1 + 3\cdot C_2 + \cdots + (n+1)\cdot C_n = (n+2)2^{n-1}$

7.7 ***Useful formulae:*** Let $C_0, C_1, C_2, \ldots, C_n$ be binomial coefficients in the expansion of $(1+x)^n$. Then

(1) $1\cdot C_1 + 2\cdot C_2 + 3\cdot C_3 + \cdots + n\cdot C_n = n\cdot 2^{n-1}$

(2) $a\cdot C_0 + (a+d)\cdot C_1 + (a+2d)\cdot C_2 + \cdots + (a+(n-1)d)\cdot C_n = (2a+(n-1)d)\,2^{n-1}$

(3) $\sum_{r=1}^{n} r(r-1)\cdot C_r = n(n-1)2^{n-2}$

(4) $\sum_{r=1}^{n} r^2\cdot C_r = n(n+1)2^{n-2}$

(5) $C_0^2 + C_1^2 + C_2^2 + \cdots + C_n^2 = {}^{2n}C_n = \dfrac{(2n)!}{(n!)^2}$

(6) $C_0C_r + C_1C_{r+1} + C_2C_{r+2} + \cdots + C_{n-r}C_n = \dfrac{(2n)!}{(n-r)!\cdot(n+r)!}$

7.8 ***Most useful result:*** If C_r is the binomial coefficient in the expansion of $(1+x)^n$, and $x \neq 0, -1, -2, -3, \ldots$, then

$$\frac{C_0}{x} - \frac{C_1}{x+1} + \frac{C_2}{x+2} - \frac{C_3}{x+3} + \cdots + (-1)^n\frac{C_n}{x+n}$$
$$= \frac{n!}{x(x+1)(x+2)\cdots(x+3)}$$

For example,

(1) When $x = 1$, then

$$\frac{C_0}{1} - \frac{C_1}{2} + \frac{C_2}{3} - \cdots = \frac{1}{n+1}$$

(2) When $x = 2$, then

$$\frac{C_0}{2} - \frac{C_1}{3} + \frac{C_2}{4} - \cdots = \frac{1}{(n+1)(n+2)}$$

7.9 ***Greatest term:*** Consider the expansion of $(1+x)^n$.

Let

$$p = \frac{(n+1)|x|}{1+|x|}$$

Then

(1) pth and $(p+1)$th terms are numerically equal and they are the numerically greatest terms in the expansion of $(1+x)^n$, if p is an integer.

(2) If p is not an integer and $[p]$ denotes the integral part of p, then $([p]+1)$th term is numerically greatest term.

7.10 ***Greatest value of*** $^n\mathbf{C}_r$**:**

Greatest value of

$${}^nC_r = \begin{cases} {}^nC_{n/2} & \text{if } n \text{ is even} \\ {}^nC_{(n-1)/2} = {}^nC_{(n+1)/2} & \text{if } n \text{ is odd} \end{cases}$$

Binomial Theorem for Rational Index

7.11 ***Theorem:*** If n is a rational number and $-1 < x < 1$, then

$$(1+x)^n = 1 + \frac{n}{1!}x + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + \infty$$

the general term is

$$\frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}x^r$$

7.12 ***Useful expansions:***

(1) $(1-x)^n = 1-\dfrac{n}{1!}x+\dfrac{n(n-1)}{2!}x^2 - \dfrac{n(n-1)(n-2)}{3!}x^3\cdots\infty$

(2) $(1+x)^{-n} = 1-\dfrac{n}{1!}x+\dfrac{n(n+1)}{2!}x^2 - \dfrac{n(n+1)(n+2)}{3!}x^3\cdots\infty$

(3) $(1-x)^{-n} = 1+\dfrac{n}{1!}x+\dfrac{n(n+1)}{2!}x^2+\cdots+\dfrac{n(n+1)(n+2)}{3!}x^3+\cdots+\infty$

(4) $(1-x)^{-1} = 1+x+x^2+x^3+\cdots+\infty$

(5) $(1+x)^{-1} = 1-x+x^2-x^3+\cdots+\infty$

(6) $(1-x)^{-2} = 1+2x+3x^2+4x^3+\cdots+\infty$

(7) $(1+x)^{-2} = 1-2x+3x^2-4x^3+\cdots+\infty$

EXERCISES

Single Correct Choice Type Questions

1. The coefficient of x^4 in the expansion of $(x^2-x-2)^5$ is

(A) 490 (B) -490 (C) 390 (D) 30

2. The number of non-zero terms in the expansion of $(\sqrt{5}+1)^6+(\sqrt{5}-1)^6$ is

(A) 3 (B) 4 (C) 5 (D) 2

3. $2\cdot{}^{10}C_2+3\cdot 2\cdot{}^{10}C_3+4\cdot 3\cdot{}^{10}C_4+\cdots+10\cdot 9\cdot{}^{10}C_{10}=$

(A) $35\cdot 2^9$ (B) $45\cdot 2^8$ (C) $45\cdot 2^{10}$ (D) $45\cdot 2^9$

4. $C_1-2\cdot C_2+3\cdot C_3-4\cdot C_4+\cdots+(-1)^{n-1}\cdot n\cdot C_n=$

(A) 1 (B) 0 (C) -1 (D) n

5. The numerically greatest term in the expansion of $(1-3x)^{10}$ when $x=1/2$ is

(A) ${}^{10}C_6\left(\dfrac{2}{3}\right)^6$ (B) ${}^{10}C_7\left(\dfrac{2}{3}\right)^7$

(C) ${}^{10}C_6\left(\dfrac{3}{2}\right)^7$ (D) ${}^{10}C_6\left(\dfrac{3}{2}\right)^6$

6. If $(1+x+x^2)^n = a_0+a_1x+a_2x^2+\cdots+a_{2n}x^{2n}$ and n is odd, then the value of $a_0-a_2+a_4-a_6+\cdots+a_{2n}$ is

(A) 1 (B) -1 (C) 0 (D) 2^{2n}

7. If C_r is the binomial coefficient in the expansion of $(1+x)^{10}$ and $a=\sum_{r=0}^{10}1/C_r$, then $\sum_{r=0}^{10} r/C_r$ is equal to

(A) $9a$ (B) $10a$ (C) $5a$ (D) $11a$

8. Which one of the expansions of the following will contain x^2?

(A) $(x^{-1/5}+2x^{3/5})^{25}$ (B) $(x^{3/5}-2x^{-1/5})^{23}$

(C) $(x^{3/5}+2x^{-1/5})^{22}$ (D) $(x^{3/5}+2x^{-1/5})^{24}$

9. The coefficient of x^n in the expansion of $(x+C_0)(x+C_1)(x+C_2)\cdots(x+C_n)$ where $C_r={}^{2n+1}C_r$ is

(A) 2^{n+1} (B) $2^{n+1}-1$ (C) 2^{2n} (D) 2^{n-1}

10. The first integral term other than the first term beginning from the left in the expansion of $(\sqrt{3}+\sqrt[3]{2})^9$ is

(A) second term (B) third term

(C) fourth term (D) fifth term

11. The largest term in the expansion of $(2+3x)^{25}$ when $x=2$ is its

(A) thirteenth term (B) twentieth term

(C) twenty-sixth term (D) nineteenth term

12. If n is even, then the last term in the expansion of $\cos^n\theta$ in terms of cosines of multiples of θ is

(A) ${}^nC_{(n/2)}$ (B) $\dfrac{1}{2^{n-1}}\,{}^nC_{(n/2)}$ (C) $\dfrac{1}{2^n}\,{}^nC_{(n/2)}$ (D) $\dfrac{1}{2^n}$

13. The last term in the expansion of $\sin^9\theta$ as sines of multiples of θ is

(A) $\dfrac{63}{128}\sin\theta$ (B) $\dfrac{-63}{128}\sin\theta$ (C) $\dfrac{63}{128}$ (D) $\dfrac{-63}{128}$

14. Given positive integers $n>2$, $r>1$ and the coefficients of $(3r)$th and $(r+2)$th terms in the binomial expansion of $(1+x)^{2n}$ are equal, then

(A) $n=2r$ (B) $n=3r$

(C) $n=2r+1$ (D) $n=3r+1$

15. In the expansion of $[2a-(a^2/4)]^9$, the sum of the middle terms is

(A) $\left(\dfrac{63}{32}\right)a^{14}(a+8)$ (B) $\left(\dfrac{63}{32}\right)a^{14}(a-8)$

(C) $\left(\dfrac{63}{32}\right)a^{13}(a-8)$ (D) $\left(\dfrac{63}{32}\right)a^{13}(8-a)$

16. The coefficient of x^4 in the expansion of $(1 + x + x^3 + x^4)^{10}$. is

(A) ${}^{40}C_4$ (B) ${}^{10}C_4$ (C) 210 (D) 310

17. If $(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + \cdots + a_{2n}x^{2n}$, then $a_0a_1 - a_1a_2 + a_2a_3 - a_3a_4 + \cdots$ is equal to

(A) 0 (B) 2^{2n} (C) $2^{2n} - 1$ (D) -1

18. If the coefficients of x^7 and x^8 in the expansion of $[2 + (x/3)]^n$ are equal, then n is equal to

(A) 56 (B) 55 (C) 45 (D) 15

19. If the coefficients of rth, $(r+1)$th and $(r+2)$th terms in $(1+x)^n$ are in HP, then

(A) $n + (n-2r)^2 = 0$ (B) $n - (n+2r)^2 = 0$

(C) $n - (n-2r)^2 = 0$ (D) $n + (n-r)^2 = 0$

20. If the total number of terms in the expansion of $(x + y + 2z)^n$ is 45, then n is equal to

(A) 8 (B) 9 (C) 7 (D) 22

21. If the sum of the coefficients in the expansions of $(1 - 3x + 10x^2)^n$ and $(1 + x^2)^n$ are, respectively, a and b, then

(A) $a = 2b$ (B) $a = 3b$ (C) $a = b^2$ (D) $a = b^3$

22. $\sum_{r=1}^{n} {}^nC_r \left(\sum_{p=0}^{r-1} {}^rC_p 2^p \right) =$

(A) $4^3 - 3^n + 1$ (B) $4^n - 3^n - 1$

(C) $4^n - 3^n$ (D) (B) $4^n - 3^n + 2$

23. If the fourth term in the expansion of $(\sqrt{x^{[1/(1+\log_{10} x)]}} + \sqrt[12]{x})^6$ is equal to 200 and $x > 1$, then x is equal to

(A) 10 (B) 100 (C) $10\sqrt{2}$ (D) 10^4

24. The coefficient of $(ab)^6$ in the expansion of $[a^2 - (b/a)]^{12}$ is

(A) -824 (B) 824 (C) 924 (D) -924

Multiple Choice Type Questions

1. In the expansion of $[x + (a/x^2)]^n$, $a \neq 0$, if no term is independent of x, then n may be

(A) 10 (B) 12 (C) 16 (D) 20

2. If a and b are non-zero and only one term in each of the expansions of $[x - (a/x)]^n$ and $[x + (b/x^2)]^n$ is independent of x, then n is divisible by

(A) 2 (B) 3 (C) 4 (D) 6

3. If the third, fourth and fifth terms in the expansion of $(x + a)^n$ are respectively 84, 280 and 560, then

(A) $x = 1$ (B) $a = 2$

(C) $n = 7$ (D) $x = 2,\ a = 3,\ n = 8$

4. Which of the following is (are) true?

(A) The coefficient of x^{-1} in the expansion of $[x + (1/x^2)]^8$ is 56.

(B) The coefficient of x in the expansion of $[x + (1/x^2)]^8$ is 0.

(C) The coefficient of x^9 in the expansion of $[2x^2 - (1/x)]^{20}$ is 0.

(D) The coefficient of x^{30} in the expansion of $(x^3 + 3x^2 + 3x + 1)^{15}$ is ${}^{45}C_{15}$.

5. It is given that $(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + \cdots + a_{2n}x^{2n}$. Which of the following is (are) correct?

(A) If n is odd, then $a_0 - a_2 + a_4 - a_6 + \cdots = 0$.

(B) If n is even, then $a_1 - a_3 + a_5 - a_7 + \cdots = 0$.

(C) If $n = 4K + 1$, where K is a positive integer, then $a_1 - a_3 + a_5 - a_7 + \cdots = 1$.

(D) If n is a multiple of 4, then $a_0 - a_2 + a_4 - a_6 + \cdots = 1$.

6. Which of the following statements is (are) true?

(A) There are two consecutive terms in the expansion of $(3 + 2x)^{74}$ whose coefficients are equal.

(B) For a positive integer n, the coefficients of second, third and fourth terms in the expansion of $(1 + x)^{2n}$ are not in AP.

(C) Larger of $99^{50} + 100^{50}$ and 101^{50} is 101^{50}.

(D) The sum of the coefficients in the binomial expansion of $(5x - 4y)^{21}$ is 1.

7. Which of the following are true?

(A) $({}^{10}C_0)^2 - ({}^{10}C_1)^2 + ({}^{10}C_2)^2 - ({}^{10}C_3)^2 + \cdots + ({}^{10}C_{10})^2 = \dfrac{-(10)!}{5!5!}$

(B) $({}^{11}C_0)^2 - ({}^{11}C_1)^2 + ({}^{11}C_2)^2 - ({}^{11}C_3)^2 + \cdots - ({}^{11}C_{11})^2 = \dfrac{(11)!}{6!5!}$

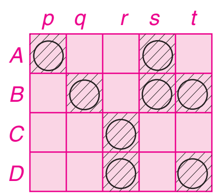
(C) $2 \cdot C_0 + 2^2 \cdot \dfrac{C_1}{2} + 2^3 \cdot \dfrac{C_2}{3} + 2^4 \cdot \dfrac{C_3}{4} + \cdots + 2^{11} \cdot \dfrac{C_{10}}{11} = \dfrac{3^{11} - 1}{11}$ (where $C_r = {}^{10}C_r$)

(D) The coefficient of x^3 in the expansion of $1 + (1+x) + (1+x)^2 + (1+x)^3 + \cdots + (1+x)^{10}$ is ${}^{11}C_4$.

Matrix-Match Type Questions

In each of the following questions, statements are given in two columns, which have to be matched. The statements in Column I are labeled as **(A)**, **(B)**, **(C)** and **(D)**, while those in Column II are labeled as **(p)**, **(q)**, **(r)**, **(s)** and **(t)**. Any given statement in Column I can have correct matching with *one or more* statements in Column II. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following example.

Example: If the correct matches are (A) → (p), (s); (B) → (q), (s), (t); (C) → (r); (D) → (r), (t); that is if the matches are (A) → (p) and (s); (B) → (q), (s) and (t); (C) → (r); and (D) → (r), (t); then the correct darkening of bubbles will look as follows:



1. Match the items in Column I with those in Column II.

Column I	*Column II*
(A) If the middle term in the expansion of $[\sqrt{x/3}+(\sqrt{3}/2x^2)]^{10}$ is ax^K, then $a=$	(p) $\dfrac{^{10}C_2}{108}$
(B) The term independent of x in the expansion of $[\sqrt{x/3}+(\sqrt{3}/2x^2)]^{10}$ is	(q) $^{10}C_4\times 5184$
(C) The term independent of x in the expansion of $[2x^2-(3/x^3)]^{10}$ is	(r) $\dfrac{^{10}C_5}{32}$
(D) The coefficient of the middle term in the expansion of $[2x^2-(3/x^3)]^{10}$ is b, then b is	(s) $-6^5\times{}^{10}C_5$

2. Match the items in Column I with those in Column II

Column I	*Column II*
(A) The coefficient of x in the expansion of $(1-2x^3+3x^5)[1+(1/x)]^8$ is	(p) 378
(B) The coefficient of x^3 in the expansion of $(1+x+2x^2)[2x^2-(1/3x)]^9$ is	(q) 154
(C) The coefficient of x^5 in the expansion of $(1+x+x^3)^9$ is	(r) 31
(D) If $(1+x-2x^2)^6 = 1+a_1x+a_2x^2+\cdots+a_{12}x^{12}$, then $a_2+a_4+a_6+a_8+a_{10}+a_{12}=$	(s) –224/24

Comprehension-Type Questions

1. Let $C_0, C_1, C_2, \ldots, C_n$ be binomial coefficients in the expansion of $(1+x)^n$.

Answer the following questions:

(i) $(C_0+C_1+C_2+\cdots+C_n)^2=$

(A) $2^{2n}+1$

(B) $1+{}^{2n}C_1+{}^{2n}C_2+\cdots+{}^{2n}C_{2n}$

(C) $2^{2n}-1$

(D) ${}^{2n}C_1+{}^{2n}C_2+{}^{2n}C_3+\cdots+{}^{2n}C_{2n}$

(ii) $\dfrac{C_0}{4}-\dfrac{C_1}{5}+\dfrac{C_2}{6}-\cdots+(-1)^n\dfrac{C_n}{n+4}=$

(A) $\dfrac{6}{(n+1)(n+2)(n+3)(n+4)}$

(B) $\dfrac{(n+2)(n+3)(n+4)}{6}$

(C) 0

(D) $\dfrac{(n+1)(n+2)(n+3)}{6}$

(iii) $-C_0 + C_1 + 3\cdot C_2 + 5\cdot C_3 + \cdots + (2n-1)\cdot C_n =$

(A) $n\cdot 2^{n-1}$ (B) $(n-1)2^{n-1}$

(C) $n\cdot 2^n$ (D) $(n-1)2^n$

2. (*Note:* This question may be attempted after studying integration):

Let $(1+x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n$ and $\int_a^b x^r\,dx = [1/(r+1)](b^{r+1} - a^{r+1})$. Using this information, answer the following questions:

(i) $\dfrac{C_0}{1} + \dfrac{C_1}{2} + \dfrac{C_2}{3} + \cdots + \dfrac{C_n}{n+1} =$

(A) $\dfrac{2^n - 1}{n+1}$ (B) $\dfrac{2^{n+1} - 1}{n+1}$ (C) $\dfrac{2^{n+1} + 1}{n+1}$ (D) $\dfrac{2^n + 1}{n+1}$

(ii) $2\cdot C_0 + 2^2\cdot\dfrac{C_1}{2} + 2^3\cdot\dfrac{C_2}{3} + \cdots + 2^{n+1}\cdot\dfrac{C_n}{n+1} =$

(A) $\dfrac{3^n - 1}{n+1}$ (B) $\dfrac{3^n + 1}{n+1}$

(C) $\dfrac{3^{n+1} + 1}{n+1}$ (D) $\dfrac{3^{n+1} - 1}{n+1}$

(iii) $C_0 - \dfrac{C_1}{2} + \dfrac{C_2}{3} - \dfrac{C_3}{4} + \cdots + (-1)^n\dfrac{C_n}{n+1} =$

(A) 0 (B) $\dfrac{2^{n+1} - 1}{n}$

(C) $\dfrac{1}{n+1}$ (D) $\dfrac{1}{(n+1)(n+2)}$

Assertion–Reasoning Type Questions

Statement I and Statement II are given in each of the questions in this section. Your answers should be as per the following pattern:

(A) If both Statements I and II are correct and II is a correct reason for I

(B) If both Statements I and II are correct and II is not a correct reason for I

(C) If Statement I is correct and Statement II is false.

(D) If Statement I is false and Statement II is correct.

1. Statement I: $\displaystyle\sum_{r=0}^{n-1} \frac{{}^nC_r}{{}^nC_r + {}^nC_{r+1}} = \frac{n}{2}$

Statement II: ${}^nC_K + {}^nC_{K-1} = {}^{(n+1)}C_K$

2. Statement I: If $\sin^7\theta$ is expressed as a series of sines of multiples of θ, then the coefficient of $\sin 5\theta$ is $7/64$.

Statement II: If $x = \cos\theta + i\sin\theta$, then

$$x^K + \frac{1}{x^K} = 2\cos K\theta$$

and $$x^K - \frac{1}{x^K} = 2\,i\,\sin K\theta$$

where K is a positive integer.

3. Statement I: If

$$(1+x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \cdots + C_n$$

then

$$\sum_{r=0}^{n} r(n-r)C_r^2 = n^{2\,(2n-2)}C_n$$

Statement II: ${}^nC_K = {}^nC_{n-K}$ and the derivative of $(x+a)^n = n(x+a)^{n-1}$.

4. Statement I: If n is a positive integer, then in the expansion of $(1+x)^n$, the coefficients of $(r+1)$th, $(r+2)$th, and $(r+3)$th terms are in G.P.

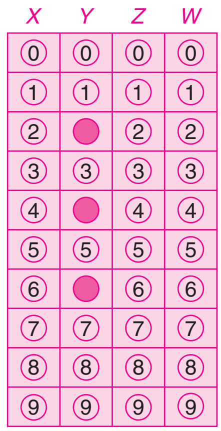
Statement II: Three non-zero numbers a, b and c are in GP if and only if $ac = b^2$.

5. Statement I: No three consecutive coefficients in the expansion of $(1+x)^n$ are in HP.

Statement II: Non-zero numbers a, b and c are in HP if $1/a$, $1/b$ and $1/c$ are in AP.

Integer Answer Type Questions

The answer to each of the questions in this section is a non-negative integer. The appropriate bubbles below the respective question numbers have to be darkened. For example, as shown in the figure, if the correct answer to the question number Y is 246, then the bubbles under Y labeled as 2, 4, 6 are to be darkened.



1. If the second term in the expansion of $(\sqrt[13]{x} + x\sqrt{x})^n$ is $14 \cdot x^{5/2}$, then ${}^nC_3 / {}^nC_2 =$ _____.

2. If P and Q are, respectively, the sum of even and odd terms in the expansion of $(x + a)^{10}$, then $(x + a)^{20} - (x - a)^{20} = k\,PQ$ where k is _____.

3. If $32 \cos^6 \theta = a_1 \cos 6\theta + a_2 \cos 4\theta + a_3 \cos 2\theta + a_4$, then a_4 is equal to _____.

4. If $256 \sin^7\theta \cdot \cos^2 \theta = a_1 \sin 9\theta + a_2 \sin 7\theta + a_3 \sin 5\theta + a_4 \sin\theta$, then a_4 is equal to _____.

5. The digit at the unit's place in the number $17^{2010} + 11^{2010} - 7^{2010}$ is _____.

ANSWERS

Single Correct Choice Type Questions

1. (D)
2. (B)
3. (D)
4. (B)
5. (D)
6. (C)
7. (C)
8. (C)
9. (C)
10. (C)
11. (B)
12. (C)
13. (A)
14. (A)
15. (D)
16. (D)
17. (A)
18. (B)
19. (A)
20. (A)
21. (D)
22. (C)
23. (A)
24. (C)

Multiple Correct Choice Type Questions

1. (A), (C), (D)
2. (A), (B), (D)
3. (A), (B), (C)
4. (A), (B), (C), (D)
5. (A), (B), (C), (D)
6. (A), (B), (C), (D)
7. (A), (C), (D)

Matrix-Match Type Questions

1. (A) → (r), (B) → (p), (C) → (q), (D) → (s)
2. (A) → (q), (B) → (s), (C) → (p), (D) → (r)

Comprehension-Type Questions

1. (i) (B), (ii) (A), (iii) (D)
2. (i) (B), (ii) (D), (iii) (C)

Assertion–Reasoning Type Questions

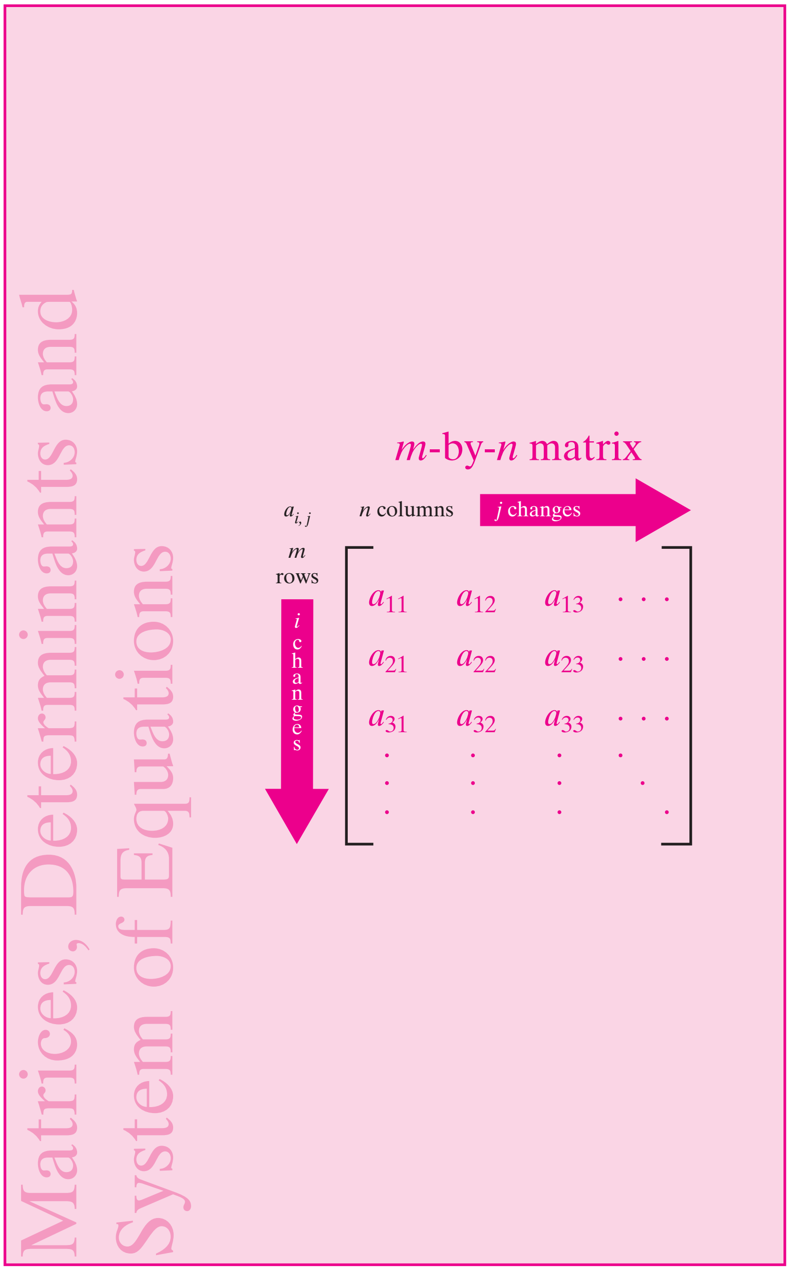
1. (A)
2. (A)
3. (A)
4. (D)
5. (A)

Integer Answer Type Questions

1. 4
2. 4
3. 10
4. 14
5. 1

Matrices, Determinants and System of Equations

8



Contents

Matrices: A matrix (plural matrices) is a rectangular array of numbers. Matrices are a key tool in linear algebra. One use of matrices is to represent linear transformations.

Determinants: The determinant is a special number associated with any square matrix. The fundamental geometric meaning of a determinant is a scale factor for measure when the matrix is regarded as a linear transformation.

The theory of matrices plays an important role in almost all branches of Mathematics and other subjects. A very important application of matrices is to find solutions of system of linear equations. Let us consider a simple situation where three students Ram, Rahim and Robert have appeared for class tests in four subjects English, Mathematics, Physics and Chemistry and the marks obtained by each of them in these subjects are given in a tabular form given below.

	English	*Mathematics*	*Physics*	*Chemistry*
Ram	80	86	78	75
Rahim	75	84	72	68
Robert	78	68	74	78

This also can be represented by an array of numbers without drawing lines and not writing the names of the subjects on the top row and the names of the students on the left most column, as given below.

$$\begin{bmatrix} 80 & 86 & 78 & 75 \\ 75 & 84 & 72 & 68 \\ 78 & 68 & 74 & 78 \end{bmatrix}$$

The brackets given on the left end and right end do not convey any meaning but just improve the presentation style. The first horizontal line of numbers shows the marks obtained by Ram in English, Mathematics, Physics and Chemistry, respectively. Similarly the second and third horizontal lines show the same for Rahim and Robert, respectively. The first vertical line of numbers shows the marks obtained in English by Ram, Rahim and Robert. The second, third and fourth vertical lines show the same for Mathematics, Physics and Chemistry, respectively.

The horizontal lines are called rows and the vertical lines are called columns. The rows are numbered from top to bottom. The top row is called the first row and the subsequent rows are called second row, third row, etc. The columns are numbered from left to right. The left most column is called the first column and the subsequent columns are called second column, third column, etc.

In this chapter, we make a detailed study of matrices whose entries are real or complex numbers.

8.1 | Matrices

In this section we shall give a formal definition of a matrix and discuss various types of matrices and their properties.

DEFINITION 8.1 **Matrix, Rows, Columns, Order** An ordered rectangular array of real or complex numbers or functions or of any kind of expressions is called a *matrix*. The horizontal lines in the array are called *rows* and the vertical lines are called *columns*. If there are m rows and n columns in a matrix A, then A is called an $m \times n$ matrix or an "m by n" matrix or a matrix of order $m \times n$.

DEFINITION 8.2 **Elements or Entries** The numbers or functions or expressions in a matrix A are called "elements" or "entries" of A. If A is $m \times n$ matrix, then there are m rows of elements and n columns of elements. In each row of an $m \times n$ matrix there are exactly n elements and in each column there are exactly m elements.

Examples

(1) Consider the matrix

$$A = \begin{bmatrix} 2 & 3 & 4 & 1 \\ -1 & 2 & 1 & 3 \\ 5 & -3 & 2 & -4 \end{bmatrix}$$

Then A is a 3×4 matrix, since there are 3 rows and 4 columns in A. Here

2	3	4	1	is the first row
–1	2	1	3	is the second row
5	–3	2	–4	is the third row

$\begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$ is the first column

$\begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix}$ is the second column

$\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$ is the third column

$\begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix}$ is the fourth column

(2) $\begin{bmatrix} 1 & 3 & 2 \\ -3 & \frac{1}{2} & 0 \\ 2 & -3 & -1 \\ \frac{1}{3} & \frac{2}{3} & 4 \end{bmatrix}$ is a 4×3 matrix, since there are 4 rows and 3 columns.

(3) $\begin{bmatrix} 2 & -1 & 3 \\ -4 & 0 & 2 \end{bmatrix}$ is a 2×3 matrix, since there are 2 rows and 3 columns.

(4) $\begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}$ is a 2×2 matrix.

(5) $[2]$ is a 1×1 matrix.

In general, an $m \times n$ matrix is of the form

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

where each a_{ij} is a number or a function or an expression. The entries in the ith row are

$$a_{i1}, a_{i2}, \ldots, a_{in}$$

and the entries in the jth column are

$$a_{1j}, a_{2j}, \ldots, a_{mj}$$

for $1 \le i \le m$ and $1 \le j \le n$. The variable a_{ij} stands for the entry which is common for the ith row and jth column. For simplicity, we write

$$A = (a_{ij})_{m \times n} \quad \text{or} \quad A = (a_{ij}) \quad \text{or} \quad A = [a_{ij}]_{m \times n}$$

to denote an $m \times n$ matrix whose entry in the ith row and jth column is a_{ij}. For convenience, a_{ij} is called the ijth entry of the matrix $A = (a_{ij})$. Further $m \times n$ is called the *order* of A.

DEFINITION 8.3 **Equality of Matrices** Two matrices are said to be *equal* if they are of the same order and for any i and j, the ijth entries of the two matrices are same. In other words, if $A = (a_{ij})$ is an $m \times n$ matrix and $B = (b_{ij})$ is a $p \times q$ matrix, then we say that A and B are equal and write $A = B$ if $m = p, n = q$ and $a_{ij} = b_{ij}$ for all $1 \le i \le m = p$ and $1 \le j \le n = q$.

Note: A 2×3 matrix can never be equal to a 3×2 matrix, since their orders are different. For example

$$\begin{bmatrix} 2 & 1 \\ -3 & 4 \\ \frac{1}{2} & 0 \end{bmatrix} \ne \begin{bmatrix} 2 & -3 & \frac{1}{2} \\ 1 & 4 & 0 \end{bmatrix}$$

DEFINITION 8.4 **Square Matrix** An $m \times n$ matrix is said to be a *square matrix* if $m = n$, that is, the number of rows is equal to the number of columns.

Example

The matrices

$$[2], \quad \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 2 & -2 \\ 3 & 4 & -1 \\ 2 & 0 & 1 \end{bmatrix}$$

are square matrices. If A is an $n \times n$ matrix, then we say that A is a square matrix of order n.

DEFINITION 8.5 Let $A = (a_{ij})$ be an $m \times n$ matrix. A is called a *vertical matrix* if $m > n$ and a *horizontal matrix* if $m < n$.

Examples

(1) $\begin{bmatrix} 2 & 3 & 1 \\ 1 & 0 & -1 \end{bmatrix}$ is a horizontal matrix, since the columns are more in number than rows.

(2) $\begin{bmatrix} 2 & 1 \\ 3 & 0 \\ 1 & -1 \end{bmatrix}$ is a vertical matrix, since the rows are more in number than the columns.

Recall that, if the rows and columns are equal in number, then the matrix is called the square matrix. Note that any matrix must be either a square matrix or a vertical matrix or a horizontal matrix.

DEFINITION 8.6 **Row Matrix and Column Matrix** A matrix is called a *row matrix* if it has only one row and is called a *column matrix* if it has only one column.

Example

The matrix $[2 \quad 3 \quad 1]$ is a row matrix and the matrix $\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$ is a column matrix.

DEFINITION 8.7 **Zero Matrix** A matrix is called a *zero matrix* or *null matrix* if all its entries are zero. A zero matrix is usually denoted by O, without mentioning its order and is to be understood as per the context.

Example

The matrices $[0]$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are all zero matrices.

DEFINITION 8.8 **Diagonal of a Matrix** Let $A = (a_{ij})$ be a square matrix of order n, that is A is a $n \times n$ matrix. Then the elements

$$a_{11}, a_{22}, a_{33}, \ldots, a_{nm}$$

are called the *diagonal elements* and the line along which these elements lie is called the *principal diagonal* or *main diagonal* or simply the *diagonal* of the matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & & a_{nn} \end{bmatrix}$$

Example

1, 0 are the diagonal elements in $\begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$ and 2, 1, –1 are the diagonal elements in $\begin{bmatrix} 2 & 3 & 0 \\ 1 & 1 & -2 \\ 4 & 0 & -1 \end{bmatrix}$.

DEFINITION 8.9 **Diagonal Matrix** A square matrix $A = (a_{ij})$ is said to be a *diagonal matrix* if $a_{ij} = 0$ for all $i \neq j$, that is, except those in the diagonal of A, all the entries in A are zeros. Note that the diagonal elements need not be zeros.

Examples

(1) $\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$ is a diagonal matrix.

(2) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$ are diagonal matrices.

Note that the zero matrix of order $n \times n$ is also a diagonal matrix for any n.

DEFINITION 8.10 **Scalar Matrix** A diagonal matrix $A = (a_{ij})$ is called a *scalar matrix* if $a_{ii} = a_{jj}$ for all i and j, that is, a matrix $A = (a_{ij})$ is a scalar matrix if all the diagonal elements are equal and the other elements are zeros.

If a is any real or complex number and n is any positive integer, then define

$$a_{ij} = \begin{cases} a & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

for any $1 \le i, j \le n$. Then (a_{ij}) is a scalar matrix

$$\begin{bmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & a \end{bmatrix}$$

and any scalar matrix is of this form.

DEFINITION 8.11 **Identity Matrix** A scalar matrix is called the *identity matrix* or a *unit matrix* if each of the diagonal element is the number 1. That is, a square matrix $A = (a_{ij})$ of order n is called the identity matrix of order n if

$$a_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

for all $1 \le i, j \le n$. The identity matrix of order n is denoted by I_n. The identity or unit matrix I_n of order n will be simply called the identity and denoted by I when there is no ambiguity about n.

Example

$$I_1 = [1], \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

DEFINITION 8.12 **Triangular Matrices** A square matrix $A = (a_{ij})$ is called an *upper triangular matrix* if

$a_{ij} = 0$ for all $i > j$ $A = (a_{ij})$ is called a *lower triangular matrix* if

$a_{ij} = 0$ for all $i < j$

Example

Upper Triangular Matrices

$$\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 2 & 3 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 4 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 2 & 3 & -2 \\ 0 & -2 & 0 & 4 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Lower Triangular Matrices

$$\begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} 3 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 4 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 3 & -2 & 5 & 0 \\ -1 & 0 & 3 & -9 \end{bmatrix}$$

Note that a square matrix is both upper and lower triangular matrix if and only it is a diagonal matrix.

DEFINITION 8.13 **Addition of Matrices** Let $A = (a_{ij})$ and $B = (b_{ij})$ be matrices of order $m \times n$. Then, we define

$$A + B = (a_{ij} + b_{ij}) \quad \text{for all } 1 \le i \le m \quad \text{and} \quad 1 \le j \le n$$

That is, the ijth entry in $A + B$ is the sum of the ijth entries in A and B. $A + B$ is called the sum of A and B and the operation + is called the addition of matrices.

Note that the addition is defined among matrices of the same order. For any $m \times n$ matrix $A = (a_{ij})$, we defined an $m \times n$ matrix $-A$ by

$$-A = (-a_{ij})$$

and for any $m \times n$ matrices A and B, we write, as usual, $A - B$ for

$$A + (-B) = (a_{ij} - b_{ij})$$

where $A = (a_{ij})$ and $B = (b_{ij})$.

Recall that two matrices $A = (a_{ij})$ and $B = (b_{ij})$ are said to equal if A and B are of the same order, say $m \times n$ and $a_{ij} = b_{ij}$ for all $1 \le i \le m$ and $1 \le j \le n$.

Example 8.1

If

$$A = \begin{bmatrix} 2 & 0 & 1 & 3 \\ 1 & -1 & 2 & 1 \\ 3 & -2 & 4 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -1 & 2 & 4 \\ -2 & 3 & 1 & 2 \\ 4 & 0 & -4 & -3 \end{bmatrix},$$

then find out $A + B$.

Solution: We have

$$A + B = \begin{bmatrix} 2+1 & 0+(-1) & 1+2 & 3+4 \\ 1+(-2) & -1+3 & 2+1 & 1+2 \\ 3+4 & (-2)+0 & 4+(-4) & 2+(-3) \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -1 & 3 & 7 \\ -1 & 2 & 3 & 3 \\ 7 & -2 & 0 & -1 \end{bmatrix}$$

We shall use this technique in proving the following, in which all matrices are considered to be over real or complex numbers.

THEOREM 8.1 Let $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$ be matrices of order $m \times n$. Then the following are true.

1. Associative law for addition: $A + (B + C) = (A + B) + C$.
2. Commutative law for addition: $A + B = B + A$.
3. $A + O = A$, where O is the $m \times n$ zero matrix and is called the additive identity.
4. $A + (-A) = O$. Here $-A$ is called the additive inverse of A.
5. *Cancellation laws* for addition: $A + B = A + C \Rightarrow B = C$ and $B + A = C + A \Rightarrow B = C$.
6. There exists unique matrix D such that $A + D = B$.

PROOF

1. For any $1 \le i \le m$ and $1 \le j \le n$,

$$\begin{aligned} ij\text{th entry in } A + (B + C) &= a_{ij} + (b_{ij} + c_{ij}) \\ &= (a_{ij} + b_{ij}) + c_{ij} \quad \text{(since + is associative for numbers)} \\ &= ij\text{th entry in } (A + B) + C \end{aligned}$$

Therefore $A + (B + C) = (A + B) + C$.

2. For any $1 \le i \le m$ and $1 \le j \le n$,

$$\begin{aligned} ij\text{th entry in } A + B &= a_{ij} + b_{ij} \\ &= b_{ij} + a_{ij} \quad \text{(since + is commutative for numbers)} \\ &= ij\text{th entry in } B + A \end{aligned}$$

Therefore $A + B = B + A$.

3.
$$\begin{aligned} A + O &= (a_{ij}) + (0) \\ &= (a_{ij} + 0) \\ &= (a_{ij}) = A \end{aligned}$$

4.
$$\begin{aligned} A + (-A) &= (a_{ij}) + (-a_{ij}) \\ &= (a_{ij} + (-a_{ij})) \\ &= (0) = O \end{aligned}$$

5. Suppose that $A + B = A + C$. Then, for any $1 \le i \le m$ and $1 \le j \le n$,

$$\begin{aligned} a_{ij} + b_{ij} &= ij\text{th entry in } A + B \\ &= ij\text{th entry in } A + C \\ &= a_{ij} + c_{ij} \end{aligned}$$

Hence $b_{ij} = c_{ij}$. Therefore $B = (b_{ij}) = (c_{ij}) = C$. Also

$$\begin{aligned} B + A = C + A &\Rightarrow A + B = A + C \\ &\Rightarrow B = C \quad [\text{by } (2)] \end{aligned}$$

6. Put $D = B - A$. Then

$$A + D = A + (B - A) = (B - A) + A = B + (-A + A) = B + O = B$$

■

DEFINITION 8.14 A real or complex number is called a *scalar*. For any matrix $A = (a_{ij})$ of numbers and for any scalar k, we define the matrix kA as the one whose ijth entry is obtained by multiplying the ijth entry of A by k, that is

$$kA = (ka_{ij})$$

Example

If

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 9 & 4 \\ 5 & -6 & 7 \\ 8 & -5 & 2 \end{bmatrix},$$

then

$$4A = \begin{bmatrix} 4 & 8 & -12 \\ 0 & 36 & 16 \\ 20 & -24 & 28 \\ 32 & -20 & 8 \end{bmatrix}$$

Hereon, all matrices are assumed to be with entries as real or complex numbers and we would not specify this any further.

THEOREM 8.2 Let $A = (a_{ij})$ and $B = (b_{ij})$ be $m \times n$ matrices and s and t be scalars. Then the following properties are satisfied:

1. $s(A + B) = sA + sB$
2. $(s + t)A = sA + tA$
3. $s(tA) = (st)A = t(sA)$
4. $(-s)A = -(sA) = s(-A)$
5. $0A = O$ (0 on the left side is the scalar zero and O on the right is the zero matrix)
6. $sO = O$

PROOF Let $1 \le i \le m$ and $1 \le j \le n$.

1. $$\begin{aligned} ij\text{th entry in } s(A + B) &= s(a_{ij} + b_{ij}) \\ &= sa_{ij} + sb_{ij} \\ &= ij\text{th entry in } sA + sB \end{aligned}$$

   Therefore, $s(A + B) = sA + sB$.

2. $$\begin{aligned} ij\text{th entry in } (s + t)A &= (s + t)a_{ij} \\ &= sa_{ij} + ta_{ij} \\ &= ij\text{th entry in } sA + tA \end{aligned}$$

   Therefore, $(s + t)A = sA + tA$.

3. $s(ta_{ij}) = (st)a_{ij} = (ts)a_{ij} = t(sa_{ij})$. Therefore, $s(tA) = (st)A = t(sA)$.
4. $(-s)a_{ij} = -(sa_{ij}) = s(-a_{ij})$. Therefore, $(-s)A = -(sA) = s(-A)$.
5. and 6. Since $0 \cdot a_{ij} = 0 = s \cdot 0$, therefore $OA = O = sO$. ■

Example 8.2

Let A and B be 2×2 matrices. Find A and B such that

$$3A + 2B = \begin{bmatrix} 9 & 4 \\ -2 & -6 \end{bmatrix}$$

and

$$2A + 5B = \begin{bmatrix} 17 & -1 \\ 6 & -15 \end{bmatrix}$$

Solution: Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

Then, from the hypothesis,

$$\begin{bmatrix} 9 & 4 \\ -2 & -6 \end{bmatrix} = 3A + 2B = 3\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + 2\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$= \begin{bmatrix} 3a_{11} + 2b_{11} & 3a_{12} + 2b_{12} \\ 3a_{21} + 2b_{21} & 3a_{22} + 2b_{22} \end{bmatrix}$$

and therefore, by equating the corresponding ijth entries on both sides, we get

$$9 = 3a_{11} + 2b_{11} \tag{8.1}$$

$$4 = 3a_{12} + 2b_{12} \tag{8.2}$$

$$-2 = 3a_{21} + 2b_{21} \tag{8.3}$$

$$-6 = 3a_{22} + 2b_{22} \tag{8.4}$$

Similarly, by using

$$2A + 5B = \begin{bmatrix} 17 & -1 \\ 6 & -15 \end{bmatrix}$$

we get that

$$17 = 2a_{11} + 5b_{11} \tag{8.5}$$

$$-1 = 2a_{12} + 5b_{12} \tag{8.6}$$

$$6 = 2a_{21} + 5b_{21} \tag{8.7}$$

$$-15 = 2a_{22} + 5b_{22} \tag{8.8}$$

By solving Eqs. (8.1) and (8.5), we can find a_{11} and b_{11} as $a_{11} = 1$ and $b_{11} = 3$. Similarly, by solving Eqs. (8.2) and (8.6), Eqs. (8.3) and (8.7), and Eqs. (8.4) and (8.8), we get

$$a_{12} = 2 \quad \text{and} \quad b_{12} = -1$$

$$a_{21} = -2 \quad \text{and} \quad b_{21} = 2$$

and

$$a_{22} = 0 \quad \text{and} \quad b_{22} = -3$$

Therefore

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & -1 \\ 2 & -3 \end{bmatrix}$$

Example 8.3

Evaluate the following:

$$\cos\theta \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \sin\theta \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$$

Solution: We have

$$\cos\theta \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \sin\theta \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ -\cos\theta\sin\theta & \cos^2\theta \end{bmatrix}$$

$$+ \begin{bmatrix} \sin^2\theta & -\sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \cos^2\theta + \sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

Example 8.4

Compute the matrix X, if it is given that $2X + 3A = 3B$, where

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$$

Solution: Suppose that $2X + 3A = 3B$. Then $2X = 3B - 3A = 3(B - A)$. Therefore

$$X = \frac{3}{2}[B - A]$$

$$= \frac{3}{2}\left(\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}\right)$$

$$= \frac{3}{2}\begin{bmatrix} 3-1 & 5-(-1) \\ 4-0 & 6-2 \end{bmatrix}$$

$$= \frac{3}{2}\begin{bmatrix} 2 & 6 \\ 4 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 9 \\ 6 & 6 \end{bmatrix}$$

In the following, we define the product AB of two matrices A and B only when the number of columns in A is equal to the number of rows in B.

DEFINITION 8.15 **Multiplication of Matrices** Let A be an $m \times n$ matrix and B an $n \times p$ matrix. If $A = (a_{ij})$ and $B = (b_{ij})$, then the product AB is defined as the $m \times p$ matrix (c_{ij}), where

$$c_{ij} = \sum_{r=1}^{n} a_{ir} b_{rj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

for all $1 \le i \le m$ and $1 \le j \le p$.

Example 8.5

Let A be a 4×3 matrix and B be a 3×2 matrix given by

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 4 \\ 0 & 2 & -2 \\ -3 & 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -3 \\ -1 & 4 \\ 3 & 1 \end{bmatrix}$$

Find the product AB.

Solution: We get the product AB as a 4×2 matrix given by

$$AB = (c_{ij})$$

where $c_{ij} = \sum_{r=1}^{3} a_{ir} b_{rj}$ for all $1 \le i \le 4$ and $1 \le j \le 2$ and $A = (a_{ij})$ and $B = (b_{ij})$. Now

$$\begin{aligned} c_{11} &= a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} \\ &= 2 \times 2 + 3 \times (-1) + (-1) \times 3 \\ &= 4 - 3 - 3 = -2 \end{aligned}$$

$$\begin{aligned} c_{12} &= a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ &= 2 \times (-3) + 3 \times 4 + (-1)1 \\ &= -6 + 12 - 1 = 5 \end{aligned}$$

$$\begin{aligned} c_{21} &= a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} \\ &= 1 \times 2 + 0 \times (-1) + 4 + 3 \\ &= 2 + 0 + 12 = 14 \end{aligned}$$

$$\begin{aligned} c_{22} &= a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \\ &= 1 \times (-3) + 0 \times 4 + 4 \times 1 \\ &= -3 + 0 + 4 = 1 \end{aligned}$$

$$\begin{aligned} c_{31} &= a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} \\ &= 0 \times 2 + 2 \times (-1) + (-2)(3) \\ &= 0 - 2 - 6 = -8 \end{aligned}$$

$$\begin{aligned} c_{32} &= a_{31}\, b_{12} + a_{32}\, b_{22} + a_{33}\, b_{32} \\ &= 0 \times (-3) + 2 \times 4 + (-2) \times 1 \\ &= 0 + 8 - 2 = 6 \end{aligned}$$

$$\begin{aligned} c_{41} &= a_{41}b_{11} + a_{42}b_{21} + a_{43}b_{31} \\ &= (-3)2 + 1(-1) + 2 \times 3 \\ &= -6 - 1 + 6 = -1 \end{aligned}$$

$$\begin{aligned} c_{42} &= a_{41}b_{12} + a_{42}b_{22} + a_{43}b_{32} \\ &= (-3)(-3) + 1 \times 4 + 2 \times 1 \\ &= 9 + 4 + 2 = 15 \end{aligned}$$

Therefore

$$AB = (c_{ij}) = \begin{bmatrix} -2 & 5 \\ 14 & 1 \\ -8 & 6 \\ -1 & 15 \end{bmatrix}$$

Note: The ijth entry in the product AB is simply obtained by multiplying the ith row of A and jth column of B. Note that the ith row of A and jth column of B are both n-tuples

$$(a_{i1}, a_{i2}, a_{i3}, \ldots, a_{in}) \quad \text{and} \quad \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{pmatrix}$$

respectively and the ijth entry in AB is the sum of the products $a_{i1} b_{1j}, a_{i2} b_{2j}, \ldots, a_{in} b_{nj}$.

THEOREM 8.3 Let $A = (a_{ij})$ be an $m \times n$ matrix, $B = (b_{ij})$ an $n \times p$ matrix and $C = (c_{ij})$ a $p \times q$ matrix. Then

$$A(BC) = (AB)C$$

PROOF Note that BC is an $n \times q$ matrix and AB is an $m \times p$ matrix. Let $BC = (d_{ij})$, for $1 \le i \le n, 1 \le j \le q$, and $AB = (u_{ij})$, for $1 \le i \le m, 1 \le j \le p$. Then

$$d_{ij} = \sum_{r=1}^{p} b_{ir} c_{rj} \tag{8.9}$$

and

$$u_{ij} = \sum_{s=1}^{n} a_{is} c_{sj} \tag{8.10}$$

Both $A(BC)$ and $(AB)C$ are $m \times q$ matrices. For any $1 \le i \le m$ and $1 \le j \le q$,

$$\begin{aligned} ij\text{th entry of } A(BC) &= \sum_{t=1}^{n} a_{it} d_{tj} \\ &= \sum_{t=1}^{n} a_{it} \left(\sum_{r=1}^{p} b_{tr} c_{rj} \right) \quad [\text{by Eq. (8.9)}] \\ &= \sum_{t=1}^{n} \sum_{r=1}^{p} (a_{it} b_{tr} c_{rj}) \\ &= \sum_{r=1}^{p} \left(\sum_{t=1}^{n} a_{it} b_{tr} \right) c_{rj} \\ &= \sum_{r=1}^{p} u_{ir} c_{rj} \\ &= ij\text{th entry in } (AB)C \quad [\text{by Eq. (8.10)}] \end{aligned}$$

Thus $(AB)C = (AB)C$. ■

QUICK LOOK 1

We have proved earlier that $A + B = B + A$ for any matrices A and B of the same order; that is the addition is commutative. However, the multiplication of matrices is not commutative. In fact, if $A \cdot B$ is defined, then $B \cdot A$ may not be defined.

Consider the following examples.

Examples

(1) Let A be a 3×2 matrix and B a 2×4 matrix. Then $A \cdot B$ is defined, since the number of columns in A is 2 which is same as the number of rows in B. However $B \cdot A$ is not defined, since the number of columns in $B(=4)$ is not equal to the number of rows in $A(=3)$.

(2) Even if $A \cdot B$ and $B \cdot A$ are defined, they may not be of same order. For example, let A be a 2×3 matrix and B be a 3×2 matrix. Then $A \cdot B$ and $B \cdot A$ are both defined. However, $A \cdot B$ is a 2×2 matrix and $B \cdot A$ is 3×3 matrix.

Even if $A \cdot B$ and $B \cdot A$ are defined and of same order, $A \cdot B$ and $B \cdot A$ may not be equal matrices.

Example 8.6

Let

$$A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

Show that $A \cdot B$ and $B \cdot A$ may not be equal matrices.

Solution: We have

$$A \cdot B = \begin{bmatrix} 1 \cdot 2 + 0(-1) & 1 \cdot 1 + 0 \cdot 0 \\ 2 \cdot 2 + (-1)(-1) & 2 \cdot 1 + (-1)0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix}$$

and $B \cdot A = \begin{bmatrix} 2 \cdot 1 + 1 \cdot 2 & 2 \cdot 0 + 1(-1) \\ (-1)1 + 0 \cdot 2 & (-1)0 + 0(-1) \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ -1 & 0 \end{bmatrix}$

Therefore $A \cdot B \neq A \cdot B$.

THEOREM 8.4 Let A and B be any matrices. Then AB and BA are both defined and are of same order if and only if both A and B are square matrices of same order.

PROOF Let A be an $m \times n$ matrix and B be a $p \times q$ matrix. Suppose that $A \cdot B$ and $B \cdot A$ are defined and of same order. Then $n = p$ and $q = m$. Also, $A \cdot B$ is of order $m \times q$ and $B \cdot A$ is of order $p \times n$. Since $A \cdot B$ and $B \cdot A$ are of same order, we have $m = p$ and $q = n$. Thus

$$m = q = n \quad \text{and} \quad p = n = q$$

Therefore, A and B are square matrices of same order $m \times m$. Converse is clear. ∎

Examples

(1) If A is a 3×4 matrix and B is 4×3 matrix, then $A \cdot B$ is defined and is a matrix of order 3×3. Also, $B \cdot A$ is defined and is a matrix of order 4×4.

(2) If A and B are square matrices each of order 3×3, then AB and BA defined and each of them is of order 3×3.

THEOREM 8.5 The multiplication of matrices is distributive over addition in the following sense.

1. If A is an $m \times n$ matrix and B and C are $n \times p$ matrices, then
$$A(B + C) = A \cdot B + A \cdot C$$
2. If A and B are $m \times n$ matrices and C is an $n \times p$ matrix, then
$$(A + B)C = AC + BC$$

PROOF Let $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$.

1. Suppose that A is of order $m \times n$ and B and C are of order $n \times p$. Then $A(B + C)$ and $A \cdot B + A \cdot C$ are both of order $m \times p$. For any $1 \leq i \leq m$ and $1 \leq j \leq p$,

$$ij\text{th entry in } A(B + C) = \sum_{r=1}^{n} a_{ir}(b_{rj} + c_{rj})$$

$$= \sum_{r=1}^{n} a_{ir} b_{rj} + \sum_{r=1}^{n} a_{ir} c_{rj}$$

$$= ij\text{th entry in } A \cdot B + A \cdot C$$

Therefore, $A(B+C) = AB + AC$. Note that we have used the fact that the multiplication of numbers is distributive over addition of numbers.

2. This can be proved on similar lines. ■

In Definition 8.15, we have defined the multiplication of a matrix by a scalar. A scalar (i.e., a real or complex number) can be identified with a scalar matrix (see Definition 8.10) and the scalar multiplication of a matrix A is actually a multiplication of A with a scalar matrix, as we see below.

THEOREM 8.6 Let A be an $m \times n$ matrix and k a scalar. Then $BA = kA = AC$, where B is the $m \times m$ scalar matrix and C is the $n \times n$ scalar matrix with k as diagonal entries.

PROOF Let B be the $m \times m$ diagonal matrix

$$\begin{bmatrix} k & 0 & 0 & \cdots & 0 \\ 0 & k & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & k \end{bmatrix}$$

and C be the $n \times n$ diagonal matrix

$$\begin{bmatrix} k & 0 & 0 & \cdots & 0 \\ 0 & k & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & k \end{bmatrix}$$

Then BA, kA and AC are all $m \times n$ matrices. For any $1 \le i \le m$ and $1 \le j \le n$,

$$ij\text{th entry in } BA = \sum_{r=1}^{m} b_{ir} a_{rj} = k a_{ij}$$

where $B = (b_{ij})$ and $b_{ij} = k$ if $i \neq j$ and $b_{ij} = 0$ if $i \neq j$. Therefore $BA = kA$. Similarly $AC = kA$. ■

COROLLARY 8.1 If A is any $n \times n$ square matrix and k is an $n \times n$ scalar matrix, then $A \cdot k = k \cdot A$.

The converse of this is proved in the following, that is, we prove next that the scalar matrices are the only matrices which commute with all similar matrices (i.e., matrices of same order).

THEOREM 8.7 Let A be an $n \times n$ square matrix such that $AB = BA$ for all $n \times n$ matrices B. Then A is a scalar matrix.

PROOF Let $A = (a_{ij})$. We shall prove that $a_{ii} = a_{jj}$ for all $1 \le i, j \le n$ and $a_{ij} = 0$ for all $1 \le i \neq j \le n$. Let $1 \le i$, $j \le n$ be fixed and define $B = (b_{st})$ by

$$b_{st} = \begin{cases} 1 & \text{if } \quad s = i \text{ and } t = j \\ 0 & \text{otherwise} \end{cases}$$

Then, since $AB = BA$, by taking ijth entries both in AB and BA, we get that

$$\sum_{r=1}^{n} a_{ir} b_{rj} = \sum_{r=1}^{n} b_{ir} a_{rj}$$

By substituting for b_{rj} and b_{ir}, we get

$$a_{ii} = a_{jj}$$

Therefore, all the diagonal elements in A are same. Also, let $1 \le i \ne j \le n$ and define $C = (c_{st})$ as

$$c_{st} = \begin{cases} 1 & \text{if } s = j = t \\ 0 & \text{if } s \ne j \quad \text{or} \quad t \ne j \end{cases}$$

Again, from $AC = CA$, we get that

$$\sum_{r=1}^{n} a_{ir} c_{rj} = \sum_{r=1}^{n} c_{ir} a_{rj}$$

Substituting for both sides, we have $a_{ij} = 0$ ($c_{ir} = 0$ for all r, since $i \ne j$). Thus, $a_{ij} = 0$ for all $i \ne j$ and hence A is a scalar matrix of order $n \times n$. ■

Theorem 8.8 Let $A = (a_{ij})$ be an $m \times n$ matrix and I_m and I_n be unit matrices of order $m \times m$ and $n \times n$, respectively. Then

$$I_m\, A = A = A\, I_n$$

Proof Note that I_m is the square matrix of order $m \times m$ in which each of the diagonal entries is 1 and all the non-diagonal entries are 0; that is,

$$I_m = (e_{ij})$$

where

$$e_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \ne j \end{cases}$$

Now, for any $1 \le i \le m$ and $1 \le j \le n$, the ijth entry of $I_m A$ is

$$\sum_{r=1}^{m} e_{ir} a_{rj} = a_{ij} \qquad (\text{since } e_{ir} = 0 \text{ for all } r \ne i)$$

Therefore $I_m \cdot A = A$. Similarly $A \cdot I_n = A$. ■

Corollary 8.2 If A is square matrix of order $n \times n$, then $I_n \cdot A = A = A \cdot I_n$.

DEFINITION 8.16 Recall from Definition 8.11 that the matrix I_n is called the *identity matrix* or *unit matrix* order n. In view of Corollary 8.2, I_n is also called the *multiplicative identity* of order n. When there is no ambiguity about n, I_n is simply denoted by I and one has to take the order of I depending on the context where it is used.

When we multiply a matrix A with I, from right or left, A is duplicated. If we multiply A with the zero matrix, we get the zero matrix as in the case of number systems.

Theorem 8.9 Let A be an $m \times n$ matrix and O_m and O_n be zero matrices of order $m \times m$ and $n \times n$ respectively. Then

$$O_m A = O_{m \times n} = A \cdot O_n$$

where $O_{m \times n}$ is the zero matrix of order $m \times n$.

PROOF Recall that O_m is a square matrix of order $m \times m$ in which all the entries are zero. Now, consider

$$\begin{aligned} O_m \cdot A &= (O_m + O_m)A \\ &= O_m A + O_m A \quad \text{[by part (2) of Theorem 8.5]} \end{aligned}$$

Therefore, we have

$$O_m A + O_m A = O_m A = O_m A + O_{m\times n} \quad \text{[by part (3) of Theorem 8.1]}$$

By the cancellation law [part (5) of Theorem 8.1], we get

$$O_m A = O_{m\times n}$$

Similar arguments yield

$$AO_n = O_{m\times n}$$

■

THEOREM 8.10 Let A be $m \times n$ matrix and B and C be $n \times p$ matrices. Then the following hold:

1. $A(-B) = -(AB) = (-A)B$
2. $A(B - C) = AB - AC$
3. $(A - D)B = AB - DB$ for any $m \times n$ matrix D.

PROOF

1. Consider the zero matrix $O_{n\times p}$. Then, we have

$$\begin{aligned} AB + A(-B) &= A[B + (-B)] && \text{[by part (1), Theorem 8.5]} \\ &= A \cdot O_{n\times p} && \text{[by part (4), Theorem 8.1]} \\ &= O_{m\times p} && \text{[by Theorem 8.9]} \\ &= AB + [-(AB)] && \text{[by part (4), Theorem 8.1]} \end{aligned}$$

By the cancellation law [part (5), Theorem 8.1], we get that

$$A(-B) = -(AB)$$

Similar argument gives us that $(-A)B = -(AB)$.

2. We have

$$\begin{aligned} A(B - C) &= A[B + (-C)] \\ &= AB + A(-C) \\ &= AB + (-AC) = AB - AC \end{aligned}$$

3. We have

$$\begin{aligned} (A - D)B &= [A + (-D)]B \\ &= AB + (-D)B \\ &= AB + (-DB) = AB - DB \end{aligned}$$

■

Unlike in the number system, product of two non-zero matrices can be zero. Consider the following example.

Example 8.7

Let A and B be non-zero matrices given by

$$A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

Show that AB and BA are zero matrices.

Solution: We have

$$AB = \begin{bmatrix} 2\times 2 + 2(-2) & 2(-2) + 2\cdot 2 \\ 2\times 2 + 2(-2) & 2(-2) + 2\cdot 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$BA = \begin{bmatrix} 2\times 2 + (-2)2 & 2\cdot 2 + (-2)2 \\ (-2)2 + 2\times 2 & (-2)2 + 2\times 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Further, for two matrices A and B, it is quite possible that $AB = 0$ without BA being zero. In the following we have such an instant.

Example 8.8

Let A and B be 2×2 matrices given by

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Find out AB and BA. Are these zero matrices?

Solution: We have

$$A \cdot B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0\cdot1+1\cdot0 & 0\cdot1+1\cdot0 \\ 0\cdot1+1\cdot0 & 0\cdot1+1\cdot0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$B \cdot A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1\cdot0+1\cdot0 & 1\cdot1+1\cdot1 \\ 0\cdot0+0\cdot0 & 0\cdot1+0\cdot1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$

Therefore $A \cdot B = O$ and $B \cdot A \neq O$.

DEFINITION 8.17 Let A be a square matrix of order $m \times m$. For any non-negative integer n, define A^n recursively as follows:

$$A^n = \begin{cases} I_m & \text{if } n = 0 \\ A^{n-1} \cdot A & \text{if } n > 0 \end{cases}$$

Note that

$$A^1 = A^0 \cdot A = I_m \cdot A = A; \quad A^2 = A^1 \cdot A = A \cdot A; \quad A^3 = A^2 \cdot A = (A \cdot A) \cdot A; \quad \text{etc.}$$

Also $A^2, A^3, \ldots$ are defined only when A is a square matrix.

Example 8.9

Let A be a scalar matrix

$$\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$$

where a is a given scalar. Then, find A^2 and A^n.

Solution: For given A we have

$$A^2 = (a\, I_3) \cdot (a\, I_3) = a(I_3 \cdot a) I_3$$
$$= a(a\, I_3)\, I_3 = a^2\, I_3$$
$$= \begin{bmatrix} a^2 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & a^2 \end{bmatrix}$$

Infact, for any $n \geq 0$,

$$A^n = (a\, I_3)^n = a^n I_3 = \begin{bmatrix} a^n & 0 & 0 \\ 0 & a^n & 0 \\ 0 & 0 & a^n \end{bmatrix}$$

Example

For a non-zero square matrix A and a positive integer n, A^n may be zero, but A may not be zero. For example, consider

$$A = \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}$$

Then $\quad A^2 = \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}$

$$= \begin{bmatrix} 2\cdot 2 + 4(-1) & 2\cdot 4 + 4(-2) \\ (-1)2 + (-2)(-1) & (-1)\cdot 4 + (-2)(-2) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

THEOREM 8.11 Let A and B be square matrices of order $m \times m$ and s a scalar. Then the following hold good:

1. For any integer $n \geq 0$, $(sA)^n = s^n A^n$
2. $(A + B)^2 = A^2 + AB + BA + B^2$
3. $(A - B)^2 = A^2 - AB - BA + B^2$
4. $(A + B)(A - B) = A^2 - AB + BA - B^2$
5. If $AB = BA$, then $(A + B)(A - B) = A^2 - B^2$
6. $A^n \cdot A^r = A^{n+r}$
7. $(A^n)^r = A^{nr}$
8. $(A + B)^3 = A^3 + A^2B + ABA + BAB + B^2A + AB^2 + BA^2 + B^3$
9. If $AB = BA$, then $(A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$

PROOF The proofs of **(1)**, **(6)** and **(7)** are by induction. The others are straightforward verifications and are left to the reader. ■

DEFINITION 8.18 Let $f(x) = a_0 + a_1x + a_2x^2 + \cdots a_nx^n$ be a polynomial in the indeterminate x and the coefficients a_i's be scalars. Then, for any $m \times m$ matrix A, we define

$$f(A) = a_0 + a_1A + a_2A^2 + \cdots + a_nA^n$$

where each a_i is treated as the scalar matrix of order $m \times m$ in which each diagonal entry is a_i and the other entries are 0. Note that $f(A)$ is again an $m \times m$ matrix. $f(A)$ is said to be a *matrix polynomial*.

Example

For any square matrix A and a scalar s

$$(s + A)^n = \sum_{r=0}^{n} {}^nC_r A^{n-r} s^r$$

This is a matrix polynomial and is equal to $f(A)$ where

$$f(x) = (s + x)^n = \sum_{r=0}^{n} {}^nC_r\, A^{n-r} s^r$$

Example

The product of any two diagonal (scalar) matrices of the same order is again a diagonal (scalar) matrix.

If $A = (a_{ij})$ and $B = (b_{ij})$ are diagonal matrices of order $n \times n$, then

$$a_{ij} = 0 = b_{ij} \quad \text{for all } 1 \leq i \neq j \leq n$$

The ijth entry in the product AB is

$$\sum_{r=1}^{n} a_{ir}b_{rj} = a_{ii}b_{ij} = \begin{cases} a_{ii}b_{ii} & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Therefore, except the diagonal entries, all the other entries in AB are zero. Therefore AB is a diagonal matrix. If A and B are scalar matrices, then

$$a_{ii} = b_{jj} \quad \text{and} \quad b_{ii} = b_{jj} \quad \text{for all } 1 \leq i, j \leq n$$

and hence $a_{ii}b_{ii} = a_{jj} \cdot b_{jj}$ and this shows that AB is also a scalar matrix.

Example 8.10

Solve the matrix equation

$$XA = B$$

where

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

Solution: Since A and B are 2×2 matrices, to satisfy the equation $XA = B$, X also must be a 2×2 matrix. Let

$$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

From $XA = B$, we have

$$\begin{bmatrix} a+2b & 2a+3b \\ c+2d & 2c+3d \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

Equating the corresponding entries on both sides, we have

$$a + 2b = 2; \quad c + 2d = 1; \quad 2a + 3b = 1; \quad 2c + 3d = 3$$

Solving these, we get that $a = -4$, $b = 3$, $c = 3$ and $d = -1$. Therefore

$$X = \begin{bmatrix} -4 & 3 \\ 3 & -1 \end{bmatrix}$$

Example 8.11

Let A be the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Prove that

$$A^n = \begin{bmatrix} 4^{n-1} & 4^{n-1} & 4^{n-1} & 4^{n-1} \\ 4^{n-1} & 4^{n-1} & 4^{n-1} & 4^{n-1} \\ 4^{n-1} & 4^{n-1} & 4^{n-1} & 4^{n-1} \\ 4^{n-1} & 4^{n-1} & 4^{n-1} & 4^{n-1} \end{bmatrix}$$

for any positive integer n.

Solution: We shall use induction on n. If $n = 1$, it is clear. Let $m > 1$ and assume that result is true for $n = m - 1$. Then

$$A^m = A^{m-1} \cdot A = \begin{bmatrix} a & a & a & a \\ a & a & a & a \\ a & a & a & a \\ a & a & a & a \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4a & 4a & 4a & 4a \\ 4a & 4a & 4a & 4a \\ 4a & 4a & 4a & 4a \\ 4a & 4a & 4a & 4a \end{bmatrix} \quad (\text{where } a = 4^{m-2})$$

$$= \begin{bmatrix} 4^{m-1} & 4^{m-1} & 4^{m-1} & 4^{m-1} \\ 4^{m-1} & 4^{m-1} & 4^{m-1} & 4^{m-1} \\ 4^{m-1} & 4^{m-1} & 4^{m-1} & 4^{m-1} \\ 4^{m-1} & 4^{m-1} & 4^{m-1} & 4^{m-1} \end{bmatrix}$$

Example 8.12

Two persons X and Y wanted to purchase onions, tomatoes and potatoes for each of their families. The quantities in kilograms of each of the items required by X and Y are given in the table below:

	Onions	Tomatoes	Potatoes
X	12	6	6
Y	10	4	5

There are two markets in the town, market I and market II. The costs per kilogram of each item in the two markets are given in rupees in the table in the right column:

	Market I	Market II
Onions	10	9
Tomatoes	6	7
Potatoes	8	9

Prepare a comparison table, showing the probable expenditures of the persons X and Y in the two markets I, and II, and their preferences of I and II. It is given that they can go to only one market each.

Solution: Let A be the person–item matrix, that is

$$A = \begin{bmatrix} 12 & 6 & 6 \\ 10 & 4 & 5 \end{bmatrix}$$

and B be the item–market matrix, that is

$$B = \begin{bmatrix} 10 & 9 \\ 6 & 7 \\ 8 & 9 \end{bmatrix}$$

We have to compute person–market matrix, that is, we have to calculate the product $A \cdot B$. Now A is a 2×3 matrix (2 person – 3 items) and B is a 3×2 matrix (3 items – 2 markets) and therefore $A \cdot B$ is a 2×2 matrix (2 person – 2 markets) given by

$$A \cdot B = \begin{bmatrix} 12 \cdot 10 + 6 \cdot 6 + 6 \cdot 8 & 12 \cdot 9 + 6 \cdot 7 + 6 \cdot 9 \\ 10 \cdot 10 + 4 \cdot 6 + 5 \cdot 8 & 10 \cdot 9 + 4 \cdot 7 + 5 \cdot 9 \end{bmatrix}$$

$$= \begin{bmatrix} 204 & 204 \\ 164 & 163 \end{bmatrix}$$

Therefore the required table is

	Market I	Market II
X	Rs. 204	Rs. 204
Y	164	163

Then X understands that both markets are equally preferable, while Y decides to go to market II, when it is given that the qualities are same in markets I and II.

DEFINITION 8.19 For any matrix A, the *transpose* of A is defined to be the matrix obtained by interchanging the rows and columns in A. The transpose of A is denoted by A^{T} or A'. If $A = (a_{ij})$ is an $m \times n$ matrix, then the transpose A^{T} is an $n \times m$ matrix given by

$$A^{\mathrm{T}} = (a'_{ij})$$

where $a'_{ij} = a_{ji}$. That is, the ijth entry in A becomes the jith entry in A^{T}.

Examples

(1) If $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \end{bmatrix}$, then $A^{\mathrm{T}} = \begin{bmatrix} 1 & 3 \\ 2 & -1 \\ -1 & 2 \end{bmatrix}$

(2) If $A = \begin{bmatrix} 2 & 3 & 4 \\ -4 & -2 & 3 \\ 1 & 6 & 5 \\ 2 & -3 & -1 \end{bmatrix}$, then $A^{\mathrm{T}} = \begin{bmatrix} 2 & -4 & 1 & 2 \\ 3 & -2 & 6 & -3 \\ 4 & 3 & 5 & -1 \end{bmatrix}$

THEOREM 8.12 Let A and B be $m \times n$ matrices and s a scalar. Then

1. $(A + B)^{\mathrm{T}} = A^{\mathrm{T}} + B^{\mathrm{T}}$
2. $(sA)^{\mathrm{T}} = s \cdot A^{\mathrm{T}}$
3. $(-A)^{\mathrm{T}} = -A^{\mathrm{T}}$

PROOF Let $A = (a_{ij})$ and $B = (b_{ij})$. Then, A, B and $A + B$ are matrices of order $m \times n$ and therefore $A^{\mathrm{T}}, B^{\mathrm{T}}, A^{\mathrm{T}} + B^{\mathrm{T}}$ and $(A + B)^{\mathrm{T}}$ are all $n \times m$ matrices. For any $1 \le i \le n$ and $1 \le j \le m$, ijth entry in

$$\begin{aligned} (A + B)^{\mathrm{T}} &= ji\text{th entry in } A + B \\ &= a_{ji} + b_{ji} \\ &= ij\text{th entry in } A^{\mathrm{T}} + ij\text{th entry in } B^{\mathrm{T}} \\ &= ij\text{th entry in } A^{\mathrm{T}} + B^{\mathrm{T}} \end{aligned}$$

Therefore

$$(A + B)^{\mathrm{T}} = A^{\mathrm{T}} + B^{\mathrm{T}}$$

Similarly, we can prove that $(sA)^{\mathrm{T}} = s \cdot A^{\mathrm{T}}$ and deduce, by taking $s = -1$, that $(-A)^{\mathrm{T}} = -A^{\mathrm{T}}$. ■

Example 8.13

Consider the following matrices:

$$A = \begin{bmatrix} 2 & 3 & 1 \\ -4 & -1 & 5 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & 2 & -3 \\ -2 & 1 & 4 \end{bmatrix}$$

Compute $A^{\mathrm{T}}, B^{\mathrm{T}}, A^{\mathrm{T}}+B^{\mathrm{T}}, A+B$ and $(A+B)^{\mathrm{T}}$.

Solution: We have

$$A^{\mathrm{T}} = \begin{bmatrix} 2 & -4 \\ 3 & -1 \\ 1 & 5 \end{bmatrix} \quad \text{and} \quad B^{\mathrm{T}} = \begin{bmatrix} -1 & -2 \\ 2 & 1 \\ -3 & 4 \end{bmatrix}$$

Now

$$A+B = \begin{bmatrix} 2+(-1) & 3+2 & 1+(-3) \\ -4+(-2) & -1+1 & 5+4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 5 & -2 \\ -6 & 0 & 9 \end{bmatrix}$$

Therefore

$$(A+B)^{\mathrm{T}} = \begin{bmatrix} 1 & -6 \\ 5 & 0 \\ -2 & 9 \end{bmatrix}$$

Also

$$A^{\mathrm{T}}+B^{\mathrm{T}} = \begin{bmatrix} 2+(-1) & -4+(-2) \\ 3+2 & -1+1 \\ 1+(-3) & 5+4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -6 \\ 5 & 0 \\ -2 & 9 \end{bmatrix} = (A+B)^{\mathrm{T}}$$

THEOREM 8.13 Let A be an $m \times n$ matrix and B an $n \times p$ matrix. Then

$$(AB)^{\mathrm{T}} = B^{\mathrm{T}} \cdot A^{\mathrm{T}}$$

PROOF First note that AB is defined, since A and B are of order $m \times n$ and $n \times p$, respectively, and that AB is of order $m \times p$ and hence $(AB)^{\mathrm{T}}$ is of order $p \times m$. Also, since B^{T} and A^{T} are of order $p \times n$ and $n \times m$, respectively; $B^{\mathrm{T}} \cdot A^{\mathrm{T}}$ is defined and is of order $p \times m$. Therefore $(AB)^{\mathrm{T}}$ and $B^{\mathrm{T}}\, A^{\mathrm{T}}$ are both of order $p \times m$. For any $1 \le i \le p$ and $1 \le j \le m$, we have

$$\begin{aligned} ij\text{th entry in } (AB)^{\mathrm{T}} &= ji\text{th entry in } AB \\ &= \sum_{r=1}^{n} a_{jr} b_{ri} \\ &= \sum_{r=1}^{n} b_{ri} \cdot a_{jr} \\ &= \sum_{r=1}^{n} b'_{ir} \cdot a'_{rj} \quad [\text{where } A^{\mathrm{T}} = (a'_{rs}) \text{ and } B^{\mathrm{T}} = (b'_{rs})] \\ &= ij\text{th entry in } B^{\mathrm{T}} \cdot A^{\mathrm{T}} \end{aligned}$$

Thus $(AB)^{\mathrm{T}} = B^{\mathrm{T}} \cdot A^{\mathrm{T}}$. ■

Example 8.14

Consider the following matrices:

$$A = \begin{bmatrix} 2 & -3 & 1 \\ -1 & 4 & -2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & -3 & 1 \\ 2 & -1 & -2 \\ 3 & -2 & -1 \end{bmatrix}$$

Compute $A^{\mathrm{T}}, B^{\mathrm{T}}, A^{\mathrm{T}} \cdot B^{\mathrm{T}}, A \cdot B$ and $(AB)^{\mathrm{T}}$.

Solution: Since A and B are of order 2×3 and 3×3, respectively, AB is defined and is of order 2×3. We have

$$A^{\mathrm{T}} = \begin{bmatrix} 2 & -1 \\ -3 & 4 \\ 1 & -2 \end{bmatrix} \quad \text{and} \quad B^{\mathrm{T}} = \begin{bmatrix} 4 & 2 & 3 \\ -3 & -1 & -2 \\ 1 & -2 & -1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 5 & -5 & 7 \\ -2 & 3 & -7 \end{bmatrix}$$

$$(AB)^{\mathrm{T}} = \begin{bmatrix} 5 & -2 \\ -5 & 3 \\ 7 & 7 \end{bmatrix}$$

Also

$$B^{\mathrm{T}}\, A^{\mathrm{T}} = \begin{bmatrix} 8-6+3 & -4+8-6 \\ -6+3-2 & 3-4+4 \\ 2+6-1 & -1-8+2 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -5 & 3 \\ 7 & -7 \end{bmatrix}$$

Example 8.15

Recall Example 8.12 where two persons X and Y wanted to purchase the items onions, tomatoes and potatoes in markets I or II. The matrix A is given as the person–item matrix and B as the item–market matrix. Instead, suppose we are given the item–person matrix and the market–item matrix. Then we have the following tables.

	X	Y
Onions	12	10
Tomatoes	6	4
Potatoes	6	5

	Onions	Tomatoes	Potatoes
Market I	10	6	8
Market II	9	7	9

These are simply A^{T} and B^{T}, respectively. B^{T} is a 2×3 matrix and A^{T} is the 3×2 matrix. If we take product $B^{\mathrm{T}} \cdot A^{\mathrm{T}}$, then we get the market–person matrix. Now

$$B^{\mathrm{T}} \cdot A^{\mathrm{T}} = \begin{bmatrix} 10 & 6 & 8 \\ 9 & 7 & 9 \end{bmatrix} \begin{bmatrix} 12 & 10 \\ 6 & 4 \\ 6 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 10 \cdot 12 + 6 \cdot 6 + 8 \cdot 6 & 10 \cdot 10 + 6 \cdot 4 + 8 \cdot 5 \\ 9 \cdot 12 + 7 \cdot 6 + 9 \cdot 6 & 9 \cdot 10 + 7 \cdot 4 + 9 \cdot 5 \end{bmatrix}$$

$$= \begin{bmatrix} 204 & 164 \\ 204 & 163 \end{bmatrix} = (A \cdot B)^{\mathrm{T}}$$

The required table is

	X	Y
Market I	204	164
Market II	204	163

DEFINITION 8.20 A matrix A is said to be *symmetric* if it is equal to its transpose, that is, $A = A^{\mathrm{T}}$. If $A = (a_{ij})$, then A is called symmetric if $a_{ij} = a_{ji}$ for all i and j. Note that if A is an $m \times n$ matrix, then A^{T} is an $n \times m$ matrix and therefore A can be symmetric only if A is a square matrix.

Examples

(1) $\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 4 \\ 0 & 4 & 3 \end{bmatrix}$ is a symmetric matrix, because when we interchange the rows and columns, we get the same matrix.

(2) $\begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$ is not a symmetric matrix

(3) $\begin{bmatrix} 2 & 3 & 1 \\ 4 & -2 & -3 \end{bmatrix}$ is not symmetric, since a symmetric matrix is necessarily a square matrix.

QUICK LOOK 2

1. The zero matrix of order $n \times n$ and the identity matrix are both symmetric matrices.
2. Any diagonal matrix is always symmetric.

Note that a square matrix is symmetric if and only if the entries on the lower side of the diagonal are precisely the reflections of those on the upper side of in the diagonal as shown in Figure 8.1.

$$\begin{bmatrix} 4 & 2 & 0 & 1 \\ 2 & -1 & 3 & -2 \\ 0 & 3 & -4 & -1 \\ 1 & -2 & -1 & 5 \end{bmatrix} \leftarrow \text{Diagonal}$$

FIGURE 8.1 Symmetric square matrix.

DEFINITION 8.21 A square matrix $A = (a_{ij})$ is said to *skew-symmetric* if $a_{ij} = -a_{ji}$ for all i and j. In other words, A is skew-symmetric if and only if $A = -A^{\mathrm{T}}$.

Examples

(1) The matrix $\begin{bmatrix} 0 & 2 & 3 & 4 \\ -2 & 0 & 1 & -3 \\ -3 & -1 & 0 & -1 \\ -4 & 3 & 1 & 0 \end{bmatrix}$ is skew-symmetric.

(2) The matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & 0 & 1 & 3 \\ -3 & -1 & 0 & 2 \\ -4 & -3 & -2 & 0 \end{bmatrix}$ is not skew-symmetric since $a_{11} \neq -a_{11}$.

In the following we derive certain important properties of symmetric matrices and skew-symmetric matrices.

THEOREM 8.14 Let A and B be any $n \times n$ square matrices.

1. If A and B are symmetric, then so is $A \pm B$.
2. If A and B are skew-symmetric, then so is $A \pm B$.
3. If $AB = BA$ and A and B are symmetric (skew-symmetric), then AB is symmetric.
4. If A is symmetric (skew-symmetric), then so is sA for any scalar s.

PROOF Recall that A is symmetric if and only if $A = A^{\mathrm{T}}$ and that A is skew-symmetric if and only if $A = -A^{\mathrm{T}}$.

1. Suppose that A and B are symmetric. Then, by Theorem 8.12, we have
$$(A \pm B)^{\mathrm{T}} = A^{\mathrm{T}} \pm B^{T} = A \pm B$$
and therefore $A \pm B$ is also symmetric.
2. If A and B are skew-symmetric matrices, then
$$(A \pm B)^{\mathrm{T}} = A^{\mathrm{T}} \pm B^{T} = -A \mp B = -(A \pm B)$$
and therefore $A \pm B$ is skew-symmetric.
3. **Case I:** Suppose A and B are symmetric and $AB = BA$. Then
$$(AB)^{\mathrm{T}} = B^{\mathrm{T}} A^{\mathrm{T}} = B \cdot A = AB$$
and therefore AB is symmetric.
Case II: If A and B are skew-symmetric and $AB = BA$, then
$$(AB)^{\mathrm{T}} = B^{\mathrm{T}} A^{\mathrm{T}} = (-B)(-A) = BA = AB$$
and therefore AB is symmetric.
4. **Case I:** If A is symmetric and s is a scalar, then
$$(sA)^{\mathrm{T}} = sA^{\mathrm{T}} = sA$$
and hence sA is symmetric.
Case II: If A is skew-symmetric, then
$$(sA)^{\mathrm{T}} = sA^{\mathrm{T}} = s(-A) = -(sA)$$
and hence sA is skew-symmetric. ■

Note that, if A and B are skew-symmetric and $AB = BA$, then AB is not a skew-symmetric; however, AB is symmetric. In this context, we have the following.

THEOREM 8.15 Let A and B be square matrices of same order such that $AB = BA$. If one of A and B is symmetric and the other is skew-symmetric, then AB is skew-symmetric.

PROOF Suppose that A is symmetric and B is skew-symmetric (there is no loss of generality, since $AB = BA$). Then

$$(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}} = (-B)A = -(BA) = -(AB)$$

and therefore AB is skew-symmetric. ■

THEOREM 8.16 Let A be a square matrix. Then A is symmetric (skew-symmetric) if and only if A^{T} is symmetric (skew-symmetric).

PROOF This follows from the fact that $(A^{\mathrm{T}})^{\mathrm{T}} = A$. Also since A is symmetric

$$A = A^{\mathrm{T}} \Rightarrow A^{\mathrm{T}} = A = (A^{\mathrm{T}})^{\mathrm{T}}$$

and

$$A = -A^{\mathrm{T}} \Rightarrow A^{\mathrm{T}} = -A = -(A^{\mathrm{T}})^{\mathrm{T}}$$ ■

THEOREM 8.17 If A is a skew-symmetric matrix, then all the diagonal entries in A are zero.

PROOF Let $A = (a_{ij})$ be a skew-symmetric matrix. Then $a_{ij} = -a_{ji}$ for all i and j. In particular, $a_{ii} = -a_{ii}$ and hence $2a_{ii} = 0$ or $a_{ii} = 0$ for all i. Therefore all the diagonal entries a_{ii} are zero. ■

Note: The converse of Theorem 8.17 is not true. For example the matrix

$$\begin{bmatrix} 0 & 1 & 2 \\ 3 & 0 & 4 \\ 1 & -1 & 0 \end{bmatrix}$$

is not skew-symmetric.

THEOREM 8.18 For any square matrix A, $A + A^{\mathrm{T}}$ is symmetric and $A - A^{\mathrm{T}}$ is skew-symmetric.

PROOF Let A be a square matrix. Then

$$(A + A^{\mathrm{T}})^{\mathrm{T}} = A^{\mathrm{T}} + (A^{\mathrm{T}})^{\mathrm{T}} = A^{\mathrm{T}} + A = A + A^{\mathrm{T}}$$

and hence $A + A^{\mathrm{T}}$ is symmetric. Also,

$$\begin{aligned}(A - A^{\mathrm{T}}) &= [A + (-A^{\mathrm{T}})]^{\mathrm{T}} \\ &= A^{\mathrm{T}} + (-A^{\mathrm{T}})^{\mathrm{T}} = A^{\mathrm{T}} + [-(A^{\mathrm{T}})^{\mathrm{T}}] \\ &= A^{\mathrm{T}} + (-A) \\ &= A^{\mathrm{T}} - A = -(A - A^{\mathrm{T}})\end{aligned}$$

Therefore $A - A^{\mathrm{T}}$ is skew-symmetric. ■

THEOREM 8.19 Any square matrix can be uniquely expressed as a sum of a symmetric matrix and a skew-symmetric matrix.

PROOF Let A be any square matrix. Then, by Theorem 8.18, $A + A^{\mathrm{T}}$ is symmetric and $A - A^{\mathrm{T}}$ is skew-symmetric. Also, by part (4) of Theorem 8.14 we have

$$\frac{1}{2}(A + A^{\mathrm{T}}) \text{ is symmetric}$$

and

$$\frac{1}{2}(A - A^{\mathrm{T}}) \text{ is skew-symmetric}$$

Now, we have

$$A = \frac{1}{2}(A + A^{\mathrm{T}}) + \frac{1}{2}(A - A^{\mathrm{T}}) \tag{8.11}$$

To prove the uniqueness of this expression, let $A = B + C$, where B is a symmetric matrix and C is a skew-symmetric matrix. Then

$$\begin{aligned} A + A^{\mathrm{T}} &= (B + C) + (B + C)^{\mathrm{T}} \\ &= B + C + B^{\mathrm{T}} + C^{\mathrm{T}} \\ &= B + C + B - C \\ &= 2B \end{aligned}$$

and therefore

$$B = \frac{1}{2}(A + A^{\mathrm{T}})$$

Also,

$$\begin{aligned} A - A^{\mathrm{T}} &= (B + C) - (B + C)^{\mathrm{T}} \\ &= B + C - (B^{\mathrm{T}} + C^{\mathrm{T}}) \\ &= B + C - (B - C) \\ &= 2C \end{aligned}$$

and therefore

$$C = \frac{1}{2}(A - A^{\mathrm{T}})$$

Thus, Eq. (8.11) is an unique expression of A as a sum of a symmetric matrix and a skew-symmetric matrix. ■

THEOREM 8.20 Let A and B be symmetric matrices of the same order. Then the following hold:

1. A^n is symmetric for all positive integers n.
2. AB is symmetric if and only if $AB = BA$.
3. $AB + BA$ is symmetric.
4. $AB - BA$ is skew-symmetric.

PROOF

1. For any positive integer n,

$$(A^n)^{\mathrm{T}} = (A \cdots A)^{\mathrm{T}} = A^{\mathrm{T}} \cdots A^{\mathrm{T}} = A \cdots A = A^n$$

and therefore A^n is symmetric.

2. $$\begin{aligned} AB \text{ is symmetric} &\Leftrightarrow (AB)^{\mathrm{T}} = AB \\ &\Leftrightarrow B^{\mathrm{T}} A^{\mathrm{T}} = AB \\ &\Leftrightarrow BA = AB \end{aligned}$$

3. $$\begin{aligned} (AB + BA)^{\mathrm{T}} &= (AB)^{\mathrm{T}} + (BA)^{\mathrm{T}} \\ &= B^{\mathrm{T}} A^{\mathrm{T}} + A^{\mathrm{T}} B^{\mathrm{T}} \\ &= BA + AB = AB + BA \end{aligned}$$

Therefore $AB + BA$ is symmetric.

4. $$\begin{aligned} (AB - BA)^{\mathrm{T}} &= (AB)^{\mathrm{T}} - (BA)^{\mathrm{T}} \\ &= B^{\mathrm{T}} A^{\mathrm{T}} - A^{\mathrm{T}} B^{\mathrm{T}} \\ &= BA - AB = -(AB - BA) \end{aligned}$$

Therefore $AB - BA$ is skew-symmetric. ■

THEOREM 8.21 For any square matrix A, AA^{T} and $A^{\mathrm{T}}A$ are both symmetric.

PROOF We have

$$(AA^{\mathrm{T}})^{\mathrm{T}} = (A^{\mathrm{T}})^{\mathrm{T}} A^{\mathrm{T}} = AA^{\mathrm{T}}$$

and

$$(A^{\mathrm{T}}A)^{\mathrm{T}} = A^{\mathrm{T}}(A^{\mathrm{T}})^{\mathrm{T}} = A^{\mathrm{T}}A$$

Therefore AA^{T} and $A^{\mathrm{T}}A$ are both symmetric. ■

Example 8.16

Express the matrix A as a sum of a symmetric matrix and a skew-symmetric matrix.

$$A = \begin{bmatrix} 2 & 3 & 1 \\ -1 & -2 & 4 \\ 5 & -3 & -5 \end{bmatrix}$$

Solution: To do this, we should compute

$$\frac{1}{2}(A + A^{\mathrm{T}}) \quad \text{and} \quad \frac{1}{2}(A - A^{\mathrm{T}})$$

Now transpose of A is given by

$$A^{\mathrm{T}} = \begin{bmatrix} 2 & -1 & 5 \\ 3 & -2 & -3 \\ 1 & 4 & -5 \end{bmatrix}$$

For A^{T}, first row of A becomes the first column of A^{T}, the ith row of A becomes the ith column of A^{T}. Now

$$A + A^{\mathrm{T}} = \begin{bmatrix} 2+2 & 3+(-1) & 1+5 \\ -1+3 & -2+(-2) & 4+(-3) \\ 5+1 & -3+4 & -5+(-5) \end{bmatrix} = \begin{bmatrix} 4 & 2 & 6 \\ 2 & -4 & 1 \\ 6 & 1 & -10 \end{bmatrix}$$

$$\frac{1}{2}(A + A^{\mathrm{T}}) = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -2 & \frac{1}{2} \\ 3 & \frac{1}{2} & -5 \end{bmatrix} \tag{8.12}$$

Again

$$A - A^{\mathrm{T}} = \begin{bmatrix} 2-2 & 3-(-1) & 1-5 \\ -1-3 & -2-(-2) & 4-(-3) \\ 5-1 & -3-4 & -5-(-5) \end{bmatrix} = \begin{bmatrix} 0 & 4 & -4 \\ -4 & 0 & 7 \\ 4 & -7 & 0 \end{bmatrix}$$

$$\frac{1}{2}(A - A^{\mathrm{T}}) = \begin{bmatrix} 0 & 2 & -2 \\ -2 & 0 & \frac{7}{2} \\ 2 & \frac{-7}{2} & 0 \end{bmatrix} \tag{8.13}$$

By Theorem 8.19 and using Eqs. (8.12) and (8.13) we have

$$A = \frac{1}{2}(A + A^{\mathrm{T}}) + \frac{1}{2}(A - A^{\mathrm{T}})$$

$$= \underbrace{\begin{bmatrix} 2 & 1 & 3 \\ 1 & -2 & \frac{1}{2} \\ 3 & \frac{1}{2} & -5 \end{bmatrix}}_{\text{symmetric}} + \underbrace{\begin{bmatrix} 0 & 2 & -2 \\ -2 & 0 & \frac{7}{2} \\ 2 & \frac{-7}{2} & 0 \end{bmatrix}}_{\text{skew-symmetric}}$$

Example

Let A and B be square matrices of same order. If B is symmetric (skew-symmetric), then so is ABA^{T}. That is

$$(ABA^{\mathrm{T}})^{\mathrm{T}} = (A^{\mathrm{T}})^{\mathrm{T}} B^{\mathrm{T}} A^{\mathrm{T}}$$

$$= AB^{\mathrm{T}}A^{\mathrm{T}}$$

$$= \begin{cases} ABA^{\mathrm{T}} & \text{if } B \text{ is symmetric} \\ A(-B)A^{\mathrm{T}} & \text{if } B \text{ is skew-symmetric} = -ABA^{\mathrm{T}} \end{cases}$$

DEFINITION 8.22 A square matrix A is said to be an *orthogonal matrix* if $A^{\mathrm{T}} \cdot A = I$, where I is the identity matrix of order same as that of A.

Example 8.17

Prove that the following matrices are orthogonal:

(1) $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$

(2) $A = \begin{bmatrix} \sin\alpha & \cos\alpha \\ -\cos\alpha & \sin\alpha \end{bmatrix}$

Solution:

(1) Consider the matrix

$$A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

Then

$$A^{\mathrm{T}} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

and

$$A^{\mathrm{T}} \cdot A = \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & \cos\alpha\sin\alpha - \sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha - \cos\alpha\sin\alpha & \sin^2\alpha + \cos^2\alpha \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Therefore, A is an orthogonal matrix.

(2) Let

$$A = \begin{bmatrix} \sin\alpha & \cos\alpha \\ -\cos\alpha & \sin\alpha \end{bmatrix}$$

Then

$$A^{\mathrm{T}} = \begin{bmatrix} \sin\alpha & -\cos\alpha \\ \cos\alpha & \sin\alpha \end{bmatrix}$$

and

$$A^{\mathrm{T}} A = \begin{bmatrix} \sin^2\alpha + \cos^2\alpha & \sin\alpha\cos\alpha - \cos\alpha\sin\alpha \\ \cos\alpha\sin\alpha - \sin\alpha\cos\alpha & \cos^2\alpha + \sin^2\alpha \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Therefore, A is an orthogonal matrix.

Example 8.18

Let

$$A = \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}$$

Find the values of a, b and c such that A is an orthogonal matrix.

Solution: For the given matrix we have

$$A^{\mathrm{T}} = \begin{bmatrix} 0 & a & a \\ 2b & b & -b \\ c & -c & c \end{bmatrix}$$

Multiplying A^{T} with A we get

$A^{\mathrm{T}} A$

$$= \begin{bmatrix} (0 + a^2 + a^2) & (0 \cdot 2b + ab - ab) & (0 \cdot c - ca + ca) \\ (2b \cdot 0 + ba - ba) & [2b \cdot 2b + b \cdot b + (-b)(-b)] & [2b \cdot c + b(-c) + (-b)c] \\ [c \cdot 0 + (-c)a + c \cdot a] & [2b \cdot c + (-c)b + c(-b)] & [c^2 + (-c)(-c) + c^2] \end{bmatrix}$$

$$= \begin{bmatrix} 2a^2 & 0 & 0 \\ 0 & 6b^2 & 0 \\ 0 & 0 & 3c^2 \end{bmatrix}$$

A is orthogonal if and only if

$$A^{\mathrm{T}} A = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

that is,

$$a = \pm\frac{1}{\sqrt{2}}, \quad b = \pm\frac{1}{\sqrt{6}} \quad \text{and} \quad c = \pm\frac{1}{\sqrt{3}}$$

DEFINITION 8.23 The following operations on matrices are called *elementary transformations* or *elementary operations:*

1. The interchange of any two rows (or columns).
2. The multiplication of any row (or column) by a non-zero scalar.
3. The addition to the entries of a row (or column) the corresponding entries of any other row (or column) multiplied by a non-zero scalar.

There are totally six types of elementary transformations on a matrix, three types are due to rows and three types due to columns. An elementary transformation is called a *row transformation* or a *column transformation* according as it applies to rows or columns, respectively. We follow a fixed notation to describe these six elementary transformations as detailed below.

1. We denote the elementary transformations of interchanging the ith row and jth row by $R_i \leftrightarrow R_j$. For example, for a matrix

$$A = \begin{bmatrix} 2 & 3 & 1 & 4 \\ -1 & 2 & -3 & 2 \\ 3 & 4 & 2 & 1 \end{bmatrix}$$

applying the elementary transformation $R_2 \leftrightarrow R_3$, that is, interchanging the second row and third row, we get the matrix

$$\begin{bmatrix} 2 & 3 & 1 & 4 \\ 3 & 4 & 2 & 1 \\ -1 & 2 & -3 & 2 \end{bmatrix}$$

2. The elementary transformation of interchanging the ith column and jth column is denoted by $C_i \leftrightarrow C_j$. For example, by applying the transformation $C_1 \leftrightarrow C_3$ to the matrix A given above we get the matrix

$$\begin{bmatrix} 1 & 3 & 2 & 4 \\ -3 & 2 & -1 & 2 \\ 2 & 4 & 3 & 1 \end{bmatrix}$$

3. The elementary transformation of multiplying the entries in the ith row by a non-zero scalar s is denoted by $R_i \to sR_i$. For example, application of the transformation $R_3 \to 2R_3$ to the matrix

$$A = \begin{bmatrix} 2 & 3 & 1 & 4 \\ -1 & 2 & -3 & 2 \\ 3 & 4 & 2 & 1 \end{bmatrix}$$

gives us the matrix

$$\begin{bmatrix} 2 & 3 & 1 & 4 \\ -1 & 2 & -3 & 2 \\ 6 & 8 & 4 & 2 \end{bmatrix}$$

4. The elementary transformation of multiplying the entries of the ith column by a non-zero scalar s is denoted by $C_i \leftrightarrow sC_i$. For example, the application at $C_3 \leftrightarrow 2C_3$ to the matrix A given above gives us the matrix

$$\begin{bmatrix} 2 & 3 & 2\cdot 1 & 4 \\ -1 & 2 & 2(-3) & 2 \\ 3 & 4 & 2\cdot 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 2 & 4 \\ -1 & 2 & -6 & 2 \\ 3 & 4 & 4 & 1 \end{bmatrix}$$

5. The transformation of adding to the entries in ith row, the corresponding entries of the jth row multiplied by a non-zero scalar s is denoted by $R_i \to R_i + sR_j$. For example, when we apply the transformation $R_2 \to R_2 + 3R_1$ to the matrix

$$A = \begin{bmatrix} 2 & 3 & 1 & 4 \\ -1 & 2 & -3 & 2 \\ 3 & 4 & 2 & 1 \end{bmatrix}$$

we get the matrix

$$\begin{bmatrix} 2 & 3 & 1 & 4 \\ -1+3\cdot 2 & 2+3\cdot 3 & -3+3\cdot 1 & 2+3\cdot 4 \\ 3 & 4 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 & 4 \\ 5 & 11 & 0 & 14 \\ 3 & 4 & 2 & 1 \end{bmatrix}$$

6. Lastly, the transformation of adding to the entries in ith column, the corresponding entries in the jth column multiplied by a non-zero scalar s is denoted by $C_i \to C_i + sC_j$. For example, the application of $C_2 \to C_2 + 3C_1$ to the matrix A in (5) gives us the matrix

$$\begin{bmatrix} 2 & 3+3\cdot 2 & 1 & 4 \\ -1 & 2+3(-1) & -3 & 2 \\ 3 & 4+3\cdot 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 9 & 1 & 4 \\ -1 & -1 & -3 & 2 \\ 2 & 13 & 2 & 1 \end{bmatrix}$$

DEFINITION 8.24 Let A and B be two matrices of the same order and *et* be an elementary transformation (i.e., *et* is any six transformations described above). Then we write $A \xrightarrow{et} B$ to denote that B is obtained by applying the elementary transformation *et* to A. For example

$$A \xrightarrow{R_1 \leftrightarrow R_2} B$$

denotes that B is obtained by interchanging the ith row and jth row in A. It can be easily seen that

$$A \xrightarrow{et} B \text{ implies } B \xrightarrow{f} A$$

where f is another elementary transformation, which may be called the *inverse transformation* of *et*.

For example, $R_i \leftrightarrow R_j$ (or $C_i \leftrightarrow C_j$) is inverse of itself. $R_i \to (1/s)R_i$ $[C_i \to (1/s)C_i]$ is the inverse of $R_i \to sR_i (C_i \to sC_i)$. Also, $R_i \to R_i + sR_j$ is the inverse of $R_i \to R_i + (-s)R_j$ and $C_i \to C_i + sC_j$ is the inverse of $C_i \to C_i + (-s)C_j$. In other words, the inverse of an elementary transformation is again an elementary transformation.

DEFINITION 8.25 A square matrix A is said to be an *elementary row* (*column*) *matrix* if it is obtained by applying an elementary row (column) transformation to the identity matrix I, that is, $I \xrightarrow{et} A$.

Examples

(1) The matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

is an elementary matrix, since it is obtained by interchanging the second and third rows in the identity matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(2) Let

$$E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Then E is an elementary column matrix, since $I \xrightarrow{C_1 \leftrightarrow C_3} E$.

(3) Let

$$E = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then $I \xrightarrow{C_3 \to C_3 + 3C_1} E$ and hence E is an elementary column matrix.

(4) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to 4R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
and therefore this is an elementary row matrix.

The following theorem is a straightforward verification and is left for the reader.

Try it out

THEOREM 8.22 Let A and B square matrices of same order.

1. For any elementary row transformation *ert*, $A \xrightarrow{ert} B$ if and only if $EA = B$, where E is the elementary matrix for which $I \xrightarrow{ert} E$.
2. For any elementary column transformation *ect*, $A \xrightarrow{ect} B$ if and only if $B = AE$, where E is the elementary matrix for which $I \xrightarrow{ect} E$.

Examples

(1) Let

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 2 & 3 & -2 \\ -1 & -2 & -3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 1 & 4 \\ 2 & 3 & -2 \\ 3 & 4 & -7 \end{bmatrix}$$

Then $A \xrightarrow{R_3 \to R_3 + 2R_2} B$ and

$$B = \begin{bmatrix} 3 & 1 & 4 \\ 2 & 3 & -2 \\ 3 & 4 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 4 \\ 2 & 3 & -2 \\ -1 & -2 & -3 \end{bmatrix} = EA$$

where

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \quad \text{and} \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 + 2R_1} E$$

(2) Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 11 & 3 \\ 4 & 23 & 6 \\ 7 & 35 & 9 \end{bmatrix}$$

Then $A \xrightarrow{C_2 \to C_2 + 3C_3} B$ and

$$B = \begin{bmatrix} 1 & 11 & 3 \\ 4 & 23 & 6 \\ 7 & 35 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} = A \cdot E$$

where

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \quad \text{and} \quad I \xrightarrow{C_2 \to C_2 + 3C_3} E$$

DEFINITION 8.26 Two matrices A and B of same order are said to be *similar* or *equivalent* if one can be obtained from the other by applying a finite number of elementary transformations. If A and B are similar or equivalent, we denote this by $A \sim B$.

In other words $A \sim B$ if there exist finite number of matrices $B_1, B_2, \ldots, B_n = B$ such that

$$A \xrightarrow{f_1} B_1 \xrightarrow{f_2} B_2 \xrightarrow{f_3} B_3 \xrightarrow{f_4} \cdots \longrightarrow B_{n-1} \xrightarrow{f_n} B_n = B$$

where $f_1, f_2, \ldots, f_n$ are some elementary transformations. We assume the validity of the following theorem without going to the intracacies of the proof.

THEOREM 8.23 Being similar is an equivalence relation on the class of all matrices, that is, for any matrices A, B and C, the following hold:

1. $A \sim A$
2. $A \sim B$ and $B \sim C \Rightarrow A \sim C$
3. $A \sim B \Rightarrow B \sim A$

Examples

(1) Let $A = \begin{bmatrix} 3 & 5 & 9 \\ 2 & 6 & 4 \\ 1 & 2 & 3 \end{bmatrix}$. Then

$$A \xrightarrow{C_3 \to C_3 - 3C_1} \begin{bmatrix} 3 & 5 & 0 \\ 2 & 6 & -2 \\ 1 & 4 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 3 & 5 & 0 \\ -1 & 1 & -2 \\ 1 & 4 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2 \to R_2 + R_3} \begin{bmatrix} 3 & 5 & 0 \\ 0 & 5 & -2 \\ 1 & 4 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 - R_3} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 5 & -2 \\ 1 & 4 & 0 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - 2R_1} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 5 & -2 \\ -3 & 2 & 0 \end{bmatrix}$$

$$\xrightarrow{C_2 \to C_2 + \frac{5}{2}C_3} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & -2 \\ -3 & 2 & 0 \end{bmatrix} = B(\text{say})$$

Therefore $A \sim B$.

(2) $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 + 3R_1} \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\xrightarrow{C_3 \to C_3 + 4C_1} \begin{bmatrix} 1 & 0 & 4 \\ 3 & 1 & 12 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + 2R_2} \begin{bmatrix} 1 & 0 & 4 \\ 3 & 1 & 12 \\ 6 & 2 & 25 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 + R_3} \begin{bmatrix} 7 & 2 & 29 \\ 3 & 1 & 12 \\ 6 & 2 & 25 \end{bmatrix} = A, \text{ say.}$$

Therefore $I_3 \sim A$.

(3) Consider

$$A = \begin{bmatrix} 7 & 2 & 29 \\ 3 & 1 & 12 \\ 6 & 2 & 25 \end{bmatrix} \xrightarrow{R_1 \to R_1 - R_3} \begin{bmatrix} 1 & 0 & 4 \\ 3 & 1 & 12 \\ 6 & 2 & 25 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - 2R_2} \begin{bmatrix} 1 & 0 & 4 \\ 3 & 1 & 12 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{C_3 \to C_3 - 4C_1} \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 \to R_2 - 3R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

Therefore $A \sim I_3$.

DEFINITION 8.27 A square matrix A of order n is said to be *invertible* or *non-singular* if there exists a square matrix B of order n such that

$$AB = I_n = BA$$

where I_n is the identity matrix of order n, B is called *inverse* of A and is denoted by A^{-1}.

THEOREM 8.24 For any $n \times n$ matrix A, there is atmost one inverse of A.

PROOF Suppose that B and C are inverses of A. Then $AB = I_n = BA$ and $AC = I_n = CA$. Now,

$$B = I_n B = (CA)B = C(AB) = CI_n = C$$

Therefore $B = C$ or there is atmost one inverse of A. ■

Example

Let

$$A = \begin{bmatrix} 2 & -1 & 3 \\ -5 & 3 & 1 \\ -3 & 2 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -7 & -9 & 10 \\ -12 & -15 & 17 \\ 1 & 1 & -1 \end{bmatrix}$$

Then

AB

$$= \begin{bmatrix} [2(-7)+(-1)(-12)+3\cdot 1] & [2(-9)+(-1)(-15)+3\cdot 1] & [2\cdot 10+(-1)17+3(-1)] \\ [(-5)(-7)+3(-12)+1\cdot 1] & [(-5)(-9)+3(-15)+1\cdot 1] & [(-5)\cdot 10+3\cdot 17+1(-1)] \\ [(-3)(-7)+2(-12)+3\cdot 1] & [(-3)(-9)+2(-15)+1\cdot 1] & [(-3)\cdot 10+2\cdot 17+3(-1)] \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

Similarly, $BA = I_3 = AB$. Therefore A is invertible and $A^{-1} = B$.

Before going to find an algorithm to find the inverse, if it exists, of a square matrix, we have the following.

THEOREM 8.25 Let A and B be square matrices of the same order.

1. For any elementary row transformation f,

$$f(AB) = f(A)B$$

2. For any elementary column transformation g

$$g(AB) = A \cdot g(B)$$

PROOF **1.** By part (1) of Theorem 8.22, we have

$$f(AB) = E(AB) = (EA)B = f(A)B$$

where $E = f(I)$.

2. Again by part (2) of Theorem 8.22, we have

$$g(AB) = (AB)E = A(BE) = Ag(B)$$

where $E = g(I)$ ■

THEOREM 8.26 Let A and B be invertible square matrices of the same order. Then AB is invertible and

$$(AB)^{-1} = B^{-1} \cdot A^{-1}$$

PROOF We have $A \cdot A^{-1} = I = A^{-1} \cdot A$ and $BB^{-1} = I = B^{-1}B$. Now

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

and $$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$$

and hence AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$. ■

In the following, we prove that, for any invertible matrix, the operations of taking transpose and inverse commute with each other.

Theorem 8.27 A square matrix A is invertible if and only if its transpose is invertible and, in this case

$$(A^{\mathrm{T}})^{-1} = (A^{-1})^{\mathrm{T}}$$

That is, the transpose of the inverse of A is the inverse of transpose of A.

Proof Suppose that A is invertible. Then

$$A \cdot A^{-1} = I = A^{-1} \cdot A$$

Taking transposes, we get

$$(A^{-1})^{\mathrm{T}} A^{\mathrm{T}} = (AA^{-1})^{\mathrm{T}} = I^{\mathrm{T}} = I = (A^{-1}A)^{\mathrm{T}} = A^{\mathrm{T}} \cdot (A^{-1})^{\mathrm{T}}$$

Therefore, A^{T} is invertible and its inverse is $(A^{-1})^{\mathrm{T}}$ and therefore $(A^{\mathrm{T}})^{-1} = (A^{-1})^{\mathrm{T}}$. The converse follows from the facts that $(A^{\mathrm{T}})^{\mathrm{T}} = A$ and $(A^{-1})^{-1} = A$. ■

Theorem 8.28 Every elementary matrix is invertible.

Proof First recall, from the discussion made after Definition 8.24, that any elementary row (or column) transformation has an inverse which is again an elementary row (respectively column) transformation. If E is an elementary matrix of order $n \times n$, then

$$E = f(I)$$

for a suitable elementary row (or column) transformation f, where I is the identity matrix of order n. Then f^{-1} is also an elementary row (or column) transformation. Let

$$F = f^{-1}(I)$$

Suppose that f is an elementary row transformation. Then, by part (1) of Theorem 8.25, we have

$$E \cdot F = f(I)F = f(IF) = f(F) = f(f^{-1}(I)) = I$$

and $$FE = f^{-1}(I)E = f^{-1}(IE) = f^{-1}(E) = f^{-1}(f(I)) = I$$

If f is a column transformation, again by part (2) of Theorem 8.25, we have

$$E \cdot F = E \cdot f^{-1}(I) = f^{-1}(EI) = f^{-1}(E) = f^{-1}(f(I)) = I$$

and $$F \cdot E = F \cdot f(I) = f(FI) = f(F) = f(f^{-1}(I)) = I$$

Thus $EF = I = FE$ and hence E is invertible and F is the inverse of E. ■

Theorem 8.29 Let A be an invertible matrix. Then in each row (and in each column) there is atleast one non-zero entry.

Proof Let $A = (a_{ij})$ and $A^{-1} = (b_{ij})$. Then, for each i, the ith diagonal entry (i.e., iith entry) in $A \cdot A^{-1} (= I)$ is 1 and hence

$$1 = \sum_{r=1}^{n} a_{ir}\, a_{ri}$$

Therefore $a_{ir} \neq 0$ for some r (otherwise, the above sum becomes 0). Similarly, for each j,

$$a_{rj} \neq 0 \quad \text{for some } r$$ ■

Theorem 8.30 Let A be a square matrix of order n. Then A is invertible if and only if A and I_n are similar.

Proof Suppose that A and I_n are similar. Then there exist finite number of matrices $B_1, B_2, \ldots, B_m = I_n$ such that

$$A \xrightarrow{f_1} B_1 \xrightarrow{f_2} B_2 \xrightarrow{f_3} \cdots \xrightarrow{f_{m-1}} B_{m-1} \xrightarrow{f_m} B_m = I_n$$

where each f_i is either an elementary row transformation or an elementary column transformation. Then, by Theorem 8.22,

$$B_i = R\,B_{i-1} \quad \text{or} \quad B_i = B_{i-1}C$$

for each i, where R is an elementary row matrix or C is an elementary column matrix, according as whether f_i is a row transformation or column transformation, respectively. Therefore, there exists elementary row matrices $R_1, R_2, \ldots, R_s$ and elementary column matrices $C_1, C_2, \ldots, C_t$ such that

$$I_n = (R_1\,R_2 \cdots R_s)\,A\,(C_1\,C_2 \cdots C_t), \quad s+t=m$$

Put $R = R_1\,R_2 \cdots R_s$ and $C = C_1\,C_2 \cdots C_t$. Then, by Theorems 8.28 and 8.26, R and C are invertible matrices and $I_n = RAC$. Hence

$$A = R^{-1}\,I_n C^{-1} = R^{-1}C^{-1} = (CR)^{-1}$$

Thus, A is invertible.

Conversely, suppose that A is invertible and let $A = (a_{ij})$. If $a_{11} = 0$, then some entry, say a_{i1} in the first column of A, is non-zero and we interchange the first row and ith row, by applying the row transformation $R_1 \leftrightarrow R_i$, to get a matrix whose (1-1)th entry is not zero. Then, by applying the transformation $R_1 \to (1/a_{11})R_1$, we get a matrix whose (1-1)th entry is 1. Then apply $R_2 \to R_2 - a_{21}R_1, R_3 \to R_3 - a_{31}R_1, \ldots, R_n \to R_n - a_{n1}R_1$ successively to get a matrix whose (1-1)th entry is 1 and the other entries in the first column are zeroes. Next, in the resulting matrix, consider second diagonal element (i.e., (2-2)th entry). For some $r \geq 2$, $a_{r2} \neq 0$ (see the following remark) and then exchange the rth row with the second row. (2-2)th entry is not zero. Make it 1 by applying a row transformation and then make $a_{r2} = 0$ for all $r \neq 2$. Now take up (3-3)th element and make it 1 and other r3th elements $(r \neq 3)$ zeroes. We can continue the process until we get I_n. We thus have a sequence

$$A \xrightarrow{f_1} A_1 \xrightarrow{f_2} A_2 \xrightarrow{f_3} A_3 \longrightarrow \cdots \longrightarrow A_m = I_n$$

of elementary row operations. Therefore A and I_n are similar. ■

The reader is advised to go through Examples 8.19 and 8.20 to get a better understanding of the above proof.

Remark: Let $A = (a_{ij})$ be an $n \times n$ invertible matrix and $1 \leq r \leq n$ satisfying the following:

$$\begin{aligned} a_{ii} &= 1 \quad \text{for all } i < r \\ a_{ij} &= 0 \quad \text{for all } 1 \leq i \leq n, \text{ and } i \neq j < r \end{aligned}$$

Then there exists $i \geq r$ such that $a_{ir} \neq 0$. For, suppose that $a_{ir} = 0$ for all $i \geq r$. Consider

$$A \xrightarrow{C_r \to C_r - a_{1r}C_1} A_1 \xrightarrow{C_r \to C_r - a_{2r}C_2} A_2 \longrightarrow \cdots \longrightarrow A_{r-1} \xrightarrow{C_r \to C_r - a_{r-1,r}C_{r-1}} A_r$$

Now A and A_r are similar and hence A_r is similar to I_n so that A_r is also invertible. But the rth column of A_r contains only zeroes, which is a contradiction to Theorem 8.29. Therefore $a_{ir} \neq 0$ for some $i \geq r$.

COROLLARY 8.3 Let A and B be similar square matrices of the same order. Then A is invertible if and only B is invertible.

PROOF Since $A \sim B$, we have $A \sim I_n \Leftrightarrow B \sim I_n$. Now, we can use the above theorem to get the required result. ■

Note that, in the second part of the proof of Theorem 8.30, we have reduced a given invertible matrix to I_n by using certain row transformations only. A similar procedure can be followed using the column transformations only. Therefore, we have the following.

COROLLARY 8.4 Any invertible matrix can be expressed as a product of finite number of elementary row matrices as well as a product of finite number of elementary column matrices.

Example 8.19

Consider the matrix

$$\begin{bmatrix} 2 & 3 & 1 \\ -3 & 4 & 5 \\ -5 & -2 & 1 \end{bmatrix}$$

Reduce it to I_3 using elementary row transformations.

Solution: We have

$$A = \begin{bmatrix} 2 & 3 & 1 \\ -3 & 4 & 5 \\ -5 & -2 & 1 \end{bmatrix} \xrightarrow{R_1 \to \frac{1}{2}R_1} \begin{bmatrix} 1 & \frac{3}{2} & \frac{1}{2} \\ -3 & 4 & 5 \\ -5 & -2 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 \to R_2 + 3R_1} \begin{bmatrix} 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & \frac{17}{2} & \frac{13}{2} \\ -5 & -2 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + 5R_1} \begin{bmatrix} 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & \frac{17}{2} & \frac{13}{2} \\ 0 & \frac{11}{2} & \frac{7}{2} \end{bmatrix}$$

$$\xrightarrow{R_2 \to \frac{2}{17}R_2} \begin{bmatrix} 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & 1 & \frac{13}{17} \\ 0 & \frac{11}{2} & \frac{7}{2} \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 - \frac{3}{2}R_2} \begin{bmatrix} 1 & 0 & \frac{-22}{34} \\ 0 & 1 & \frac{13}{17} \\ 0 & \frac{11}{2} & \frac{7}{2} \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - \frac{11}{2}R_2} \begin{bmatrix} 1 & 0 & \frac{-22}{34} \\ 0 & 1 & \frac{13}{17} \\ 0 & 0 & \frac{-24}{34} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \frac{-11}{17} \\ 0 & 1 & \frac{13}{17} \\ 0 & 0 & \frac{-12}{17} \end{bmatrix}$$

$$\xrightarrow{R_3 \to -\frac{17}{2}R_3} \begin{bmatrix} 1 & 0 & \frac{-11}{17} \\ 0 & 1 & \frac{13}{17} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 \to R_2 - \frac{13}{17}R_3} \begin{bmatrix} 1 & 0 & \frac{-11}{17} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 + \frac{11}{17}R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

Example 8.20

Reduce the matrix given in Example 8.19 to I_3 using column transformations only.

Solution: We have

$$A = \begin{bmatrix} 2 & 3 & 1 \\ -3 & 4 & 5 \\ -5 & -2 & 1 \end{bmatrix} \xrightarrow{C_1 \to \frac{1}{2}C_1} \begin{bmatrix} 1 & \frac{3}{2} & \frac{1}{2} \\ -3 & 4 & 5 \\ -5 & -2 & 1 \end{bmatrix}$$

$$\xrightarrow{C_2 \to C_2 - \frac{3}{2}C_1} \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ -3 & \frac{17}{2} & 5 \\ -5 & \frac{11}{2} & 1 \end{bmatrix}$$

$$\xrightarrow{C_3 \to C_3 - \frac{1}{2}C_1} \begin{bmatrix} 1 & 0 & 0 \\ -3 & \frac{17}{2} & \frac{13}{2} \\ -5 & \frac{11}{2} & \frac{7}{2} \end{bmatrix}$$

$$\xrightarrow{C_2 \to \frac{2}{17}C_2} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & \frac{13}{2} \\ -5 & \frac{11}{17} & \frac{7}{2} \end{bmatrix}$$

$$\xrightarrow{C_1 \to C_1 + 3C_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{13}{2} \\ -\frac{52}{17} & \frac{11}{17} & \frac{7}{2} \end{bmatrix}$$

$$\xrightarrow{C_3 \to C_3 - \frac{13}{2}C_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{52}{17} & \frac{11}{17} & \frac{-24}{34} \end{bmatrix}$$

$$\xrightarrow{C_3 \to -\frac{34}{24}C_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{52}{17} & \frac{11}{17} & 1 \end{bmatrix}$$

$$\xrightarrow{C_2 \to C_2 - \frac{11}{17}C_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{-52}{17} & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{C_1 \to C_1 + \frac{52}{17}C_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

In the above discussion, we have given a procedure to reduce an invertible matrix into the identity matrix using only the row transformations (only the column transformations). This can be used as an algorithm to find the inverse of a given invertible matrix.

8.1.1 Algorithm to Find Inverse Using Only the Row Transformations (Gauss–Jordan Method)

Let A be an invertible matrix of order $n \times n$. Then we get matrices $A_1, A_2, \ldots, A_s = I_n$ and elementary row transformations $f_1, f_2, \ldots, f_s$ such that

$$A \xrightarrow{f_1} A_1 \xrightarrow{f_2} A_2 \xrightarrow{f_3} \cdots \xrightarrow{f_{s-1}} A_{s-1} \xrightarrow{f_s} A_s = I_n$$

Then $I_n = A_s = f_s(A_{s-1}) = (f_s \cdot f_{s-1})(A_{s-2}) \cdots = (f_s f_{s-1} \cdots f_1)(A)$. If $R_1, R_2, \ldots, R_s$ are elementary row matrices corresponding to $f_1, f_2, \ldots, f_s$, respectively, then

$$I_n = R_s \cdot R_{s-1} \cdots R_1 \cdot A$$

and therefore $R_s R_{s-1} \cdots R_1$ is the inverse of A. Note that this is same as $f_s(I) f_{s-1}(I) \cdots f_1(I)$.

This procedure can be easily remembered by the following method. Consider the equation

$$A = I_n A$$

Apply successively $f_1, f_2, \ldots, f_s$ to get

$$\begin{aligned} f_1(A) &= f_1(IA) = f_1(I)A \\ f_2(f_1(A)) &= f_2(f_1(I)A) = f_2(f_1(I)) \cdot A \\ &\vdots \\ I &= (f_s f_{s-1} \cdots f_1)(A) = (f_s f_{s-1} \cdots f_2 f_1)(I) \cdot A \end{aligned}$$

Therefore $(f_s f_{s-1} \cdots f_2 f_1)(I)$ is the inverse of A.

Example 8.21

Find the inverse of the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

by using elementary row transformations.

Solution: We have $A = IA$. That is

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \quad (\text{by } R_1 \leftrightarrow R_2)$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A \quad (\text{by } R_3 \to R_3 - 3R_1)$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A \quad (\text{by } R_1 \to R_1 - 2R_2)$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 5 & -3 & 1 \end{bmatrix} A \quad (\text{by } R_3 \to R_3 + 5R_2)$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix} A \quad \left(\text{by } R_3 \to \frac{1}{2}R_3\right)$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ -4 & 3 & -1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix} A \quad (\text{by } R_2 \to R_2 - 2R_3)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix} A \quad (\text{by } R_1 \to R_1 - R_3)$$

Therefore

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}$$

8.1.2 Algorithm to Find Inverse Using Only the Column Transformations

Let A be an invertible matrix of order $n \times n$. Then by Corollary 8.4, we get matrices $B_1, B_2, \ldots, B_r = I_n$ and elementary column transformations $g_1, g_2, \ldots, g_r$ such that

$$A \xrightarrow{g_1} B_1 \xrightarrow{g_2} B_2 \xrightarrow{g_3} B_3 \xrightarrow{g_4} \cdots \xrightarrow{g_{r-1}} B_{r-1} \xrightarrow{g_r} B_r = I_n$$

Then $I_n = B_r = g_r(B_{r-1}) = g_r(g_{r-1}(B_{r-2})) = \cdots = (g_r g_{r-1} \cdots g_1)(A)$. If $C_1, C_2, \ldots, C_r$ are elementary column matrices corresponding to $g_1, g_2, \ldots, g_r$ respectively, then

$$B_i = g_i(B_{i-1}) = g_i(B_{i-1} \cdot I) = B_{i-1} g_i(I) = B_{i-1} C_i$$

for all $1 \le i \le r$ (where $I = I_n$ and $B_0 = A$). Therefore

$$I = B_r = B_{r-1}C_r = B_{r-2}C_{r-1}C_r = \cdots = AC_1C_2 \cdots C$$

and hence $C_1\, C_2 \cdots C_r$ is the inverse of A. This procedure can be easily remembered as follows. Let us consider the equation

$$A = AI$$

Apply $g_1, g_2, \ldots, g_r$ successively to get

$$g_1(A) = Ag_1(I) = AC_1 = AC_1 I$$
$$g_2(g_1(A)) = AC_1 g_2(I) = AC_1C_2 = AC_1C_2 I$$
$$\vdots$$
$$I = g_r g_{r-1} \cdots g_1(A) = AC_1C_2 \cdots C_r$$

and hence $C_1 C_2 \cdots C_r$ is the inverse of A.

Example 8.22

Find the inverse of the matrix given in Example 8.21 by using elementary column transformations.

Solution: We have $A = AI$ where

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

So we have

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \\ 1 & 3 & 1 \end{bmatrix} = A \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (\text{by } C_1 \leftrightarrow C_2)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -1 \\ 1 & 3 & -1 \end{bmatrix} = A \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix} \quad (\text{by } C_3 \to C_3 - 2C_1)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ -5 & 3 & -1 \end{bmatrix} = A \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix} \quad (\text{by } C_1 \to C_1 - 2C_2)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 3 & 2 \end{bmatrix} = A \begin{bmatrix} -2 & 1 & 1 \\ 1 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix} \quad (\text{by } C_3 \to C_3 + C_2)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 3 & 1 \end{bmatrix} = A \begin{bmatrix} -2 & 1 & \frac{1}{2} \\ 1 & 0 & -1 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \quad \left(\text{by } C_3 \to \frac{1}{2} C_3\right)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix} = A \begin{bmatrix} -2 & -\frac{1}{2} & \frac{1}{2} \\ 1 & 3 & -1 \\ 0 & -\frac{3}{2} & \frac{1}{2} \end{bmatrix} \quad (\text{by } C_2 \to C_2 - 3C_3)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix} \quad (\text{by } C_1 \to C_1 + 5C_3)$$

Therefore

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}$$

The process of finding the inverse of A by the elementary column transformations, as demonstrated in Example 8.22, is abstracted in the following.

8.2 | Determinants

Let us recall that a system of two equations in two unknowns, for example,

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

has a unique solution if $a_{11}a_{22} - a_{21}a_{12} \neq 0$. This system of equations can be expressed in matrix form as

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

If the 2×2 matrix on the left side is invertible, then we can multiply the above matrix equation by the inverse of this matrix on the left side to get that

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

The existence of the inverse of this 2×2 matrix depends on whether the number $a_{11}a_{22} - a_{21}a_{12}$ is non-zero. This number is known as the determinant of the matrix. In this section, we shall define the concept of the determinant of any square matrix and study its properties. As usual we take matrices over real or complex numbers only.

DEFINITION 8.28 Let A be a square matrix of order $n \times n$. Then the *determinant of* A, which is denoted by $|A|$ or det A, is defined inductively as follows.

1. If A is a 1×1 matrix, say $A = [a_{11}]$, then

$$\det A = a_{11}$$

2. If A is a 2×2 matrix, say

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

then we define

$$\det A = a_{11}a_{22} - a_{21}a_{12}$$

3. If A is a 3×3 matrix, say

$$A = (a_{ij}) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

then det A is defined as

$$a_{11} \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

4. In general, let $A = (a_{ij})$ be an $n \times n$ matrix, say

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

For any $1 \le i, j \le n$, let B_{ij} be the matrix of order $(n-1) \times (n-1)$ obtained from A by deleting the ith row and jth column. Then determinant of A is given by

$$\det A = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det B_{1j} = a_{11} \det B_{11} - a_{12} \det B_{12} + \cdots + (-1)^{n+1} a_{1n} \det B_{1n}$$

Example 8.23

Find the determinant of the following matrices:

(1) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

(2) $A = \begin{bmatrix} 2 & 1 & 3 \\ -3 & 4 & -2 \\ -1 & 2 & 1 \end{bmatrix}$

Solution:

(1) For the given matrix, the determinant is

$$\det A = 1\cdot 4 - 3\cdot 2 = 4 - 6 = -2$$

(2) For the given matrix, the determinant can be calculated as

$$B_{11} = \begin{bmatrix} 4 & -2 \\ 2 & 1 \end{bmatrix}, B_{12} = \begin{bmatrix} -3 & -2 \\ -1 & 1 \end{bmatrix} \text{ and } B_{13} = \begin{bmatrix} -3 & 4 \\ -1 & 2 \end{bmatrix}$$

(B_{11} is obtained from A by deleting first row and first column, B_{12} is obtained by deleting first row and second column from A and B_{13} is obtained by deleting first row and third column from A.) Now

$$\begin{aligned}\det A &= (-1)^{1+1} a_{11} \det B_{11} + (-1)^{1+2} a_{12} \det B_{12} \\ &\quad + (-1)^{1+3} a_{13} \det B_{13} \\ &= 2(4\cdot 1 - 2(-2)) - 1\cdot(-3\cdot 1 - (-2)(-1)) \\ &\quad + 3(-3\cdot 2 - (-1)4) \\ &= 2(4+4) - (-3-2) + 3(-6+4) \\ &= 16 + 5 - 6 = 15\end{aligned}$$

NOTATION 8.1 For a square matrix $A = (a_{ij})$, we write $|A|$ or $|a_{ij}|$ also for the det A. For example, if

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ -4 & -1 & 3 \end{bmatrix}$$

then

$$\det A = \begin{vmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ -4 & -1 & 3 \end{vmatrix} = |A|$$

QUICK LOOK

1. The determinant is defined only for a square matrix.
2. For any square matrix $A = (a_{ij})$ and $1 \le i, j \le n$, let B_{ij} be the matrix obtained from A by deleting the ith row and jth column in A. Then, it can be proved that

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det B_{ij} = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det B_{ij}$$

That is, the determinant will remain the same on expanding it along any row or any column.

3. In particular, $|A| = |A^T|$; that is, the determinant of a square matrix A is same as that of its transpose.

Example

Let

$$A = \begin{bmatrix} 2 & 1 & 3 \\ -3 & 4 & -2 \\ -1 & 2 & 1 \end{bmatrix}$$

Then

$$A^{\mathrm{T}} = \begin{bmatrix} 2 & -3 & -1 \\ 1 & 4 & 2 \\ 3 & -2 & 1 \end{bmatrix}$$

and its determinant is given by

$$|A^{\mathrm{T}}| = 2\begin{vmatrix} 4 & 2 \\ -2 & 1 \end{vmatrix} - (-3)\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} + (-1)\begin{vmatrix} 1 & 4 \\ 3 & -2 \end{vmatrix}$$

$$\begin{aligned} &= 2(4-(-4)) + 3(1-6) - (-2-12) \\ &= 16 - 15 + 14 \\ &= 15 = |A| \quad [\text{see part } (2), \text{Example } 8.23]\end{aligned}$$

Also, expanding det A along the second column of A we get

$$\begin{aligned}&(-1)^{1+2}\cdot 1\cdot(-3-2) + (-1)^{2+2}\cdot 4\cdot(2+3) + (-1)^{2+3}\cdot 2\cdot(-4+9) \\ &= -(-5) + 20 - 10 = 15 = |A|\end{aligned}$$

Again, expanding det A along the third row we get

$$\begin{aligned}&(-1)^{3+1}(-1)(-2-12) + (-1)^{3+2}\cdot 2\cdot(-4+9) \\ &\quad + (-1)^{3+3}\cdot 1\cdot(8+3) \\ &= 14 - 10 + 11 = 15 = |A|\end{aligned}$$

The reader can check that det $A^{\mathrm{T}} = -40 = \det A$ [part (3), Example 8.23].

8.2.1 Evaluation of the Determinant of a 3 × 3 Matrix (Sarrus Diagram)

Let

$$A = (a_{ij}) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Then

$$\begin{aligned} |A| &= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22}) \\ &= a_{11}a_{22}a_{23} + a_{12}a_{31}a_{23} + a_{13}a_{21}a_{32} - (a_{11}a_{32}a_{23} + a_{12}a_{21}a_{33} + a_{13}a_{31}a_{22}) \end{aligned}$$

There is an easy of way of remembering this to evaluate the determinant of A. Write down the columns of the matrix A. Write down the first and the second columns on the right side and draw broken lines as shown in Figure 8.2. Put + sign before the products of the triplets on the downward arrows and – sign before the products of the triplets on the upward arrows. The diagram in Figure 8.2 is called the *Sarrus diagram*.

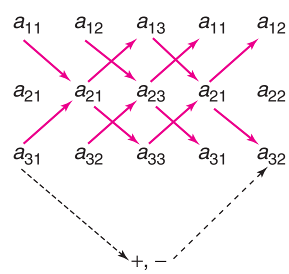


FIGURE 8.2 Sarrus diagram.

Example 8.24

Let

$$A = \begin{bmatrix} 4 & 2 & 3 \\ 1 & -3 & 0 \\ -2 & 5 & 2 \end{bmatrix}$$

Find the determinant of A by using the Sarrus diagram.

Solution:

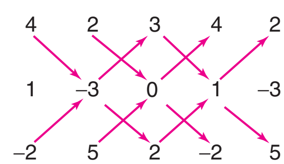


FIGURE 8.3 Sarrus diagram for Example 8.24.

From Figure 8.3 we have

$$\begin{aligned} |A| &= 4(-3)2 + 2\cdot 0(-2) + 3\cdot 1\cdot 5 \\ &\quad -(-2)(-3)3 - 5\cdot 0\cdot 4 - 2\cdot 1\cdot 2 \\ &= -24 + 0 + 15 - 18 - 0 - 4 = -31 \end{aligned}$$

In the following theorems, we state certain properties of determinants of matrices. The reader is advised to assume these for the present and verify these in simpler cases of 2×2 and 3×3 matrices. Let us begin with the following definition.

DEFINITION 8.29 Let $A = (a_{ij})$ be a square matrix of order $n\times n$. For any $1 \le i, j \le n$, let M_{ij} be the determinant of the $(n-1)\times(n-1)$ matrix obtained from A by deleting the ith row and jth column in A. Note that the ith row is one in which a_{ij} occurs and the jth column is one in which a_{ij} occurs. Then M_{ij} is called the *minor* of a_{ij}. There will be n^2 number of minors corresponding to an $n\times n$ matrix, for each entry in the matrix. If A is 3×3 matrix then there will be 9 minors associated with A.

Example 8.25

Find the minors M_{11}, M_{12} and M_{23} of the following matrix:

$$A = \begin{bmatrix} 4 & 3 & 2 \\ 5 & 1 & 3 \\ 2 & 0 & -4 \end{bmatrix}$$

Solution: The minors associated with the given matrix are:

$$M_{11} = \begin{vmatrix} 1 & 3 \\ 0 & -4 \end{vmatrix} = -4$$

$$M_{12} = \begin{vmatrix} 5 & 3 \\ 2 & -4 \end{vmatrix} = 5(-4) - 2 \cdot 3 = -26$$

$$M_{23} = \begin{vmatrix} 4 & 3 \\ 2 & 0 \end{vmatrix} = -6$$

Try it out Find all the minors of the matrix A given in Example 8.25.

DEFINITION 8.30 Let $A = (a_{ij})$ be an $n \times n$ matrix. For any $1 \le i, j \le n$ the *cofactor* of a_{ij} is defined by $(-1)^{i+j} M_{ij}$, where M_{ij} is the minor of a_{ij}; that is, the determinant of the $(n-1) \times (n-1)$ matrix obtained from A by deleting the ith row and jth column in A, multiplied by $(-1)^{i+j}$. The cofactor of a_{ij} is denoted by A_{ij} and is given by

$$\text{Cofactor of } a_{ij} = A_{ij} = (-1)^{i+j} M_{ij}$$

8.2.2 Formula for Determinant in Cofactors

Let $A = (a_{ij})$ be an $n \times n$ matrix. Then

$$\det A = |A| = \sum_{j=1}^{n} a_{ij} A_{ij} \quad \text{for any } 1 \le i \le n$$

$$= \sum_{i=1}^{n} a_{ij} A_{ij} \quad \text{for any } 1 \le j \le n$$

That is,

$$|A| = a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n} = a_{11}A_{11} + a_{21}A_{21} + \cdots + a_{2n}A_{2n}$$

Example 8.26

Find the cofactors of the following matrices:

(1) $A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$

(2) $A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 2 & 6 \\ -1 & 0 & -3 \end{bmatrix}$

Solution:

(1) For the given matrix, the cofactors are

$$A_{11} = (-1)^{1+1} 4 = 4; \quad A_{12} = (-1)^{1+2} \cdot 1 = -1$$
$$A_{21} = (-1)^{2+1} \cdot 2 = -2; \quad A_{22} = (-1)^{2+2} \cdot 3 = 3$$

(2) For the given matrix, the cofactors are

$$A_{11} = (-1)^{1+1} \begin{vmatrix} 2 & 6 \\ 0 & -3 \end{vmatrix} = -6; \quad A_{12} = (-1)^{1+2} \begin{vmatrix} 4 & 6 \\ -1 & -3 \end{vmatrix} = 6$$

$$A_{13} = (-1)^{1+3} \begin{vmatrix} 4 & 2 \\ -1 & 0 \end{vmatrix} = 2; \quad A_{21} = (-1)^{2+1} \begin{vmatrix} 3 & 1 \\ 0 & -3 \end{vmatrix} = 9$$

$$A_{22} = (-1)^{2+2} \begin{vmatrix} 2 & 1 \\ -1 & -3 \end{vmatrix} = -5; \quad A_{23} = (-1)^{2+3} \begin{vmatrix} 2 & 3 \\ -1 & 0 \end{vmatrix} = -3$$

$$A_{31} = (-1)^{3+1} \begin{vmatrix} 3 & 1 \\ 2 & 6 \end{vmatrix} = 16; \quad A_{32} = (-1)^{3+2} \begin{vmatrix} 2 & 1 \\ 4 & 6 \end{vmatrix} = -8$$

$$A_{33} = (-1)^{3+3} \begin{vmatrix} 2 & 3 \\ 4 & 2 \end{vmatrix} = -8$$

Theorem 8.31 Let $A = (a_{ij})$ be an $n \times n$ matrix. Let B be the matrix obtained by interchanging two rows (or column) in A. Then $|B| = -|A|$.

Proof We shall verify the theorem in special case when $n = 3$. Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{bmatrix}$$

Now B is obtained from A by applying the fundamental row transformation $R_1 \leftrightarrow R_3$. Then

$$\begin{aligned} |B| &= a_{31}(a_{22}a_{13} - a_{12}a_{23}) - a_{32}(a_{21}a_{13} - a_{11}a_{23}) + a_{33}(a_{21}a_{12} - a_{11}a_{22}) \\ &= a_{31}a_{22}a_{13} - a_{31}a_{12}a_{23} - a_{32}a_{21}a_{13} + a_{32}a_{11}a_{23} + a_{33}a_{21}a_{12} - a_{33}a_{11}a_{22} \\ &= -a_{11}(a_{22}a_{33} - a_{32}a_{23}) + a_{12}(a_{21}a_{33} - a_{31}a_{23}) - a_{13}(a_{21}a_{32} - a_{31}a_{22}) \\ &= -|A| \end{aligned}$$

Similar proof works for interchanging two columns. ■

Corollary 8.5 Let A be an $n \times n$ matrix in which any two rows (or two columns) are identical. Then $|A| = 0$.

Proof Let ith and kth rows in A be identical and B be the matrix obtained from A by interchanging the ith row and kth row, that is $A \xrightarrow{R_i \leftrightarrow R_k} B$. Then, by Theorem 8.31

$$|A| = -|B|$$

But, since the ith and kth rows are identical, $A = B$, and hence $|A| = |B|$. Therefore $|A| = 0$. Similarly when two columns are identical, $|A| = 0$. ■

Examples

(1) Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 4 \\ 1 & 2 & 3 \end{bmatrix}$$

Then $|A| = 0$, since first and third rows in A are identical. We can check this, by actual evaluation of $|A|$:

$$\begin{aligned} |A| &= 1(1 \cdot 3 - 2 \cdot 4) - 2(3 \cdot 3 - 1 \cdot 4) + 3(3 \cdot 2 - 1 \cdot 1) \\ &= -5 - 10 + 15 = 0 \end{aligned}$$

(2) Let

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 3 \\ 3 & 4 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & 3 & 1 \\ 3 & 1 & 2 \\ 2 & 4 & 3 \end{bmatrix}$$

Then $A \xrightarrow{C_1 \leftrightarrow C_3} B$. We have

$$\begin{aligned} |A| &= 1(1 \cdot 2 - 4 \cdot 3) - 3(2 \cdot 2 - 3 \cdot 3) + 4(2 \cdot 4 - 3 \cdot 1) \\ &= -10 + 15 + 20 = 25 \\ |B| &= 4(1 \cdot 3 - 4 \cdot 2) - 3(3 \cdot 3 - 2 \cdot 2) + 1(3 \cdot 4 - 2 \cdot 1) \\ &= -20 - 15 + 10 = -25 \end{aligned}$$

Therefore $|A| = -|B|$.

Theorem 8.32 Let $A = (a_{ij})$ be a square matrix of order $n \times n$ and s a scalar (i.e., s is a real or complex number). Let B be the matrix obtained from A by multiplying all the entries in a row (or a column) by s. That is, for some $1 \le k \le n$, if

$$A \xrightarrow{R_k \to sR_k} B \quad \text{or} \quad A \xrightarrow{C_k \to sC_k} B$$

then $|B| = s|A|$.

PROOF Suppose that $A \xrightarrow{R_k \to sR_k} B$. Let us evaluate the determinant of B along the kth row of B. Note that, for any $1 \le j \le n$, the kjth cofactor of B (i.e., the cofactor of the kjth entry in B) is same as that of A. Now,

$$\begin{aligned}|B| &= \sum_{j=1}^{n} s\,a_{kj} \cdot B_{kj} \\ &= \sum_{j=1}^{n} s\,a_{kj} \cdot A_{kj} \\ &= s\left(\sum_{j=1}^{n} a_{kj} \cdot A_{kj}\right) = s|A|\end{aligned}$$

Example 8.27

Let

$$A = \begin{bmatrix} 4 & 3 & 2 \\ 1 & 2 & -3 \\ -2 & 1 & -1 \end{bmatrix} \quad \text{and} \quad A \xrightarrow{R_2 \to 4R_2} B$$

Find B, $|A|$, $|B|$ and s such that $|B| = s|A|$.

Solution: By hypothesis we get

$$B = \begin{bmatrix} 4 & 3 & 2 \\ 4 & 8 & -12 \\ -2 & 1 & -1 \end{bmatrix}$$

Now

$$\begin{aligned}|A| &= 4[2(-1) - 1(-3)] - 3[1(-1) \\ &\quad -(-2)(-3)] + 2[1 \cdot 1 - (-2)2] \\ &= 4(-2+3) - 3(-1-6) + 2(1+4) \\ &= 4 + 21 + 10 = 35\end{aligned}$$

$$\begin{aligned}|B| &= 4[8(-1) - 1(-12)] - 3[4(-1) \\ &\quad -(-2)(-12)] + 2[4 \cdot 1 - (-2)8] \\ &= 4(-8+12) - 3(-4-24) + 2(4+16) \\ &= 16 + 84 + 40 = 140\end{aligned}$$

This gives

$$|B| = 140 = 4 \cdot 35 = 4|A|$$

Therefore $s = 4$.

COROLLARY 8.6 For any square matrix A of order $n \times n$ and for any scalar s,

$$|sA| = s^n |A|$$

THEOREM 8.33 Let $A = (a_{ij})$ be a square matrix of order $n \times n$. For a fixed k, let each entry in the kth row of A be a sum of two terms b_{kj} and c_{kj}, that is,

$$a_{kj} = b_{kj} + c_{kj} \quad \text{for each } 1 \le j \le n$$

Let $B = (b_{ij})$ and $C = (c_{ij})$, where

$$b_{ij} = a_{ij} = c_{ij} \quad \text{for all } i \ne k$$

Then $|A| = |B| + |C|$.

PROOF Without loss of generality we can assume that $k = 1$ we are given that

$$A = \begin{bmatrix} b_{11}+c_{11} & b_{12}+c_{12} & \cdots & b_{1n}+c_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Let us evaluate the determinants along the first rows. Note that, for any $1 \le j \le n$, $A_{1j} = B_{1j} = C_{1j}$, that is,

$$(1j)\text{th cofactor of } A = (1j)\text{th cofactor of } B = (1j)\text{th cofactor of } C$$

Therefore

$$\begin{aligned} |A| &= \sum_{j=1}^{n} (b_{1j} + c_{1j}) A_{1j} \\ &= \sum_{j=1}^{n} b_{1j} A_{1j} + \sum_{j=1}^{n} c_{1j} A_{1j} \\ &= \sum_{j=1}^{n} b_{1j} B_{1j} + \sum_{j=1}^{n} c_{1j} C_{1j} \\ &= |B| + |C| \end{aligned}$$

■

Try it out

Let

$$A = \begin{bmatrix} 2+3 & 4+5 & 1+3 \\ 2 & 6 & -4 \\ -3 & -4 & -2 \end{bmatrix}$$

Show that

$$|A| = \begin{vmatrix} 2 & 4 & 1 \\ 2 & 6 & -4 \\ -3 & -4 & -2 \end{vmatrix} + \begin{vmatrix} 3 & 5 & 3 \\ 2 & 6 & -4 \\ -3 & -4 & -2 \end{vmatrix}$$

Also, verify that

$$\begin{vmatrix} 2+3 & 4 & 5 \\ 1+2 & -3 & 6 \\ 4+5 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 4 & 5 \\ 1 & -3 & 6 \\ 4 & -2 & 1 \end{vmatrix} + \begin{vmatrix} 3 & 4 & 5 \\ 2 & -3 & 6 \\ 5 & -2 & 1 \end{vmatrix}$$

COROLLARY 8.7 The value of the determinant of a matrix does not change when any row (or column) is multiplied by a scalar s and then added to any other row (or column); that is, if $A \xrightarrow{R_i \to R_i + sR_k} B$ or $A \xrightarrow{C_i \to C_i + sC_k} B$, then $|A| = |B|$.

PROOF Let $A \xrightarrow{R_1 \to R_1 + sR_3} B$ and $A = (a_{ij})$. Then

$$B = \begin{bmatrix} a_{11}+sa_{31} & a_{12}+sa_{32} & \cdots & a_{1n}+sa_{3n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix}$$

By Theorem 8.33,

$$|B| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & & a_{nn} \end{vmatrix} + \begin{vmatrix} sa_{31} & sa_{32} & \cdots & sa_{3n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \vdots & a_{nn} \end{vmatrix}$$

$$= |A| + s\begin{vmatrix} a_{31} & a_{32} & \cdots & a_{3n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \quad \text{(by Theorem 8.32)}$$

$$= |A| + s\cdot 0 = |A| \quad (\because \text{the first and third rows are identical})$$

■

Example 8.28

Let

$$A = \begin{bmatrix} 4 & 3 & 2 \\ 2 & 1 & -3 \\ -1 & 2 & 5 \end{bmatrix} \quad \text{and} \quad A \xrightarrow{R_2 \to R_2 + 3R_1} B$$

Find out B and show that $|B| = |A|$.

Solution: By hypothesis we get B as

$$B = \begin{bmatrix} 4 & 3 & 2 \\ 2+3\cdot 4 & 1+3\cdot 3 & -3+3\cdot 2 \\ -1 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 4 & 3 & 2 \\ 14 & 10 & 3 \\ -1 & 2 & 5 \end{bmatrix}$$

Determinant of B is

$$|B| = 4(10\cdot 5 - 2.3) - 3[14\cdot 5 - (-1)3] + 2[14\cdot 2 - (-1)10]$$
$$= 4\times 44 - 3\times 73 + 2\times 38 = 33$$

Now

$$|A| = 4[1\cdot 5 - 2(-3)] - 3[2\cdot 5 - (-1)(-3)] + 2[2\cdot 2 - (-1)1]$$
$$= 4\times 11 - 3\times 7 + 2\times 5 = 33$$

Therefore

$$|B| = |A|$$

Example 8.29

Let A be a 3×3 matrix, in which each row is in geometric progression. That is,

$$A = \begin{bmatrix} a & ar & ar^2 \\ b & bs & bs^2 \\ c & ct & ct^2 \end{bmatrix}$$

Then show that $|A| = abc(r-s)(s-t)(t-r)$.

Solution: We have

$$|A| = a\begin{vmatrix} 1 & r & r^2 \\ b & bs & bs^2 \\ c & ct & ct^2 \end{vmatrix} = ab\begin{vmatrix} 1 & r & r^2 \\ 1 & s & s^2 \\ c & ct & ct^2 \end{vmatrix} = abc\begin{vmatrix} 1 & r & r^2 \\ 1 & s & s^2 \\ 1 & t & t^2 \end{vmatrix}$$

$$= abc\begin{vmatrix} 1 & r & r^2 \\ 0 & s-r & s^2-r^2 \\ 0 & t-r & t^2-r^2 \end{vmatrix}$$ (applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$ and using Theorem 8.32)

$$= abc(s-r)\begin{vmatrix} 1 & r & r^2 \\ 0 & 1 & s+r \\ 0 & t-r & t^2-r^2 \end{vmatrix} \quad \text{(by Theorem 8.32)}$$

$$= abc(s-r)(t-r)\begin{vmatrix} 1 & r & r^2 \\ 0 & 1 & s+r \\ 0 & 1 & t+r \end{vmatrix}$$

$$= abc(s-r)(t-r)\begin{vmatrix} 1 & s+r \\ 1 & t+r \end{vmatrix}$$

$$= abc(s-r)(t-r)[t+r-(s+r)]$$

$$= abc(s-r)(t-r)(t-s)$$

$$= abc(r-s)(s-t)(t-r)$$

Example 8.30

Let a, b, c be in AP. Evaluate the determinant of

$$A = \begin{bmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{bmatrix}$$

Solution: Since a, b, c are in AP, $b-a=c-b$ or $2b = a+c$. Let d be the common difference $b-a=c-b$. By first applying $R_2 \to R_2 - R_1$ and then $R_3 \to R_3 - R_1$, we get

$$|A| = \begin{vmatrix} x+1 & x+2 & x+a \\ 1 & 1 & b-a \\ 2 & 2 & c-a \end{vmatrix}$$

$$= 2\begin{bmatrix} x+1 & x+2 & x+a \\ 1 & 1 & d \\ 1 & 1 & d \end{bmatrix} = 0 \quad (\text{since } R_2 = R_3)$$

Let us recall that, for any square matrix $A = (a_{ij})$, the ijth minor is defined as the determinant of the matrix obtained by deleting the ith row and the jth column from A. It is denoted by M_{ij}. Also, the (ij)th cofactor is defined as $(-1)^{i+j}M_{ij}$ and is denoted by A_{ij}. Further recall that the determinant of A is defined by

$$|A| = \sum_{j=1}^{n} a_{ij}A_{ij} = \sum_{j=1}^{n} (-1)^{i+j} a_{ij}M_{ij}$$

for any $1 \le i \le n$ and $1 \le j \le n$.

DEFINITION 8.31 Let $A = (a_{ij})$ be a square matrix. Then *adjoint* of A is defined as the transpose of the matrix (A_{ij}), where A_{ij} is the ijth cofactor in A. The adjoint of A is denoted by adjoint A or adj A. The ijth entry in adj A is the jith cofactor in A.

Note that adj A is also a square matrix whose order is same as that of A.

Example 8.31

Find adj A where A is given by

$$A = \begin{bmatrix} 4 & 3 & 2 \\ 3 & 1 & -2 \\ -1 & 2 & -4 \end{bmatrix}$$

Solution: The cofactors of A are

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 1 & -2 \\ 2 & -4 \end{vmatrix} = 0; \quad A_{12} = (-1)^{1+2}\begin{vmatrix} 3 & -2 \\ -1 & -4 \end{vmatrix} = 14$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 3 & 1 \\ -1 & 2 \end{vmatrix} = 7; \quad A_{21} = (-1)^{2+1}\begin{vmatrix} 3 & 2 \\ 2 & -4 \end{vmatrix} = 16$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 4 & 2 \\ -1 & -4 \end{vmatrix} = -14; \quad A_{23} = (-1)^{2+3}\begin{vmatrix} 4 & 3 \\ -1 & 2 \end{vmatrix} = -11$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} 3 & 2 \\ 1 & -2 \end{vmatrix} = -8; \quad A_{32} = (-1)^{3+2}\begin{vmatrix} 4 & 2 \\ 3 & -2 \end{vmatrix} = 14$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 4 & 3 \\ 3 & 1 \end{vmatrix} = -5$$

Therefore

$$\text{adj } A = \begin{bmatrix} 0 & 14 & 7 \\ 16 & -14 & -11 \\ -8 & 14 & -5 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 0 & 16 & -8 \\ 14 & -14 & 14 \\ 7 & -11 & -5 \end{bmatrix}$$

THEOREM 8.34 Let A be a square matrix of order n. Then

$$A\cdot(\text{adj } A) = (\text{adj } A)\cdot A = |A|\, I_n$$

PROOF Let $A = (a_{ij})$ and A_{ij} be the cofactor of a_{ij}. Then, it can be verified that, for any $1 \le i, k \le n$,

$$\sum_{j=1}^{n} a_{ij} A_{kj} = \begin{cases} |A| & \text{if } i = k \\ 0 & \text{if } i \ne k \end{cases}$$

We have

$$\text{adj } A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \dots & \vdots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} A_{11} & A_{21} & A_{31} & \dots & A_{n1} \\ A_{12} & A_{22} & A_{32} & \dots & A_{n2} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ A_{1n} & A_{2n} & A_{3n} & \dots & A_{nn} \end{bmatrix}$$

Therefore

$$A\cdot\text{adj } A = \begin{bmatrix} |A| & 0 & 0 & \dots & 0 \\ 0 & |A| & 0 & \dots & 0 \\ 0 & 0 & |A| & \dots & 0 \\ \vdots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & |A| \end{bmatrix} = |A|\cdot I_n$$

Similarly, $(\text{adj } A)\cdot A = |A|\cdot I_n$. ■

Try it out Verify Theorem 8.34 by considering a 3×3 matrix.

COROLLARY 8.8 A square matrix A is invertible if and only if the determinant $|A|$ is non-zero and, in this case,

$$A^{-1} = \frac{1}{|A|}(\text{adj } A)$$

PROOF If $|A| = 0$, then by Theorem 8.34, we have $(\text{adj } A)\cdot A = O$, the zero matrix. Therefore, we cannot find a matrix B such that $AB = I_n$ (otherwise adj $A = O$ and $A = O$) and so A is not invertible. Conversely, if $|A| \ne 0$, then

$$\left(\frac{1}{|A|}\text{adj } A\right)\cdot A = I_n = A\cdot\left(\frac{1}{|A|}\text{adj } A\right)$$

Hence A is invertible and

$$A^{-1} = \frac{1}{|A|}(\text{adj } A)$$ ■

Let us recall that an invertible matrix is also called a *non-singular matrix*. A matrix which is not invertible is called a *singular matrix*. From the above result, a matrix is non-singular if and only if its determinant is non-zero. The following theorem is stated without proof, as it is beyond the scope of this book. However, the reader can assume this and use it freely in any instance.

Theorem 8.35 For any square matrices A and B of the same order

$$|AB| = |A||B|$$

That is, the determinant of the product is equal to the product of the determinants. ■

Corollary 8.9 Let A and B be square matrices of the same order. Then the product AB is non-singular if and only if both A and B are non-singular.

Proof Since $|A|$ and $|B|$ are real numbers, we have

$$|AB| = |A||B| \neq 0 \Leftrightarrow |A| \neq 0 \quad \text{and} \quad |B| \neq 0$$

and the result follows from Corollary 8.8. ■

Corollary 8.10 A square matrix is non-singular if and only if its transpose is non-singular.

Proof Let A be a square matrix and A^{T} be its transpose. Then, we know that $|A| = |A^{\mathrm{T}}|$ [see part (4), Quick Look 3]. Therefore A is non-singular if and only if A^{T} is non-singular. ■

Theorem 8.36 Let A be a non-singular matrix. Then A is symmetric if and only if A^{-1} is symmetric.

Proof Recall that a square matrix is called symmetric if it is equal to its transpose. By Theorem 8.27, we have $(A^{\mathrm{T}})^{-1} = (A^{-1})^{\mathrm{T}}$. Therefore

$$\begin{aligned} A \text{ is symmetric} &\Leftrightarrow A = A^{\mathrm{T}} \\ &\Leftrightarrow A^{-1} = (A^{\mathrm{T}})^{-1} = (A^{-1})^{\mathrm{T}} \\ &\Leftrightarrow A^{-1} \text{ is symmetric} \end{aligned}$$ ■

Theorem 8.37 Let A be a skew-symmetric matrix of order n, where n is an odd integer. Then A is singular, that is, its determinant is zero and hence A is not invertible.

Proof Since A is skew-symmetric, we have

$$A^{\mathrm{T}} = -A$$

Therefore,

$$\begin{aligned} -|A| &= (-1)^n\,|A| \quad (\text{since } n \text{ is odd}) \\ &= |-A| \\ &= |A^{\mathrm{T}}| = |A| \end{aligned}$$

and hence $|A| + |A| = 0$ so that $|A| = 0$. Therefore, A is singular and hence not invertible. ■

Theorem 8.38 Let A be a non-singular matrix of order n. Then

$$|\text{adj } A| = |A|^{n-1}$$

and hence adj A is non-singular.

Proof In Theorem 8.34, we have proved that

$$A \cdot (\text{adj } A) = |A|\, I_n$$

Note that $|A|I_n$ is the scalar matrix, in which each of the diagonal entries is $|A|$ and each of the non-diagonal entries is zero. Now, by Theorem 8.35,

$$\begin{aligned}|A||\text{adj } A| &= |A(\text{adj } A)| \\ &= ||A|I_n| \\ &= |A|^n\end{aligned}$$

and, since $|A| \neq 0$, it follows that

$$|\text{adj } A| = |A|^{n-1} \neq 0 \quad (\text{since } |A| \neq 0)$$

and therefore adj A is non-singular. ■

Try it out

Prove that $|\text{adj } A| = |A|^{n-1}$ is also true, even if $|A| = 0$.

Hint: When $|A| = 0$, then $|\text{adj } A| = 0$.

THEOREM 8.39 Let A be a non-singular matrix of order n, where $n \geq 2$. Then

$$\text{adj}\,(\text{adj } A) = |A|^{n-2} \cdot A$$

PROOF Put $B = \text{adj } A$. By Theorem 8.34,

$$B\,(\text{adj } B) = |B|\, I_n = |A|^{n-1}\, I_n$$

Therefore,

$$A \cdot B(\text{adj } B) = A(|A|^{n-1}\, I_n) = |A|^{n-1}\,(AI_n) = |A|^{n-1}\, A$$

Also

$$\begin{aligned}|A|\, I_n(\text{adj } B) &= A \cdot (\text{adj } A) \cdot (\text{adj } B) \\ &= AB(\text{adj } B) \\ &= |A|^{n-1}\, A\end{aligned}$$

Therefore, $\text{adj}\,(\text{adj } A) = \text{adj } B = |A|^{n-2} \cdot A$. ■

THEOREM 8.40 Let A be a non-singular matrix of order n and B and C any square matrices of order n. Then

$$AB = AC \Rightarrow B = C \quad (\text{left cancellation law})$$
$$BA = CA \Rightarrow B = C \quad (\text{right cancellation law})$$

PROOF We have

$$\begin{aligned}AB = AC &\Rightarrow A^{-1}(AB) = A^{-1}(AC) \\ &\Rightarrow (A^{-1}A)B = (A^{-1}A)C \\ &\Rightarrow I_nB = I_nC \\ &\Rightarrow B = C\end{aligned}$$

Similarly, by multiplying with A^{-1} on the right side, we get

$$BA = CA \Rightarrow B = C$$ ■

THEOREM 8.41 Let A and B be non-singular matrices of same order n. Then

$$\text{adj}\,(AB) = (\text{adj } B)(\text{adj } A)$$

Proof Recall that AB is non-singular and $(AB)^{-1} = B^{-1}A^{-1}$. Now, consider

$$\begin{aligned}(AB)(\text{adj } B)(\text{adj } A) &= A(B\cdot \text{adj } B)(\text{adj } A)\\ &= A(|B|\cdot I_n)(\text{adj } A) \quad \text{(by Theorem 8.34)}\\ &= |B|(A\, I_n)(\text{adj } A)\\ &= |B|(A\cdot \text{adj } A)\\ &= |B||A|\, I_n\\ &= |A||B|\, I_n\\ &= |AB|\, I_n\\ &= (AB)[\text{adj}\,(AB)] \quad \text{(by Theorem 8.34)}\end{aligned}$$

Since AB is non-singular, by Theorem 8.40, we get that

$$(\text{adj } B)(\text{adj } A) = \text{adj}(AB)$$ ■

Theorem 8.42 Let A be non-singular matrix of order n. Then

$$(\text{adj } A)^{\mathrm{T}} = \text{adj } (A^{\mathrm{T}})$$

That is, the transpose of the adjoint is the adjoint of the transpose.

Proof Since $|A^{\mathrm{T}}| = |A| \neq 0$, A^{T} and A are both invertible. Consider

$$\begin{aligned}(\text{adj } A)^{\mathrm{T}}\cdot A^{\mathrm{T}} &= (A\cdot \text{adj } A)^{\mathrm{T}}\\ &= (|A|\, I_n)^{\mathrm{T}}\\ &= |A|\, I_n^{\mathrm{T}}\\ &= |A|\, I_n\\ &= |A^{\mathrm{T}}|\, I_n\\ &= (\text{adj } A^{\mathrm{T}})\cdot A^{\mathrm{T}}\end{aligned}$$

Therefore, by the right cancellation law,

$$(\text{adj } A)^{\mathrm{T}} = \text{adj } (A^{\mathrm{T}})$$ ■

Theorem 8.43 Let A be a non-singular matrix of order n. Then A is symmetric if and only if adj A is symmetric.

Proof We have

$$\begin{aligned}A \text{ is symmetric} &\Rightarrow A^{\mathrm{T}} = A\\ &\Rightarrow \text{adj}(A^{\mathrm{T}}) = \text{adj } A\\ &\Rightarrow (\text{adj } A)^{\mathrm{T}} = \text{adj } A\\ &\Rightarrow \text{adj } A \text{ is symmetric}\end{aligned}$$

Conversely,

$$\begin{aligned}\text{adj } A \text{ is symmetric} &\Rightarrow \text{adj}(\text{adj } A) \text{ is symmetric}\\ &\Rightarrow |A|^{n-2}\cdot A \text{ is symmetric} \quad \text{(by Theorem 8.39)}\\ &\Rightarrow A \text{ is symmetric}\end{aligned}$$ ■

If B is symmetric, then $s\,B$ is also symmetric for any non-zero scalar s. Recall that, for any real number s, the scalar matrix, in which each diagonal entry is s and each of the other entries is zero, is also denoted by s. By writing $s\,A$, we mean the product of the scalar matrix s with the matrix A. For example, if

$$A = \begin{bmatrix} 2 & 3 & 1 \\ -2 & 1 & -3 \\ -1 & 2 & 4 \end{bmatrix}$$

then

$$sA = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ -2 & 1 & -3 \\ -1 & 2 & 4 \end{bmatrix}$$

Also, $s\,A$ is the matrix in which each entry is obtained by multiplying the corresponding entry in A with s. If A is a non-singular matrix of order n, then sA is also non-singular for any non-zero scalar s and $|s\,A| = s^n|A|$, since the determinant of a scalar matrix s is equal to s^n. Infact, the determinant of a diagonal matrix or a triangular matrix is equal to the product of the diagonal entries.

THEOREM 8.44 Let $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_mx^m$ be a polynomial in x, where $a_0, a_1, a_2, \ldots, a_m$ are real numbers, such that $a_0 \neq 0$. If A is a non-singular matrix such that $f(A) = 0$, then

$$A^{-1} = \frac{-1}{a_0}(a_1 + a_2A + a_3A^2 + \cdots + a_mA^{m-1})$$

PROOF Let A be a non-singular matrix and $f(A) = 0$. That is,

$$a_0 + a_1A + a_2A^2 + \cdots + a_mA^{m-1} = 0$$

By multiplying both sides with A^{-1}, we get that

$$a_0A^{-1} + a_1AA^{-1} + a_2A^2A^{-1} + \cdots + a_mA^m \cdot A^{-1} = 0$$

Therefore

$$A^{-1} = \frac{-1}{a_0}(a_1 + a_2A + a_3A^2 + \cdots + a_mA^{m-1})$$

■

Example 8.32

Find real numbers a and b such that $a + bA + A^2 = O$ where

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$$

Also find A^{-1}.

Solution: First

$$A^2 = A \cdot A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3\cdot3 + 2\cdot1 & 3\cdot2 + 2\cdot1 \\ 1\cdot3 + 1\cdot1 & 1\cdot2 + 1\cdot1 \end{bmatrix} = \begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix}$$

Now, suppose that a and b are real numbers such that

$$0 = a + bA + A^2$$

$$= \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} + \begin{bmatrix} 3b & 2b \\ b & b \end{bmatrix} + \begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} a + 3b + 11 & 2b + 8 \\ b + 4 & a + b + 3 \end{bmatrix}$$

Therefore,

$$a + 3b + 11 = 0;\ 2b + 8 = 0$$
$$b + 4 = 0;\ a + b + 3 = 0$$

Solving these we get $b = -4$ and $a = 1$. Therefore

$$1 - 4A + A^2 = 0$$
$$A^{-1} - 4AA^{-1} + A^2A^{-1} = 0$$

This gives the inverse as

$$A^{-1} = 4 - A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$

Example 8.33

Find adj A, $|A|$ and A^{-1} where

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$$

Solution: For the given matrix, we have

$$|A| = 3\cdot 1 - 2\cdot 1 = 1 \neq 0$$

Therefore, A is non-singular. The cofactors for A are

$$A_{11} = (-1)^{1+1}\cdot 1 = 1;\ A_{12} = (-1)^{1+2}\cdot 1 = -1$$
$$A_{21} = (-1)^{2+1}\cdot 2 = -2;\ A_{22} = (-1)^{2+2}\cdot 3 = 3$$

Therefore,

$$\text{adj } A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}(\text{adj } A) = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$

To find the inverse of a square matrix A, or to express A^{-1} in terms of A, the concept of a characteristic polynomial of a square matrix and the much known *Cayley–Hamilton Theorem* are useful, especially for 2×2 and 3×3 matrices. Let us begin with a definition.

DEFINITION 8.32 If A is a square matrix and I is the corresponding unit matrix, then the polynomial $|A - xI|$ in x is called *characteristic polynomial* of A and the equation $|A - xI| = 0$ is called the *characteristic equation* of the matrix A.

Example 8.34

Find the characteristic polynomial and characteristic equation of

$$\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$$

Solution:
Let

$$A = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$$

so that

$$|A - xI| = \begin{vmatrix} 2-x & 1 \\ 1 & -1-x \end{vmatrix} = (x-2)(x+1) - 1 = x^2 - x - 3$$

is the characteristic polynomial of A and its characteristic equation is

$$x^2 - x - 3 = 0$$

Example 8.35

Find the characteristic polynomial and characteristic equation for A given by

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \\ -1 & 0 & 3 \end{bmatrix}$$

Solution: We have

$$|A - xI| = \begin{vmatrix} 1-x & -1 & 2 \\ 0 & 2-x & 1 \\ -1 & 0 & 3-x \end{vmatrix}$$

$$\begin{aligned} &= (1-x)[(2-x)(3-x) - 0] + 1(0+1) + 2(2-x) \\ &= (1-x)(2-x)(3-x) + (5-2x) \\ &= (1-x)(6 - 5x + x^2) + 5 - 2x \\ &= -x^3 + 6x^2 - 13x + 11 \end{aligned}$$

This is the *characteristic polynomial* of A and its *characteristic equation* is $x^3 - 6x^2 + 13x - 11 = 0$.

The following theorem is stated without proof as it is not necessary for a student of plus two class. However, the student can assume this and use it freely whenever it is needed.

THEOREM 8.45 (CAYLEY–HAMILTON) Every square matrix satisfies its characteristic equation; that is, if A is a square matrix of order n and

$$f(x) = |A - xI| = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0$$

is its characteristic equation, then

$$f(A) = a_0 I_n + a_1 A + a_2 A^2 + \cdots + a_n A^n = 0$$

Also if $a_0 \neq 0$, then

$$A^{-1} = \frac{-1}{a_0}(a_1 I + a_2 A + a_3 A^2 + \cdots + a_n A^{n-1})$$

Note that A^{-1} exists if and only if the constant term of the characteristic of A is non-zero. ■

Example 8.36

Show that the matrix A satisfies its characteristic equation and hence find A^{-1}, where

$$A = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$$

Solution: The characteristic equation of the given matrix is $x^2 - x - 3 = 0$ (see Example 8.34). That is

$$f(x) = x^2 - x - 3$$

Now

$$A^2 = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 1 & 2 \end{bmatrix}$$

Also

$$f(A) = A^2 - A - 3 = \begin{bmatrix} 5 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} + \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 5-2-3 & 1-1+0 \\ 1-1+0 & 2+1-3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Therefore, A satisfies its characteristic equation and

$$A^2 - A = 3I \Rightarrow A^{-1} = \frac{1}{3}(A - I)$$

A few more examples of this type are discussed in worked-out problems. It is better for the reader to know few more kinds of matrices as discussed next.

DEFINITION 8.33 A square matrix A is called *idempotent matrix* if $A^2 = A$.

Example 8.37

Show that the matrix A is idempotent

$$A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

Solution: By definition, a matrix is idempotent if $A^2 = A$. Now

$$A^2 = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 4+2-4 & -4-6+8 & -8-8+12 \\ -2-3+4 & 2+9-8 & 4+12-12 \\ 2+2-3 & -2-6+6 & -4-8+9 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = A$$

Therefore, A is an idempotent matrix.

DEFINITION 8.34 A square matrix A is called *nilpotent matrix* if there exists a positive integer m such that A^m is a zero matrix. Among such positive integers m, the least positive one is called the *index* of the nilpotent matrix.

Examples

(1) $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is a nilpotent matrix of index 2.

(2) Every zero square matrix is a nilpotent matrix of index 1.

DEFINITION 8.35 A square matrix A is called *involutary* if A^2 is equal to unit matrix of same order. Note that a square matrix A is involutary if and only if $A^{-1} = A$.

Examples

(1) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is an involutary matrix.

(2) Every unit matrix is involutary.

DEFINITION 8.36 A square matrix A is said to be periodic matrix, if $A^{k+1} = A$ for some positive integer k. If k is the least positive integer such that $A^{k+1} = A$, then k is called the *period* of A.

Example 8.38

Show that A is periodic matrix and find its period.

$$A = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \quad (\text{where } i = \sqrt{-1})$$

Solution: For the given matrix we have

$$A^2 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I_2$$

Therefore $A^4 = I_2$ and $A^5 = A$. Hence A is a periodic matrix of period 4.

8.3 | Solutions of Linear Equations

In this section, we shall apply the results on matrices and determinants in solving systems of linear equations. In particular, we derive certain conditions on the coefficients for the system of equations to have a unique solution.

DEFINITION 8.37 An equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $a_1, a_2, \ldots, a_n$ and b are real numbers and $x_1, x_2, \ldots, x_n$ are unknowns, is called *linear equation* in n unknowns. Also n linear equations in n unknowns of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots \qquad \cdots \qquad \cdots \qquad &\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

is a called a *system of linear equations* in n unknowns. The above system of linear equations can be expressed in the form of a matrix equation as

$$AX = B$$

where A the $n \times n$ matrix and X and B are the $n \times 1$ matrices given by

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ b_n \end{bmatrix}$$

Recall that AX is the product of the square matrix A of order $n \times n$ and the matrix X of order $n \times 1$. Therefore both AX and B are $n \times 1$ matrices. The equation $AX = B$ means that the corresponding entries in the matrices AX and B are equal; that is, for each $1 \le i \le n$,

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i$$

A and B are given matrices and we have to find X satisfying $AX = B$. The matrix A is called the *coefficient matrix*.

DEFINITION 8.38 Let $AX = B$ be a system of n linear equations in n unknowns given by

$$\begin{aligned} &a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ &a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{aligned}$$

If $b_1 = b_2 = \cdots = b_n = 0$, then the system is called a *system of homogenous linear equations*. If atleast one $b_i \neq 0$, then the system is called a *non-homogenous system of linear equations*. A *solution* of the system $AX = B$ is defined to be an n-tuple $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ of real numbers which satisfy each of the above equations; that is,

$$a_{i1}\alpha_1 + a_{i2}\alpha_2 + \cdots + a_{in}\alpha_n = b_i \quad \text{for all } 1 \le i \le n$$

DEFINITION 8.39 A system $AX = B$ of linear equations is said to be *consistent* if there exists a solution for the system; otherwise the system is called *inconsistent*.

8.3.1 Crammer's Rule

THEOREM 8.46 (CRAMMER'S RULE) Let $AX = B$ be a system of n linear equations in n unknowns. If A is a non-singular matrix, then the system is consistent and has a unique solution.

PROOF Let

$$\begin{aligned} &a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ &a_{21}x_2 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &a_0x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{aligned}$$

be a system of linear equations. Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Then $AX = B$ represents the given system of linear equations in n unknowns $x_1, x_2, \ldots, x_n$. Suppose that A is non-singular; that is, A has a multiplicative inverse A^{-1}. Now,

$$X = I_n X = (A^{-1}A)X = A^{-1}(AX) = A^{-1}B$$

and hence $(x_1, x_2, \ldots, x_n)$ is a solution of $AX = B$. Further,

$$X = A^{-1}B = \left(\frac{\text{adj } A}{|A|}\right)B = \frac{1}{|A|}\begin{bmatrix} A_{11} & A_{21} & A_{31} & \cdots & A_{n1} \\ A_{12} & A_{22} & A_{32} & \cdots & A_{n2} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ A_{1n} & A_{2n} & A_{3n} & \cdots & A_{nn} \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

where A_{ij} is the ijth cofactor of the matrix A. This implies

$$x_1 = \sum_{i=1}^{n} \frac{b_i A_{i1}}{|A|}, \; x_2 = \sum_{i=1}^{n} \frac{b_i A_{i2}}{|A|}, \ldots, x_n = \sum_{i=1}^{n} \frac{b_i A_{in}}{|A|}$$

One can observe that $\sum_{k=1}^{n} b_k A_{ik}$ is the determinant of the matrix obtained from the matrix A by replacing its kth column with B. If we denote this determinant obtained by replacing the kth column of A by B with Δ_k, then

$$x_k = \frac{\Delta_k}{|A|}$$

Thus

$$x_1 = \frac{\Delta_1}{|A|}, \; x_2 = \frac{\Delta_2}{|A|}, \ldots, x_n = \frac{\Delta_n}{|A|}$$

which shows that the solution X is unique because $X = A^{-1}B$ always satisfies the equation $AX = B$. ■

Example 8.39

Find the solution of the system of equations $x + y + z = 6$, $x - y + z = 2$, $2x + y - z = 1$ using Crammer's rule.

Solution: Here the coefficient matrix is given by

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Now

$$|A| = 1(1-1) - 1(-1-2) + 1(1+2) = 6 \neq 0$$

Hence A is non-singular and the system will have unique solution. Also

$$\Delta_1 = \begin{vmatrix} 6 & 1 & 1 \\ 2 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = 6(1-1) - 1(-2-1) + 1(2+1) = 6$$

$$\Delta_2 = \begin{vmatrix} 1 & 6 & 1 \\ 1 & 2 & 1 \\ 2 & 1 & -1 \end{vmatrix} = 1(-2-1) - 6(-1-2) + 1(1-4) = 12$$

$$\Delta_3 = \begin{vmatrix} 1 & 1 & 6 \\ 1 & -1 & 2 \\ 2 & 1 & 1 \end{vmatrix} = 1(-1-2) - 1(1-4) + 6(1+2) = 18$$

Therefore by Crammer's rule

$$x = \frac{\Delta_1}{|A|} = \frac{6}{6} = 1, \quad y = \frac{\Delta_2}{|A|} = \frac{12}{6} = 2 \quad \text{and} \quad z = \frac{\Delta_3}{|A|} = \frac{18}{6} = 3$$

is the solution.

8.3.2 Gauss–Jordan Method

Now we are going to discuss another method of solving the equation $AX = B$ by applying elementary row operations on A, where A is a non-singular matrix and B is a column matrix, and same operations on B. The equation will be reduced to the form $IX = D$, where I is the unit matrix of the same order as A and D is a column matrix whose elements are $d_1, d_2, \ldots, d_n$. So $x_1 = d_1, x_2 = d_2, \ldots, x_n = d_n$ is the solution. This method of finding the solution of $AX = B$ $(|A| \neq 0)$ is called *Gauss–Jordan method*. In this method, we use the following theorem which is stated and whose proof is beyond the scope of this book.

THEOREM 8.47 Solution of the equation $AX = B$ will not be altered by applying elementary row operations on the equation.

Example 8.40

Using Gauss–Jordan method, solve the equations

$$x + 2z = 2, \quad y + z = 3 \quad \text{and} \quad 2x + y = 1$$

Solution: The matrix equation is

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix} X = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \quad \text{where } X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & -4 \end{bmatrix} X = \begin{bmatrix} 2 \\ 3 \\ -3 \end{bmatrix} \quad (R_3 \sim R_3 - 2R_1)$$

$$\sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & -5 \end{bmatrix} X = \begin{bmatrix} 2 \\ 3 \\ -6 \end{bmatrix} \quad (R_3 \sim R_3 - R_2)$$

$$\sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} X = \begin{bmatrix} 2 \\ 3 \\ \frac{6}{5} \end{bmatrix} \quad \left(R_3 \sim \frac{-1}{5} R_3\right)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} X = \begin{bmatrix} 2 - \frac{12}{5} \\ 3 - \frac{6}{5} \\ \frac{6}{5} \end{bmatrix} \quad \begin{pmatrix} R_1 \sim R_1 - 2R_3 \text{ and} \\ R_2 \sim R_2 - R_3 \end{pmatrix}$$

Therefore, the solution is

$$x = 2 - \frac{12}{5} = \frac{-2}{5}, \quad y = \frac{9}{5} \quad \text{and} \quad z = \frac{6}{5}$$

8.3.3 Consistent and Inconsistent Systems

Illustration

The discussion till now provides the reader the technique of solving the equation $AX = B$ when A is non-singular matrix (i.e., $|A| \neq 0$).

If $|A| = 0$, then the system may have or may not have solution.

1. If the system has no solution, then it is called *inconsistent system.*
2. If it has solution, the system is called *consistent system*.

Let us consider the system

$$AX = B \tag{8.14}$$

where $|A| = 0$.

Applying series of elementary row operations simultaneously on both sides of Eq. (8.14), suppose at a stage, we obtain zero row (i.e., all elements of the row are zeros) in the transformed matrix of A and the corresponding element in the transformed form of B is non-zero, then the system is *inconsistent* otherwise it is *consistent*. When the system is consistent, then we rewrite the equivalent system and express x, y, z in terms of a parameter(s) which shows that the system has infinite number of solutions. This process will be explained by the following examples.

Example 8.41

Check if the following system of equations is consistent or inconsistent.

$$x + y + z = 1, \quad x + 2y + 4z = 3 \quad \text{and} \quad x + 4y + 10z = 9$$

Solution: The matrix equation representing the system is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 10 \end{bmatrix} X = \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 3 & 9 \end{bmatrix} X = \begin{bmatrix} 1 \\ 2 \\ 8 \end{bmatrix} \quad (R_2 \sim R_2 - R_1 \text{ and } R_3 \sim R_3 - R_1)$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} X = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \quad (R_3 \sim R_3 - 3R_2)$$

The system is inconsistent because $0x + 0y + 0z = 2$. Note that, here

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 10 \end{vmatrix} \begin{aligned} &= 1(20-16) - 1(10-4) + 1(4-2) \\ &= 4 - 6 + 2 = 0 \end{aligned}$$

That is, in this case $|A| = 0$ and the system is inconsistent.

Example 8.42

Consider the following system of equations. Check for consistency of this system $x + y + z = 1$, $x + 2y + 4z = 2$, $x + 4y + 10z = 4$.

Solution: The matrix equation representing the system is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 10 \end{bmatrix} X = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

Proceeding in similar fashion as in Example 8.41 we get

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 3 & 9 \end{bmatrix} X = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} X = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} X = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad (R_1 \sim R_1 - R_2)$$

Therefore we get

$$x - 2z = 0, \; y + 3z = 1$$

This is a system of two equations in three variables. Let $z = k$, so that $x = 2k$, $y = 1 - 3k$. Therefore $x = 2k, y = 1 - 3k$ and $z = k$ is a solution for all real values of k. Hence the system has infinite number of solutions. Here also $|A| = 0$.

Examples 8.41 and 8.42 revealed that $AX = B$ is inconsistent in one case whereas it is consistent in other case. In both cases, the coefficient matrix is singular.

8.3.4 Homogenous System of Equations

Now let us turn our attention to homogenous system of equations which are given below. If a_{ij} $(1 \le i, j \le n)$ are real, then the system of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ \vdots \qquad \vdots \qquad\quad \vdots \quad &\;\; \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= 0 \end{aligned}$$

is called a *homogenous system* of n equations in n variables $x_1, x_2, \ldots, x_n$. If

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

then the matrix equation representing the above system of equations is $AX = O$, where

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

and O is $O_{n\times 1}$ zero matrix. Note that $x_1 = 0, x_2 = 0, \ldots, x_n = 0$ is always a solution of the system and this solution is called *trivial solution* of the system. Any solution in which at least one $x_i \neq 0$ is called non-trivial solution or non-zero solution.

THEOREM 8.48 If A is a non-singular matrix with real entries, then $X = O$ is the only solution of $AX = O$. If X is a non-zero solution, then A is a singular matrix.

PROOF Suppose A is non-singular matrix so that A^{-1} exists. Therefore

$$\begin{aligned} AX = O &\Rightarrow A^{-1}(AX) = O \\ &\Rightarrow (A^{-1}A)X = O \\ &\Rightarrow X = O \end{aligned}$$

Now suppose $X = X_1$ is a non-zero solution. If A is non-singular, then $A^{-1}(AX_1) = O$ and hence $X_1 = O$ which is a contradiction. Hence A must be singular matrix. ■

In solving $AX = O$, we employ the same technique as in the case of $AX = B$ which is explained in illustration in Section 8.3.3.

Example 8.43

Find the solution of the system of equations

$x + 3y - 2z = 0, \quad 2x - y + 4z = 0 \quad$ and $\quad x - 11y + 14z = 0$

Solution: The matrix equation representing the given system is

$$\begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix} X = O$$

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & -14 & 16 \end{bmatrix} X = O \quad (R_2 \sim R_2 - 2R_1 \text{ and } R_3 \sim R_3 - R_1)$$

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{bmatrix} X = O \quad (R_3 \sim R_3 - 2R_2)$$

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & \frac{-8}{7} \\ 0 & 0 & 0 \end{bmatrix} X = O \quad \left(R_2 \sim \frac{-1}{7} R_2\right)$$

$$\begin{bmatrix} 1 & 0 & \frac{10}{7} \\ 0 & 1 & \frac{-8}{7} \\ 0 & 0 & 0 \end{bmatrix} X = O \quad (R_1 \sim R_1 - 3R_2)$$

Therefore

$$x + \frac{10}{7} z = 0 \quad \text{and} \quad y - \frac{8}{7} z = 0$$

If we put $z = k$, then the solution is

$$x = \frac{-10}{7} k, \; y = \frac{8}{7} k, \; z = k$$

WORKED-OUT PROBLEMS

Single Correct Choice Type Questions

1. Let

$$A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$$

where $ab \neq 0$. Then

(A) $A^2 = A$ (B) $A^2 = O$ (C) $A^2 = I$ (D) $A^3 = A$

Solution: We have

$$A^2 = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}\begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$$

$$= \begin{bmatrix} a^2b^2 - a^2b^2 & ab^3 - ab^3 \\ -a^3b + a^3b & -a^2b^2 + a^2b^2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

Answer: (B)

2. Let

$$A = \begin{bmatrix} 0 & -\tan\theta/2 \\ \tan\theta/2 & 0 \end{bmatrix} \quad (\theta \neq n\pi)$$

$$B = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then the matrix $I + A$ is equal to

(A) $(I - A)B$ (B) $(I - A)^2 B$

(C) $(I + A)^2 B$ (D) $(I - A)^2$

Solution: Put $\tan(\theta/2) = a$ so that

$$B = \begin{bmatrix} \dfrac{1-a^2}{1+a^2} & \dfrac{-2a}{1+a^2} \\ \dfrac{2a}{1+a^2} & \dfrac{1-a^2}{1+a^2} \end{bmatrix}$$

Therefore

$$(I - A)B = \begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix}\begin{bmatrix} \dfrac{1-a^2}{1+a^2} & \dfrac{-2a}{1+a^2} \\ \dfrac{2a}{1+a^2} & \dfrac{1-a^2}{1+a^2} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{1-a^2}{1+a^2} + \dfrac{2a^2}{1+a^2} & \dfrac{-2a}{1+a^2} + \dfrac{a(1-a^2)}{1+a^2} \\ \dfrac{-a(1-a^2)}{1+a^2} + \dfrac{2a}{1+a^2} & \dfrac{2a^2}{1+a^2} + \dfrac{1-a^2}{1+a^2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -a \\ a & 1 \end{bmatrix} = I + A$$

Answer: (A)

3. If

$$\begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \end{bmatrix}$$

then

(A) $x = 3, y = 5$ (B) $x = 4, y = 3$

(C) $x = 4, y = 5$ (D) $x = 5, y = 3$

Solution: We have

$$\begin{bmatrix} 3 \\ 11 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x - 4y \\ x + 2y \end{bmatrix}$$

Therefore we get

$$3x - 4y = 3 \quad \text{and} \quad x + 2y = 11$$

Solving these equations, we have $x = 5$ and $y = 3$.

Answer: (D)

4. If

$$A = \begin{bmatrix} -4 & 1 \\ 3 & 2 \end{bmatrix} \quad \text{and} \quad f(x) = x^2 - 2x + 3$$

then $f(A)$ is

(A) O (B) $\begin{bmatrix} 30 & 4 \\ -12 & 6 \end{bmatrix}$ (C) $\begin{bmatrix} 30 & -4 \\ -12 & 6 \end{bmatrix}$ (D) $\begin{bmatrix} 30 & -4 \\ 12 & 6 \end{bmatrix}$

Solution: We have $f(A) = A^2 - 2A + 3I$ where

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Now

$$A^2 = \begin{bmatrix} -4 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} -4 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 19 & -2 \\ -6 & 7 \end{bmatrix}$$

$$-2A = \begin{bmatrix} 8 & -2 \\ -6 & -4 \end{bmatrix}$$

$$3I = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

Substituting these in the equation for $f(A)$ we get

$$f(A)=\begin{bmatrix} 30 & -4 \\ -12 & 6 \end{bmatrix}$$

Answer: (C)

5. Let

$$A=\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

and n be a positive integer. Then

$$A^n=\begin{bmatrix} \lambda^{n-1} & \lambda^{n-1} & \lambda^{n-1} \\ \lambda^{n-1} & \lambda^{n-1} & \lambda^{n-1} \\ \lambda^{n-1} & \lambda^{n-1} & \lambda^{n-1} \end{bmatrix}$$

where λ equals

(A) 2 (B) 3 (C) 9 (D) 6

Solution: For the given matrix A we have

$$A^2=\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}=\begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix}=\begin{bmatrix} 3^{2-1} & 3^{2-1} & 3^{2-1} \\ 3^{2-1} & 3^{2-1} & 3^{2-1} \\ 3^{2-1} & 3^{2-1} & 3^{2-1} \end{bmatrix}$$

Also

$$A=\begin{bmatrix} 3^{1-1} & 3^{1-1} & 3^{1-1} \\ 3^{1-1} & 3^{1-1} & 3^{1-1} \\ 3^{1-1} & 3^{1-1} & 3^{1-1} \end{bmatrix}$$

Therefore for $n=1,2$

$$A^n=\begin{bmatrix} 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \end{bmatrix}$$

is true. Assume that

$$A^m=\begin{bmatrix} 3^{m-1} & 3^{m-1} & 3^{m-1} \\ 3^{m-1} & 3^{m-1} & 3^{m-1} \\ 3^{m-1} & 3^{m-1} & 3^{m-1} \end{bmatrix}$$

Then

$$A^{m+1}=A^m\cdot A=\begin{bmatrix} 3^{m-1} & 3^{m-1} & 3^{m-1} \\ 3^{m-1} & 3^{m-1} & 3^{m-1} \\ 3^{m-1} & 3^{m-1} & 3^{m-1} \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 3\cdot 3^{m-1} & 3\cdot 3^{m-1} & 3\cdot 3^{m-1} \\ 3\cdot 3^{m-1} & 3\cdot 3^{m-1} & 3\cdot 3^{m-1} \\ 3\cdot 3^{m-1} & 3\cdot 3^{m-1} & 3\cdot 3^{m-1} \end{bmatrix}$$

$$=\begin{bmatrix} 3^m & 3^m & 3^m \\ 3^m & 3^m & 3^m \\ 3^m & 3^m & 3^m \end{bmatrix}$$

Hence by induction,

$$A^n=\begin{bmatrix} 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \end{bmatrix}$$

Answer: (B)

6. If

$$[1 \quad x \quad 1]\begin{bmatrix} 1 & 3 & 2 \\ 2 & 5 & 1 \\ 15 & 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ x \end{bmatrix}=O$$

then value of x is

(A) –2 or –14 (B) 2 or –4
(C) 2 or –14 (D) 2 or 14

Solution: We have

$$[1 \quad x \quad 1]\begin{bmatrix} 1 & 3 & 2 \\ 2 & 5 & 1 \\ 15 & 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ x \end{bmatrix}=[1 \quad x \quad 1]\begin{bmatrix} 1+6+2x \\ 2+10+x \\ 15+6+2x \end{bmatrix}$$

$$=[1 \quad x \quad 1]\begin{bmatrix} 2x+7 \\ x+12 \\ 2x+21 \end{bmatrix}$$

By hypothesis $(2x+7)+x(x+12)+(2x+21)=0$

$$(2x+7)+(x^2+12x)+2x+21=0$$

$$x^2+16x+28=0$$

$$(x+2)(x+14)=0$$

$$x=-2 \quad \text{or} \quad -14$$

Answer: (A)

7. Let

$$A=\frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ x & 2 & y \end{bmatrix}$$

If $AA^{\mathrm{T}}=I_3$, then

(A) $x=2, y=-1$ (B) $x=2, y=1$
(C) $x=-2, y=-1$ (D) $x=-2, y=1$

Solution: For the given matrix A we have

$$AA^{\mathrm{T}}=\frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ x & 2 & y \end{bmatrix}\begin{bmatrix} 1 & 2 & x \\ 2 & 1 & 2 \\ 2 & -2 & y \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 1+4+4 & 2+2-4 & x+4+2y \\ 2+2-4 & 4+1+4 & 2x+2-2y \\ x+4+2y & 2x+2-2y & x^2+4+y^2 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 9 & 0 & x+2y+4 \\ 0 & 9 & 2x-2y+2 \\ x+2y+4 & 2x-2y+2 & x^2+y^2+4 \end{bmatrix}$$

Now

$$AA^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & (x+2y+4)/9 \\ 0 & 1 & (2x-2y+2)/9 \\ (9x+2y+4)/9 & (2x-2y+2)/9 & (x^2+y^2+4)/9 \end{bmatrix}$$

$$I^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solving we get

$$x+2y+4=0 \qquad (8.15)$$

$$2x-2y+2=0 \qquad (8.16)$$

$$x^2+y^2+4=9 \qquad (8.17)$$

From Eqs. (8.15) and (8.16), we get $x = -2$, $y = -1$. These values also satisfy Eq. (8.17).

Answer: (C)

8. If x, y, z are real and

$$A = \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}$$

is such that $AA^{\mathrm{T}} = I$, then

(A) $x+y+z=1$
(B) $x^2+y^2+z^2=1$
(C) $x+y+z=xyz$
(D) $x^2+y^2+z^2=2xyz$

Solution: For the given matrix A we have

$$A^{\mathrm{T}} = \begin{bmatrix} 0 & x & x \\ 2y & y & -y \\ z & -z & z \end{bmatrix}$$

$$AA^{\mathrm{T}} = \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}\begin{bmatrix} 0 & x & x \\ 2y & y & -y \\ z & -z & z \end{bmatrix}$$

Therefore

$$= \begin{bmatrix} 0+4y^2+z^2 & 0+2y^2-z^2 & 0-2y^2+z^2 \\ 0+2y^2-z^2 & x^2+y^2+z^2 & x^2-y^2-z^2 \\ 0-2y^2+z^2 & x^2-y^2-z^2 & x^2+y^2+z^2 \end{bmatrix}$$

$$= \begin{bmatrix} 4y^2+z^2 & 2y^2-z^2 & -2y^2+z^2 \\ 2y^2-z^2 & x^2+y^2+z^2 & x^2-y^2-z^2 \\ -2y^2+z^2 & x^2-y^2-z^2 & x^2+y^2+z^2 \end{bmatrix}$$

But it is given that $AA^{\mathrm{T}} = I$, therefore

$$AA^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solving we get [taking (2-2)th entry]

$$x^2+y^2+z^2=1$$

Answer: (B)

9. If

$$A = \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$$

is idempotent matrix and $bc = 1/4$ then the value of a is

(A) 1 (B) –1 (C) 1/2 (D) –1/2

Solution: If A is idempotent matrix then $A^2 = A$. This implies

$$\begin{bmatrix} a^2+bc & b \\ c & bc+(1-a)^2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$$

Solving we get

$$a^2+bc=a \quad \text{and} \quad bc+(1-a)^2=1-a$$

Using $bc = 1/4$ we get

$$a^2+\frac{1}{4}=a$$

$$(2a-1)^2=0$$

$$a=\frac{1}{2}$$

Answer: (C)

10. If

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + B$$

then

(A) $B^2 = B$ (B) $B^2 = I$

(C) $B^2 = O$ (D) $B^2 = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$

Solution: Let

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

so that

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+1 & b \\ c & d+1 \end{bmatrix}$$

Solving we get

$$\begin{aligned} a+1 &= 1 \\ b &= 0 \\ c &= 2 \\ d+1 &= 1 \end{aligned}$$

So

$$a = 0, \quad b = 0, \quad c = 2, \quad d = 0$$

Therefore

$$B = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \quad \text{and} \quad B^2 = O$$

Answer: (C)

Note: One can observe that

$$B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$$

11. If

$$\begin{bmatrix} 3 & -2 \\ 3 & 0 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x & x \\ y & y \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3x & 3x \\ 10 & 10 \end{bmatrix}$$

then the integral part of $x + y$ is

(A) 3 (B) 2 (C) 4 (D) 1

Solution: We have

$$\begin{bmatrix} 3x-2y & 3x-2y \\ 3x & 3x \\ 2x+4y & 2x+4y \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3x & 3x \\ 10 & 10 \end{bmatrix}$$

Solving we get

$$3x - 2y = 3 \qquad (8.18)$$

$$2x + 4y = 10 \qquad (8.19)$$

Solving Eqs. (8.18) and (8.19), we get $x = 2$, $y = 3/2$. Therefore

$$x + y = \frac{7}{2} = 3\frac{1}{2}$$

Hence, the integral part of $x + y = 3$.

Answer: (A)

12. If a non-zero square matrix of order 3×3 commutes with every square matrix of order 3×3, then the matrix is necessarily

(A) a scalar matrix (B) a unit matrix

(C) an idempotent matrix (D) a nilpotent matrix

Solution: Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

and suppose A commutes with every matrix of 3×3 order. Choose

$$B = \begin{bmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{bmatrix}$$

where b_1, b_2 and b_3 are distinct. The (i, j)th element of $AB = b_i a_{ij}$, whereas (i, j)th element of $BA = b_i a_{ij}$. Now

$$AB = BA \Rightarrow b_j a_{ij} = b_i a_{ij}$$

$$\Rightarrow a_{ij} = 0 \text{ when } i \neq j \quad (\because b_i \neq b_j \text{ for } i \neq j)$$

Therefore

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

Again choose

$$D = \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix}$$

where $d_{ij} \neq 0$ for $1 \leq i, j \leq 3$. Again

$$AD = DA \Rightarrow a_{ii} d_{ij} = a_{jj} d_{ij} \Rightarrow a_{ii} = a_{jj} \quad (\because d_{ij} \neq 0)$$

Therefore

$$a_{11} = a_{22} = a_{33}$$

and hence A is a scalar matrix.

Answer: (A)

Note: For more general case, the reader is advised to see Theorem 8.7.

13. If

$$A = \begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$$

then the value of α for which $A^2 = B$ is

(A) 1 (B) –1 (C) no real value (D) 4

Solution: We have from the hypothesis that

$$\begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix} = B = A^2 = \begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \alpha^2 & 0 \\ \alpha+1 & 1 \end{bmatrix}$$

Therefore $\alpha^2 = 1$ and $\alpha + 1 = 5$ which are inconsistent. Hence there is no real value of α.

Answer: (C)

14. If A is a square matrix such that $A^3 = O$, then $I + A + A^2$ (I is the corresponding unit matrix) is

(A) $I + A$ (B) $(I + A)^{-1}$

(C) $I - A$ (D) $(I - A)^{-1}$

Solution: We have

$$(I - A)(I + A + A^2) = I - A^3 = I$$

Therefore

$$I + A + A^2 = (I - A)^{-1}$$

Answer: (D)

15. If the product of the matrices

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\cdots\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 378 \\ 0 & 1 \end{bmatrix}$$

then n is equal to

(A) 27 (B) 26 (C) 376 (D) 378

Solution: We have

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1+2 \\ 0 & 1 \end{bmatrix}$$

Again

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1+2+3 \\ 0 & 1 \end{bmatrix}$$

By induction,

$$\text{LHS} = \begin{bmatrix} 1 & \sum_{k=1}^{n} k \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 378 \\ 0 & 1 \end{bmatrix}$$

Therefore

$$\frac{n(n+1)}{2} = 378 \quad \text{or} \quad n(n+1) = 27 \times 28$$

Hence $n = 27$.

Answer: (A)

16. If P is a 2×2 matrix satisfying the relation

$$\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} P \begin{bmatrix} 3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 3 & -1 \end{bmatrix}$$

then P is equal to

(A) $\dfrac{1}{19}\begin{bmatrix} -48 & 25 \\ 70 & -42 \end{bmatrix}$ (B) $\dfrac{1}{19}\begin{bmatrix} 48 & -25 \\ -70 & 42 \end{bmatrix}$

(C) $\dfrac{-1}{19}\begin{bmatrix} 38 & -25 \\ -75 & 42 \end{bmatrix}$ (D) $\dfrac{-1}{19}\begin{bmatrix} 48 & 25 \\ 70 & 42 \end{bmatrix}$

Solution: Let

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 2 \\ 5 & -3 \end{bmatrix}$$

so that

$$A^{-1} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \quad \text{and} \quad B^{-1} = \frac{1}{19}\begin{bmatrix} 3 & 2 \\ 5 & -3 \end{bmatrix}$$

Now by hypothesis

$$\begin{aligned} APB &= \begin{bmatrix} 2 & 4 \\ 3 & -1 \end{bmatrix} \Rightarrow P = A^{-1}\begin{bmatrix} 2 & 4 \\ 3 & -1 \end{bmatrix}B^{-1} \\ &= \frac{1}{19}\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 5 & -3 \end{bmatrix} \\ &= \frac{1}{19}\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 26 & -8 \\ 4 & 9 \end{bmatrix} \\ &= \frac{1}{19}\begin{bmatrix} 48 & -25 \\ -70 & 42 \end{bmatrix} \end{aligned}$$

Answer: (B)

17. If

$$A = \begin{bmatrix} 2x & 0 \\ x & x \end{bmatrix} \quad \text{and} \quad A^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$$

then x is equal to

(A) 2 (B) 1/2 (C) 1 (D) 3

Solution: We know that $AA^{-1} = I$. Hence

$$\begin{bmatrix} 2x & 0 \\ x & x \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2x & 0 \\ 0 & 2x \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This gives $x = 1/2$.

Answer: (B)

18. Let $w \neq 1$ be a cube root of unity and

$$A = \begin{bmatrix} w & 0 \\ 0 & w \end{bmatrix}$$

Then A^{2010} is equal to

(A) A (B) A^2 (C) A^3 (D) $3A$

Solution: For the given matrix, we have

$$A^2 = \begin{bmatrix} w & 0 \\ 0 & w \end{bmatrix}\begin{bmatrix} w & 0 \\ 0 & w \end{bmatrix} = \begin{bmatrix} w^2 & 0 \\ 0 & w^2 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} w^2 & 0 \\ 0 & w^2 \end{bmatrix}\begin{bmatrix} w & 0 \\ 0 & w \end{bmatrix}$$

$$= \begin{bmatrix} w^3 & 0 \\ 0 & w^3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{2010} = (A^3)^{670} = I^{670} = I = A^3$$

Answer: (C)

19. The number of idempotent diagonal matrices of 3×3 order is

(A) 8 (B) 2 (C) 6 (D) infinite

Solution: Let

$$D = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \quad \text{and} \quad D^2 = D$$

Now

$$\begin{bmatrix} d_1^2 & 0 & 0 \\ 0 & d_2^2 & 0 \\ 0 & 0 & d_3^2 \end{bmatrix} = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$$

Solving we get

$$d_1^2 = d_1; \quad d_2^2 = d_2; \quad d_3^2 = d_3$$
$$d_1 = 0, 1; \quad d_2 = 0, 1; \quad d_3 = 0, 1$$

Answer: (A)

20. Let A be a non-singular square matrix. If B is a square matrix such that $B = -A^{-1}BA$, then the matrix $(A+B)^2$ is equal to

(A) $A + B$ (B) $A^2 + B^2$ (C) O (D) I

Solution: We have

$$AB = -AA^{-1}BA = -BA$$

$$AB + BA = O$$

Now

$$(A+B)^2 = A^2 + AB + BA + B^2 = A^2 + O + B^2 = A^2 + B^2$$

Answer: (B)

21. If A and B are square matrices of same order such that $A + B = AB$, then

(A) $A + B = -BA$ (B) $A - B = BA$
(C) $AB = BA$ (D) $A - B = 0$

Solution: We have

$$A + B = AB \Rightarrow AB - A - B + I = I$$
$$\Rightarrow (I - A)(I - B) = I$$
$$\Rightarrow I - A \text{ is invertible and its inverse is } I - B$$

Therefore

$$(I - B)(I - A) = I$$
$$I - B - A + BA = I$$
$$A + B = BA$$
$$AB = A + B = BA$$

Answer: (C)

22. Let

$$A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} -1 & 2 \\ 3 & 1 \\ 5 & 4 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 2 & -1 \\ 2 & 3 \\ -4 & 1 \end{bmatrix}$$

If

$$A - B + xC = \begin{bmatrix} 7 & -1 \\ 0 & 9 \\ -12 & -2 \end{bmatrix}$$

then the value of x is

(A) –2 (B) 3 (C) –3 (D) 2

Solution: By hypothesis

$$\begin{bmatrix} 7 & -1 \\ 0 & 9 \\ -12 & -2 \end{bmatrix} = A - B + xC = \begin{bmatrix} 3+2x & 1-x \\ -4+2x & 3+3x \\ -4-4x & -4+x \end{bmatrix}$$

Therefore

$$3 + 2x = 7; \quad 1 - x = -1; \quad -4 + 2x = 0$$
$$3 + 3x = 9; \quad -4 - 4x = -12; \quad -4 + x = -2$$

Solving any one gives $x = 2$.

Answer: (D)

23. If l, m and n are positive real numbers and the matrix

$$A = \begin{bmatrix} 0 & 2m & n \\ l & m & -n \\ l & -m & n \end{bmatrix}$$

is such that $AA^{\mathrm{T}} = I$ (unit matrix), then the ordered triple (l, m, n) may be

(A) $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{2}}\right)$ (B) $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}\right)$

(C) $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}}\right)$ (D) $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{3}}\right)$

Solution: We have for the given matrix that

$$AA^{\mathrm{T}} = \begin{bmatrix} 0 & 2m & n \\ l & m & -n \\ l & -m & n \end{bmatrix} \begin{bmatrix} 0 & l & l \\ 2m & m & -m \\ n & -n & n \end{bmatrix}$$

$$= \begin{bmatrix} 4m^2 + n^2 & 2m^2 - n^2 & -2m^2 + n^2 \\ 2m^2 - n^2 & l^2 + m^2 + n^2 & l^2 - m^2 - n^2 \\ -2m^2 + n^2 & l^2 - m^2 - n^2 & l^2 + m^2 + n^2 \end{bmatrix}$$

But $AA^{\mathrm{T}} = I$, that is

$$\begin{bmatrix} 4m^2 + n^2 & 2m^2 - n^2 & -2m^2 + n^2 \\ 2m^2 - n^2 & l^2 + m^2 + n^2 & l^2 - m^2 - n^2 \\ -2m^2 + n^2 & l^2 - m^2 - n^2 & l^2 + m^2 + n^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solving this we get

$$4m^2 + n^2 = 1 \quad (8.18)$$

$$2m^2 - n^2 = 0 \quad (8.19)$$

$$-2m^2 + n^2 = 0 \quad (8.20)$$

$$l^2 + m^2 + n^2 = 1 \quad (8.21)$$

$$l^2 - m^2 - n^2 = 0 \quad (8.22)$$

Adding Eqs. (8.18) and (8.19) we get

$$6m^2 = 1 \Rightarrow m = \frac{\pm 1}{\sqrt{6}}$$

Adding Eqs. (8.21) and (8.22) we get

$$2l^2 = 1 \Rightarrow l = \frac{\pm 1}{\sqrt{2}}$$

Equation (8.18) – Eq. (8.19) + Eq. (8.20) gives

$$3n^2 = 1 \Rightarrow n = \frac{\pm 1}{\sqrt{3}}$$

Answer: (D)

24. Let A, B and C be square matrices of order 3×3. If A is invertible and $(A - B)C = BA^{-1}$, then

(A) $C(A - B) = A^{-1}B$ (B) $C(A - B) = BA^{-1}$

(C) $(A - B)C = A^{-1}B$ (D) all the above

Solution: We have

$$(A - B)C = BA^{-1} \Rightarrow AC - BC = BA^{-1}$$

$$\Rightarrow AC - BC - BA^{-1} + AA^{-1} = I \text{ (unit matrix)}$$

$$\Rightarrow (A - B)C + (A - B)A^{-1} = I$$

$$\Rightarrow (A - B)(C + A^{-1}) = I$$

Therefore, $C + A^{-1}$ is the inverse of $A - B$. This implies

$$(C + A^{-1})(A - B) = I$$

$$C(A - B) = I - A^{-1}A + A^{-1}B$$

$$C(A - B) = A^{-1}B$$

Answer: (A)

25. If

$$A = \begin{bmatrix} \alpha & 2 \\ 2 & \alpha \end{bmatrix} \text{ and } |A^3| = 125$$

then α is equal to

(A) ± 3 (B) ± 2 (C) ± 5 (D) 0

Solution: By hypothesis

$$125 = |A^3| = |A|^3 = (\alpha^2 - 4)^3$$

Hence $\alpha^2 - 4 = 5 \Rightarrow \alpha = \pm 3$.

Answer: (A)

26. If

$$A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$$

then $|A^{2003}| - 4|A^{2002}| =$

(A) –3 (B) 0 (C) 9 (D) –9

Solution: We have that $|A| = -3 + 4 = 1$. Therefore

$$|A^{2003}| - 4|A^{2002}| = |A|^{2003} - 4|A|^{2002} = 1 - 4 = -3$$

Answer: (A)

27. Let A be 3×3 matrix such that $A^3 = \alpha A$, where $\alpha \neq 1$. Then, the matrix $A = I$ is

(A) non-singular

(B) idempotent

(C) nilpotent matrix

(D) symmetric matrix

Solution: Let $B = A + I$. Then

$$A^3 = (B - I)^3$$
$$\alpha A = (B - I)^3 = B^3 - 3B^2 + 3B - I$$
$$B^3 - 3B^2 + 3B - I = \alpha A = \alpha(B - I)$$
$$B^3 - 3B^2 + 3B - \alpha B = (1 - \alpha)I$$
$$(B^2 - 3B + (3 - \alpha)I)B = (1 - \alpha)I$$

This gives

$$\det[B^2 - 3B + (3 - \alpha)I]\det B = (1 - \alpha)^3 \neq 0$$

Hence $\det B \neq 0$. This implies $B = A + I$ is non-singular.

Answer: (A)

28. A and B are different square matrices of same order such that $A^3 = B^3$ and $A^2B = B^2A$. Then

(A) $A^2 + B^2$ is singular matrix

(B) $A^2 + B^2$ is non-singular

(C) $A^2 + B^2$ is idempotent

(D) $A^2 - B^2$ is symmetric

Solution: We know that

$$(A^2 + B^2)(A - B) = A^3 - A^2B + B^2A - B^3 = O$$

If $A^2 + B^2$ is non-singular, then

$$(A^2 + B^2)^{-1}(A^2 + B^2)(A - B) = 0$$

Therefore, $A - B = O$ or $A = B$, which is a contradiction. Hence $A^2 + B^2$ must be singular matrix.

Answer: (A)

29. If

$$P = \begin{bmatrix} \dfrac{\sqrt{3}}{2} & \dfrac{1}{2} \\ \dfrac{-1}{2} & \dfrac{\sqrt{3}}{2} \end{bmatrix}, A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } \quad Q = PAP^{\mathrm{T}}$$

and $X = P^{\mathrm{T}}Q^{2010}P$, then X is equal to

(A) $\begin{bmatrix} 4 + 2010\sqrt{3} & 8015 \\ 2010 & 4 - 2010\sqrt{5} \end{bmatrix}$

(B) $\begin{bmatrix} 2010 & 2 - \sqrt{3} \\ 2 + \sqrt{3} & 2010 \end{bmatrix}$

(C) $\begin{bmatrix} 2 + \sqrt{3} & 1 \\ 2 - \sqrt{3} & 1 \end{bmatrix}$

(D) $\begin{bmatrix} 1 & 2010 \\ 0 & 1 \end{bmatrix}$

Solution: It can be seen that $P^{\mathrm{T}} = P^{-1}$

$$Q = PAP^{\mathrm{T}} = PAP^{-1} \Rightarrow P^{-1}Q = AP^{-1}$$

Now

$$X = P^{\mathrm{T}}Q^{2010}P = P^{-1}Q^{2010}P$$
$$= (P^{-1}Q)Q^{2009}P = A(P^{-1}Q^{2009}P)$$
$$= A(P^{-1}Q)Q^{2008} \cdot P = A^2P^{-1}Q^{2008}P$$

Finally

$$X = A^{2010}P^{-1}P = A^{2010} \tag{8.23}$$

Now

$$A^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$
$$A^3 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

By induction we can see that

$$A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$$

Therefore

$$X = A^{2010} = \begin{bmatrix} 1 & 2010 \\ 0 & 1 \end{bmatrix}$$

Answer: (D)

30. The number of real roots of the equation

$$\begin{vmatrix} a & a & x \\ b & x & b \\ x & x & x \end{vmatrix} = 0$$

where a and b are distinct non-zero real numbers, is

(A) 2 (B) 3 (C) 1 (D) 0

Solution: Clearly $x = 0$ is a root. When $x = a$, first and third rows are identical and when $x = b$, the second and third rows are identical. Therefore $x = 0, a, b$ are the roots.

Answer: (B)

31. If $n \geq 3$ is even and

$$\Delta_r = \begin{vmatrix} {}^{(n-2)}C_{r-2} & {}^{(n-2)}C_{r-1} & {}^{(n-2)}C_r \\ -3 & 1 & 1 \\ 2 & -1 & 0 \end{vmatrix}$$

then $\sum_{r=2}^{n}(-2)^r \Delta_r$ is

(A) $2n - 1$ (B) $2n + 1$ (C) $2n$ (D) 3^n

Solution: We have

$$\begin{aligned}\Delta_r &= {}^{(n-2)}C_{r-2} + 2\,{}^{(n-2)}C_{r-1} + {}^{(n-2)}C_r\\ &= [{}^{(n-2)}C_{r-2} + {}^{(n-2)}C_{r-1}] + [{}^{(n-2)}C_{r-1} + {}^{(n-2)}C_r]\\ &= {}^{(n-1)}C_{r-1} + {}^{(n-1)}C_r\\ &= {}^nC_r\end{aligned}$$

Therefore we get

$$\begin{aligned}\sum_{r=2}^{n}(-2)^r\Delta_r &= \sum_{r=2}^{n}(-2)^r\,{}^nC_r\\ &= {}^nC_2\cdot 2^2 - {}^nC_3\cdot 2^3 + {}^nC_4\cdot 2^4 - \cdots (-2)^n C_n\\ &= (1-2)^n - 1 + {}^nC_1\cdot 2\\ &= (-1)^n - 1 + 2n\\ &= 2n \qquad (\because n \text{ is even})\end{aligned}$$

Answer: (C)

32. Let

$$\Delta = \begin{vmatrix} 2xy & x^2 & y^2\\ x^2 & y^2 & 2xy\\ y^2 & 2xy & x^2\end{vmatrix}$$

Then Δ is equal to

(A) $(x^2+y^2)^3$ (B) $(x^3+y^3)^2$
(C) $-(x^2+y^2)^3$ (D) $-(x^3+y^3)^2$

Solution: Adding R_2 and R_3 to R_1 and taking $(x+y)^2$ common from R_1, we get

$$\Delta = (x+y)^2\begin{vmatrix} 1 & 1 & 1\\ x^2 & y^2 & 2xy\\ y^2 & 2xy & x^2\end{vmatrix}$$

$$= (x+y)^2\begin{vmatrix} 1 & 0 & 0\\ x^2 & y^2-x^2 & 2xy-x^2\\ y^2 & 2xy-y^2 & x^2-y^2\end{vmatrix} \quad (\text{by } C_2 - C_1 \text{ and } C_3 - C_1)$$

$$\begin{aligned}&= (x+y)^2[-(x^2-y^2)^2 - xy(2y-x)(2x-y)]\\ &= -(x+y)^2[(x^2-y^2)^2 + 4x^2y^2 - 2xy(x^2+y^2) + x^2y^2]\\ &= -(x+y)^2[(x^2+y^2)^2 - 2xy(x^2+y^2) + x^2y^2]\\ &= -(x+y)^2[(x^2+y^2) - xy]^2\\ &= -(x^3+y^3)^2\end{aligned}$$

Answer: (D)

33. If

$$\begin{vmatrix} a^2 & bc & ac+c^2\\ a^2+ab & b^2 & ac\\ ab & b^2+bc & c^2\end{vmatrix} = m\,a^n\,b^n\,c^n$$

then $m+n$ value is

(A) 4 (B) 6 (C) 8 (D) 7

Solution: Let Δ be the given determinant. Taking a, b, c common from C_1, C_2 and C_3, respectively, we get

$$\Delta = abc\begin{vmatrix} a & c & a+c\\ a+b & b & a\\ b & b+c & c\end{vmatrix}$$

Now the column operation $C_3 - (C_1 + C_2)$ gives

$$\Delta = \begin{vmatrix} a & c & 0\\ a+b & b & -2b\\ b & b+c & -2b\end{vmatrix}(abc)$$

$$\begin{aligned}&= (abc)(-2b)\begin{vmatrix} a & c & 0\\ a+b & b & 1\\ b & b+c & 1\end{vmatrix}\\ &= -2b(abc)[a(b-b-c) - c(a+b-b)]\\ &= -2b(abc)(-2ac)\\ &= 4a^2b^2c^2\end{aligned}$$

Answer: (B)

34. If

$$\Delta = \begin{vmatrix} a-b-c & 2a & 2a\\ 2b & b-c-a & 2b\\ 2c & 2c & c-a-b\end{vmatrix}$$

then Δ is equal to

(A) $(a+b+c)^2$
(B) $(a+b+c)(a-b)(b-c)(c-a)$
(C) $(a^2+b^2+c^2)(ab+bc+ca)$
(D) $(a+b+c)^3$

Solution: By the operations $C_1 - C_2$ and $C_2 - C_3$ and taking $a+b+c$ common from C_1 and C_2 we get

$$\Delta = (a+b+c)^2\begin{vmatrix} -1 & 0 & 2a\\ 1 & -1 & 2b\\ 0 & 1 & c-a-b\end{vmatrix} = (a+b+c)^3$$

Answer: (D)

35. Let $A = \begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 1\\ 0 & -2 & 4\end{bmatrix}$

If

$$A^{-1} = \frac{1}{6}(A^2 + xA + yI)$$

where x, y are scalars and I is 3×3 unit matrix, then x, y are, respectively,

(A) $-11, 6$ (B) $-6, 11$
(C) $6, 11$ (D) $-6, -11$

Solution: We have $\det A = 4+2 = 6$. Identify A with

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

and represent the cofactors of a_i, b_i, c_i, respectively, with A_i, B_i and C_i. Therefore

$$\begin{aligned} &A_1 = 4+2 = 6, && B_1 = -(0-0) = 0, && C_1 = 0 \\ &A_2 = -(0-0) = 0, && B_2 = 4, && C_2 = -(-2-0) = 2 \\ &A_3 = 0, && B_3 = -1, && C_3 = 1 \end{aligned}$$

Therefore

$$A^{-1} = \frac{1}{\det A}(\text{adj } A) = \frac{1}{6}\begin{bmatrix} 6 & 0 & 0 \\ 0 & 4 & -1 \\ 0 & 2 & 1 \end{bmatrix}$$

Also

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 5 \\ 0 & -10 & 14 \end{bmatrix}$$

So by hypothesis

$$\frac{1}{6}\begin{bmatrix} 6 & 0 & 0 \\ 0 & 4 & -1 \\ 0 & 2 & 1 \end{bmatrix} = A^{-1} = \frac{1}{6}(A^2 + xA + yI)$$

$$= \frac{1}{6}\begin{bmatrix} 1+x+y & 0 & 0 \\ 0 & -1+x+y & 5+x \\ 0 & -10-2x & 14+4x+y \end{bmatrix}$$

Comparing the two sides we get

$$x+y = 5; \quad 5+x = -1; \quad -10-2x = 2; \quad 14+4x+y = 1$$

From the first two equations, we get $x = -6, y = 11$, which also satisfy the other two equations.

Answer: (B)

Second Method (Easy Method): Consider the characteristic polynomial of the matrix A, which is

$$f(x) = \begin{vmatrix} 1-x & 0 & 0 \\ 0 & 1-x & 1 \\ 0 & -2 & 4-x \end{vmatrix}$$

$$= (1-x)[(1-x)(4-x)+2]$$

$$= (1-x)[x^2 - 5x + 6]$$

$$= -x^3 + 6x^2 - 11x + 6$$

By Cayley–Hamilton theorem $f(A) = O$. Hence

$$-A^3 + 6A^2 - 11A + 6 = 0$$

$$\frac{1}{6}(A^2 - 6A + 11I)A = I$$

$$A^{-1} = \frac{1}{6}(A^2 - 6A + 11I)$$

$$x = -6, \; y = 11$$

36. The parameter on which the determinant of the following matrix does not depend is

$$A = \begin{bmatrix} 1 & a & a^2 \\ \cos(p-d)x & \cos px & \cos(p+d)x \\ \sin(p-d)x & \sin px & \sin(p+d)x \end{bmatrix}$$

(A) a (B) p (C) d (D) x

Solution: Add $C_3 - (2\cos dx)C_2$ to C_1. Using

$$\cos(A+B) + \cos(A-B) = 2\cos A \cos B$$

and $$\sin(A+B) + \sin(A-B) = 2\sin A \cos B$$

we have

$$\det A = \begin{vmatrix} 1 + a^2 - 2a\cos dx & a & a^2 \\ 0 & \cos px & \cos(p+d)x \\ 0 & \sin px & \sin(p+d)x \end{vmatrix}$$

$$= (1 + a^2 - 2a\cos dx)\,[\sin(p+d)x\cos px - \cos(p+d)x\sin px]$$

$$= (1 + a^2 - 2a\cos dx)\sin(p+d-p)x$$

$$= (1 + a^2 - 2a\cos dx)\sin dx$$

which does not contain the parameter p.

Answer: (B)

37. Let

$$f(x) = \begin{vmatrix} 2ax & 2ax-1 & 2ax+b+1 \\ b & b+1 & -1 \\ 2(ax+b) & 2(ax+b)+1 & 2ax+b \end{vmatrix}$$

where a and b are real constants and $a \neq 0$, then $f^{-1}(x)$ equals

(A) $\dfrac{x+b}{2a}$ (B) $\dfrac{x-b}{a}$ (C) $\dfrac{x-b}{2a}$ (D) $\dfrac{2x+b}{a}$

Solution: Substracting $R_1 + 2R_2$ from R_3 (i.e., $R_3 \to R_3 - R_1 - 2R_2$) we have

$$f(x)=\begin{vmatrix} 2ax & 2ax-1 & 2ax+b+1 \\ b & b+1 & -1 \\ 0 & 0 & 1 \end{vmatrix}$$

Simplifying we get

$$f(x)=2ax(b+1)-b(2ax-1)=2ax+b$$

Therefore

$$f^{-1}(x)=\frac{x-b}{2a} \quad (\because a\neq 0)$$

Answer: (C)

38. If

$$f(x)=\begin{vmatrix} 1 & x & x+1 \\ 2x & x(x-1) & x(x+1) \\ 3x(x-1) & x(x-1)(x-2) & (x-1)x(x+1) \end{vmatrix}$$

Then $f(2010)$ is equal to

(A) 1 (B) 2010 (C) 2009 (D) 0

Solution: Using the column operation $C_3-(C_1+C_2)$ we get

$$f(x)=\begin{vmatrix} 1 & x & 0 \\ 2x & x(x-1) & 0 \\ 3x(x-1) & x(x-1)(x-2) & 0 \end{vmatrix}$$

Therefore $f(x)=0$ for all real x. Hence

$$f(2010)=0$$

Answer: (D)

39. The system of equations $x-ky-z=0, kx-y-z=0, x+y-z=0$ has a non-zero solution. Then possible values of k are

(A) $-1, 2$ (B) $1, 2$ (C) $0, 1$ (D) $-1, 1$

Solution: If A is a square matrix and X is a column matrix, then the matrix equation $AX=0$ has non-zero solution if det A is equal to zero. Therefore

$$\begin{vmatrix} 1 & -k & -1 \\ k & -1 & -1 \\ 1 & 1 & -1 \end{vmatrix}=0$$

$$1(1+1)+k(-k+1)-1(k+1)=0$$

$$-k^2+1=0 \Rightarrow k=\pm 1$$

Answer: (D)

40. The number of values of k for which the system of equations

$$(k+1)x+8y=4k$$
$$kx+(k+3)y=3k-1$$

has infinitely many solutions, is (are)

(A) 0 (B) 1 (C) 2 (D) ∞

Solution: For a system of non-homogenous equations to have infinitely many solution, the determinant of the coefficient matrix is necessarily be zero. That is

$$(k+1)(k+3)-8k=0$$
$$k^2-4k+3=0$$
$$(k-1)(k-3)=0$$

(1) When $k=1$, the system reduces to one equation which is $x+4y=2$ and has infinitely many solutions.

(2) When $x=3$, the system will be $x+2y=3$ and $x+2y=3/8$ and hence is inconsistent.

Answer: (B)

41. Consider

$$2x-y+2z=2, \quad x-2y+z=-4, \quad x+y+\lambda z=4$$

Then the value of λ such that the given system has no solution is:

(A) 3 (B) 1 (C) 0 (D) -3

Solution: The given system of equations is

$$\begin{bmatrix} 2 & -1 & 2 \\ 1 & -2 & 1 \\ 1 & 1 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 2 \\ -4 \\ 4 \end{bmatrix}$$

Using row transformations R_1-2R_2 and R_3-R_2 we get

$$\begin{bmatrix} 0 & 3 & 0 \\ 1 & -2 & 1 \\ 0 & 3 & \lambda-1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 10 \\ -4 \\ 8 \end{bmatrix}$$

Again using R_3-R_1 we get $\begin{bmatrix} 0 & 3 & 0 \\ 1 & -2 & 1 \\ 0 & 0 & \lambda-1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 10 \\ -4 \\ -2 \end{bmatrix}$

If $\lambda=1$, then the system is equivalent to the system

$$3x=10, \ x-2y+z=-4 \quad \text{and} \quad 0x+0y+0z=-2$$

which is impossible. Therefore when $\lambda=1$, the system has no solution.

Answer: (B)

42. The number of values of λ for which the system of equation

$$3x-y+3z=3, \ x+2y-3z=-2, \ 6x+5y+\lambda z=-3$$

has unique solution is

(A) 2 (B) 4 (C) 8 (D) infinite

Solution: System of non-homogenous equations $AX = B$ (A is a square matrix) has unique solution if and only if A is non-singular. Here

$$A = \begin{bmatrix} 3 & -1 & 3 \\ 1 & 2 & -3 \\ 6 & 5 & \lambda \end{bmatrix}$$

Therefore the determinant is

$$\det A = 3(2\lambda + 15) + 1(\lambda + 18) + 3(5 - 12) = 7\lambda + 42$$

Now A is non-singular if $\det A \neq 0$, that is $\lambda \neq -6$.

Answer: (D)

43. If

$$\Delta_k = \begin{vmatrix} 1 & n & n \\ 2k & n^2+n+1 & n^2+n \\ 2k-1 & n^2 & n^2+n+1 \end{vmatrix}$$

and $\sum_{k=1}^{n} \Delta_k = 56$, then n is equal to

(A) 4 (B) 6 (C) 8 (D) 7

Solution: We have

$$\sum_{k=1}^{n} \Delta_k = \begin{vmatrix} \sum 1 & n & n \\ \sum_{k=1}^{n} 2k & n^2+n+1 & n^2+n \\ \sum_{k=1}^{n} (2k-1) & n^2 & n^2+n+1 \end{vmatrix}$$

$$= \begin{vmatrix} n & n & n \\ n(n+1) & n^2+n+1 & n^2+n \\ n^2 & n^2 & n^2+n+1 \end{vmatrix}$$

$$= \begin{vmatrix} n & 0 & 0 \\ n(n+1) & 1 & 0 \\ n^2 & 0 & n+1 \end{vmatrix} = n(n+1) = 56 = 7 \times 8$$

Answer: (D)

44. If x, y, z are positive and none of them is 1, then the value of the following determinant is

$$\begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix} \text{ is}$$

(A) 1 (B) 0 (C) 2 (D) −2

Solution: Let Δ be the given determinant. Then

$$\Delta = \begin{vmatrix} 1 & \dfrac{\log y}{\log x} & \dfrac{\log z}{\log x} \\ \dfrac{\log x}{\log y} & 1 & \dfrac{\log z}{\log y} \\ \dfrac{\log x}{\log z} & \dfrac{\log y}{\log z} & 1 \end{vmatrix}$$

$$= \frac{1}{(\log x)(\log y)(\log z)} \begin{vmatrix} \log x & \log y & \log z \\ \log x & \log y & \log z \\ \log x & \log y & \log z \end{vmatrix} = 0$$

(since all the three rows are the same)

Answer: (B)

Multiple Correct Choice Type Questions

1. If

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

where $i^2 = -1$, then

(A) $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (B) $B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(C) $C^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ (D) $AB + BA = O$

Solution: We have

$$A^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$B^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} -i^2 & 0 \\ 0 & -i^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C^2 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = \begin{bmatrix} i^2 & 0 \\ 0 & i^2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

$$BA = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} = -AB$$

Therefore, $AB + BA = 0$.

Answers: (A), (B), (C), (D)

2. Let

$$f(x) = \begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix}$$

If $x = -9$ is a root of $f(x) = 0$, then the other roots are

(A) 2 (B) 3 (C) 7 (D) 6

Solution: Adding R_2 and R_3 to R_1 we get

$$f(x) = \begin{vmatrix} x+9 & x+9 & x+9 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix}$$

$$= (x+9)\begin{vmatrix} 1 & 1 & 1 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix}$$

$$= (x+9)\begin{vmatrix} 1 & 0 & 0 \\ 2 & x-2 & 0 \\ 7 & -1 & x-7 \end{vmatrix}$$

$$= (x+9)(x-2)(x-7)$$

Therefore the other roots are 2 and 7.

Answers: (A), (C)

3. Let A, B, C be square matrices of same order and I the unit matrix of the same order such that $A + B + C = AB + BC + CA$. Consider the following three statements.

(i) $ABC = AC - CA$

(ii) $BCA = BA - AB$

(iii) $CAB = CB - BC$

Then

(A) (i) and (ii) equivalent

(B) (ii) and (iii) are equivalent

(C) (iii) and (i) are equivalent

(D) all the three statements are equivalent

Solution: Assume (i). That is

$$ABC = AC - CA \tag{8.24}$$

Now

$$\begin{aligned} ABC + A + B + C &= (AC - CA) + AB + BC + CA \\ &= AC + AB + BC \end{aligned}$$

Therefore

$$\begin{aligned} (A-I)(B-I)(C-I) &= ABC - (AC + AB + BC) \\ &\quad + A + B + C - I \\ &= ABC - (AC + AB + BC) \\ &\quad + AB + BC + CA - I \\ &= ABC + CA - AC - I \\ &= -I \qquad \text{[by Eq. (8.24)]} \end{aligned}$$

This implies $A - I, B - I, C - I$ are invertible matrices and the inverse of $C - I$ is $-(A - I)(B - I)$. This gives

$$(C - I)(A - I)(B - I) = -I$$

$$CAB - (CA + AB + CB) + A + B + C = 0$$

$$CAB - (CA + AB + CB) + AB + BC + CA = 0$$

$$CAB = CB - BC$$

Therefore (i) $\Rightarrow$ (iii).

Similarly, by permuting the letters A, B, C, we can show that (i), (ii) and (iii) are equivalent statements.

Answers: (A), (B), (C), (D)

4. If x is real and

$$\Delta(x) = \begin{vmatrix} x^2 + x & 2x - 1 & x + 3 \\ 3x + 1 & 2 + x^2 & x^3 - 3 \\ x - 3 & x^2 + 4 & 2x \end{vmatrix}$$

$$= a_0x^7 + a_1x^6 + a_2x^5 + \cdots + a_6x + a_7$$

then

(A) $a_7 = 21$ (B) $\sum_{k=0}^{6} a_k = 111$

(C) $\Delta(-1) = -32$ (D) $\Delta(1) = 121$

Solution: We have

$$a_7 = \Delta(0) = \begin{vmatrix} 0 & -1 & 3 \\ 1 & 2 & -3 \\ -3 & 4 & 0 \end{vmatrix}$$

$$= 1(0-9) + 3(4+6)$$

$$= -9 + 30 = 21$$

Therefore (A) is true. Again

$$\sum_{k=0}^{7} a_k = \Delta(1) = \begin{vmatrix} 2 & 1 & 4 \\ 4 & 3 & -2 \\ -2 & 5 & 2 \end{vmatrix}$$

$$= 2(6 + 10) - 1(8 - 4) + 4(20 + 6) = 32 - 4 + 104 = 132$$

Therefore

$$\sum_{k=0}^{6} a_k = 132 - a_7 = 132 - 21 = 111$$

Therefore (B) is true. Now

$$\Delta(-1) = \begin{vmatrix} 0 & -3 & 2 \\ -2 & 3 & -4 \\ -4 & 5 & -2 \end{vmatrix} = 3(4 - 16) + 2(-10 + 12)$$

$$= -36 + 4 = -32$$

Therefore (C) is true.

Answers: (A), (B), (C)

5. Let a, b, c be real numbers and

$$\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} 1 & 1 & 1 \\ b+c & c+a & a+b \\ b^2+c^2 & c^2+a^2 & a^2+b^2 \end{vmatrix}$$

Then

(A) $\Delta_1 + \Delta_2 = 0$
(B) $\Delta_1 = \Delta_2$
(C) $\Delta_2 = (ab+bc+ca)\Delta_1$
(D) $\Delta_1 = \Delta_2 = (b-c)(c-a)(a-b)$

Solution: We have

$$\Delta_1 = \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix}$$

$$= (b-a)(c^2-a^2) - (c-a)(b^2-a^2)$$

$$= (a-b)(c-a)[-(c+a)+(b+a)]$$

$$= (a-b)(b-c)(c-a)$$

Also

$$\Delta_2 = \begin{vmatrix} 1 & 1 & 1 \\ b+c & c+a & a+b \\ b^2+c^2 & c^2+a^2 & a^2+b^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ b+c & a-b & a-c \\ b^2+c^2 & a^2-b^2 & a^2-c^2 \end{vmatrix}$$

$$= (a-b)(a^2-c^2) - (a-c)(a^2-b^2)$$

$$= (a-b)(c-a)[-(c+a)+(a+b)]$$

$$= (a-b)(b-c)(c-a)$$

Therefore $\Delta_1 = \Delta_2 = (a-b)(b-c)(c-a)$. So (B) and (D) are true.

Answers: (B), (D)

6. Let a, b, c be real numbers and

$$D_1 = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}, \quad D_2 = \begin{vmatrix} a & b+c & a^2 \\ b & c+a & b^2 \\ c & a+b & c^2 \end{vmatrix}, \quad D_3 = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$$

Then

(A) $D_2 = D_1(a+b+c)$
(B) $D_3 = -D_2$
(C) $-D_2 = (a+b+c)D_1 = D_3$
(D) $D_3 = D_2 = (ab+bc+ca)D_1$

Solution: It is known that $D_1 = (a-b)(b-c)(c-a)$. Now

$$D_2 = \begin{vmatrix} a+b+c & b+c & a^2 \\ a+b+c & c+a & b^2 \\ a+b+c & a+b & c^2 \end{vmatrix} \quad \text{(by adding } C_2 \text{ to } C_1)$$

$$= \begin{vmatrix} a+b+c & -a & a^2 \\ a+b+c & -b & b^2 \\ a+b+c & -c & c^2 \end{vmatrix} \quad \text{(by } C_2 - C_1)$$

$$= -(a+b+c)\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = -(a+b+c)D_1$$

$$D_3 = \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^3 & b^3-a^3 & c^3-a^3 \end{vmatrix} \quad (C_2 - C_1 \text{ and } C_3 - C_1)$$

$$= (b-a)(c^3-a^3) - (c-a)(b^3-a^3)$$

$$= (b-a)(c-a)[(c^2+ca+a^2)-(b^2+ab+a^2)]$$

$$= (b-a)(c-a)[(c-b)(c+b)+a(c-b)]$$

$$= (b-a)(c-a)(c-b)(a+b+c)$$

$$= (a-b)(b-c)(c-a)(a+b+c)$$

$$= -D_2$$

This gives $D_3 = -D_2 = (a+b+c)D_1$. Therefore (B) and (C) are true.

Answers: (B), (C)

7. If P is any square matrix, then the sum of its principal diagonal elements is called trace of P and is denoted by $t_r(P)$. Let A and B be two square matrices of same order and λ a scalar. Which of the following are true?

(A) $t_r(A+B) = t_r(A) + t_r(B)$ (B) $t_r(\lambda A) = \lambda t_r(A)$
(C) $t_r(AB) = t_r(A)t_r(B)$ (D) $t_r(AB) = t_r(BA)$

Solution: Let $A = [a_{ij}]_{n\times n}$, $B = [b_{ij}]_{n\times n}$. Therefore

$$t_r(A+B) = \sum_{i=1}^{n}(a_{ii}+b_{ii}) = \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii}$$
$$= t_r(A) + t_r(B)$$

This implies (A) is true. Again

$$t_r(\lambda A) = \sum_{i=1}^{n} \lambda a_{ii}$$

$$= \lambda \sum_{i=1}^{n} a_{ii} = \lambda t_r(A)$$

This implies (B) is true. Let $AB = [c_{ik}]_{n\times n}$ where

$$c_{ik} = \sum_{j=1}^{n} a_{ij}\, b_{jk}$$

Therefore

$$\begin{aligned}\sum c_{ii} &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}b_{ji}\\ &= \sum_{j=1}^{n}\sum_{i=1}^{n} b_{ji}a_{ij}\\ &= t_r(BA)\end{aligned}$$

Clearly $t_r(AB) \neq (t_r A)(t_r B)$. Therefore (A), (B) and (D) are true.

Answers: (A), (B), (D)

8. Consider the system of equations $3x + my = m$ and $2x - 5y = 20$. Then

(A) the system is inconsistent (i.e., has no solution) if $m = 15/2$

(B) the system has no solution, if $2m = -15$

(C) has unique solution, if $m \neq -15/2$

(D) has solutions with $x > 0$, $y > 0$ if and only if $m \in \left(-\infty, \dfrac{-15}{2}\right) \cup (30, \infty)$

Solution: The given system is equivalent to the matrix equation

$$\begin{bmatrix} 3 & m \\ 2 & -5 \end{bmatrix} X = \begin{bmatrix} m \\ 20 \end{bmatrix}$$

where

$$X = \begin{bmatrix} x \\ y \end{bmatrix}$$

If

$$\begin{vmatrix} 3 & m \\ 2 & -5 \end{vmatrix} = 0$$

then $m = -15/2$ and hence the equations are

$$6x - 15y = -15 \quad \text{and} \quad 2x - 5y = 20$$

which are inconsistent. Therefore (B) is true.

If $m \neq -15/2$, by Crammer's rule, the system has unique solution. Therefore (C) is true.

Now using row operations $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - 2R_1$, respectively, we get

$$\begin{bmatrix} 1 & m+5 \\ 2 & -5 \end{bmatrix} X = \begin{bmatrix} m-20 \\ 20 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & +5 \\ 0 & -2m-15 \end{bmatrix} X = \begin{bmatrix} m-20 \\ -2m+60 \end{bmatrix}$$

Therefore

$$x + (m+5)y = m - 20$$

and

$$-(2m+15)y = 60 - 2m$$

Therefore

$$y = \frac{2m-60}{2m+15} \quad \text{and} \quad x = \frac{25m}{2m+15}$$

This gives

$$x > 0 \quad \text{and} \quad y > 0 \Leftrightarrow m \in \left(-\infty, \frac{-15}{2}\right) \cup (30, \infty)$$

Therefore (D) is true.

Answers: (B), (C), (D)

9. It is given that

$$\begin{vmatrix} x^2+x & x+1 & x-2 \\ 2x^2+3x-1 & 3x & 3x-3 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix} = xA + B$$

where A and B are determinants of order 3 not involving x. Then

(A) $A = \begin{vmatrix} 1 & 1 & 1 \\ -4 & 0 & 0 \\ 3 & -3 & 3 \end{vmatrix}$ (B) $B = \begin{vmatrix} 0 & 1 & -2 \\ -4 & 0 & 0 \\ 3 & -3 & 3 \end{vmatrix}$

(C) $A = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 0 & 0 \\ 3 & -3 & 3 \end{vmatrix}$ (D) $B = \begin{vmatrix} 0 & 1 & -2 \\ -4 & 0 & 0 \\ 3 & 3 & 3 \end{vmatrix}$

Solution: Let

$$\Delta = \begin{vmatrix} x^2+x & x+1 & x-2 \\ 2x^2+3x-1 & 3x & 3x-3 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix}$$

$$\Delta = \begin{vmatrix} x^2+x & x+1 & x-2 \\ -4 & 0 & 0 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix} \quad [\text{by } R_2 \to R_2 - (R_1 + R_3)]$$

$$\Delta = \begin{vmatrix} x & x+1 & x-2 \\ -4 & 0 & 0 \\ 2x+3 & 2x-1 & 2x-1 \end{vmatrix} \quad \left[\text{by } R_1 \to R_1 + \left(\frac{x^2}{4}\right)R_2 \text{ and } R_3 \to R_3 + \frac{x^2}{4}R_2\right]$$

$$\Delta = \begin{vmatrix} x & x+1 & x-2 \\ -4 & 0 & 0 \\ 3 & -3 & 3 \end{vmatrix} \quad (\text{by } R_3 \to R_3 - 2R_1)$$

$$= x\begin{vmatrix} 1 & 1 & 1 \\ -4 & 0 & 0 \\ 3 & -3 & 3 \end{vmatrix} + \begin{vmatrix} 0 & 1 & -2 \\ -4 & 0 & 0 \\ 3 & -3 & 3 \end{vmatrix}$$

$$= xA + B$$

Answers: (A), (B)

10. Let

$$f(x) = \begin{vmatrix} x & -6 & -1 \\ 2 & -3x & x-3 \\ -3 & 2x & x+2 \end{vmatrix}$$

and $\alpha < \beta < \gamma$ be the roots of $f(x) = 0$. Then

(A) $\gamma = 2$

(B) $f(x) < 0$ for $\alpha < x < \beta$

(C) $f(x) > 0$ for $\beta < x < \gamma$

(D) $f(x) > 0$ for $\alpha < x < \beta$

Solution: By the row operation $R_1 \to R_1 - R_2$ and taking $x - 2$ common from R_1, we get

$$f(x) = (x-2)\begin{vmatrix} 1 & 3 & -1 \\ 2 & -3x & x-3 \\ -3 & 2x & x+2 \end{vmatrix}$$

$$f(x) = (x-2)\begin{vmatrix} 1 & 3 & -1 \\ 0 & -3x-6 & x-1 \\ 0 & 2x+9 & x-1 \end{vmatrix}$$

(by $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 + 3R_1$)

$$f(x) = (x-2)\begin{vmatrix} 1 & 0 & 0 \\ 0 & -3x-6 & x-1 \\ 0 & 2x+9 & x-1 \end{vmatrix}$$

(by $C_2 \to C_2 - 3C_1$, $C_3 \to C_3 + C_1$)

Therefore

$$f(x) = (x-2)(x-1)(-5x-15)$$
$$f(x) = -3(x+3)(x-1)(x-2)$$

Hence we get

$$\alpha = -3, \beta = 1, \gamma = 2$$

Answers: (A), (B), (C)

Matrix-Match Type Questions

1. Match the items of Column I with those of Column II. Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & 1 \\ -2 & 0 \end{bmatrix}$$

Column I	*Column II*
(A) $A + A^{\mathrm{T}}$	(p) $\begin{bmatrix} -5 & -11 \\ 1 & 3 \end{bmatrix}$
(B) $(A+B)^{\mathrm{T}}$	(q) $\begin{bmatrix} 2 & 5 \\ 5 & 8 \end{bmatrix}$
(C) $(AB)^{\mathrm{T}}$	(r) $\begin{bmatrix} 0 & 1 \\ 3 & 4 \end{bmatrix}$
(D) $B^{\mathrm{T}}A^{\mathrm{T}}$	(s) $\begin{bmatrix} 4 & 1 \\ 3 & 0 \end{bmatrix}$

Solution:

(A) $$A + A^{\mathrm{T}} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 5 & 8 \end{bmatrix}$$

Answer: (A) → (q)

(B) $$A + B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 1 & 4 \end{bmatrix}$$

Therefore

$$(A+B)^{\mathrm{T}} = \begin{bmatrix} 0 & 1 \\ 3 & 4 \end{bmatrix}$$

Answer: (B) → (r)

(C) $$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} -1-4 & 1+0 \\ -3-8 & 3+0 \end{bmatrix} = \begin{bmatrix} -5 & 1 \\ -11 & 3 \end{bmatrix}$$

Therefore

$$(AB)^{\mathrm{T}} = \begin{bmatrix} -5 & -11 \\ 1 & 3 \end{bmatrix}$$

Answer: (C) → (p)

(D) $(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$

Answer: (D) → (p)

2. Match the items of Column I with those of Column II.

Column I	*Column II*
(A) $\begin{vmatrix} \cos(A-P) & \cos(A-Q) & \cos(A-R) \\ \cos(B-P) & \cos(B-Q) & \cos(B-R) \\ \cos(C-P) & \cos(C-Q) & \cos(C-R) \end{vmatrix}$ equals	(p) 1
(B) If α, β, γ are roots of $x^3 + bx + c = 0$, then the value of the determinant $\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix}$ is	(q) -1
(C) $\begin{vmatrix} b+c & a & 1 \\ c+a & b & 1 \\ a+b & c & 1 \end{vmatrix}$ is equal to	(r) 0
(D) If the system of equations $x + y = 3z, (1+\lambda)x + (2+\lambda)y = 8z$, $x - (1+\lambda)y = -(\lambda+2)$ has infinitely many solutions, then value of λ is	(s) $-5/3$

Solution:

(A) Given determinant is

$$\begin{vmatrix} \cos A & \sin A & 0 \\ \cos B & \sin B & 0 \\ \cos C & \sin C & 0 \end{vmatrix} \times \begin{vmatrix} \cos P & \sin P & 0 \\ \cos Q & \sin Q & 0 \\ \cos R & \sin R & 0 \end{vmatrix} = 0 \times 0 = 0$$

Answer: (A) → (r)

(B) $$\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix} = \begin{vmatrix} \alpha+\beta+\gamma & \alpha+\beta+\gamma & \alpha+\beta+\gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 0 \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix} \qquad (\because \alpha+\beta+\gamma = 0)$$

$$= 0$$

Answer: (B) → (r)

(C) $$\begin{vmatrix} b+c & a & 1 \\ c+a & b & 1 \\ a+b & c & 1 \end{vmatrix} = \begin{vmatrix} a+b+c & a & 1 \\ a+b+c & b & 1 \\ a+b+c & c & 1 \end{vmatrix} = 0$$

Answer: (C) → (r)

(D) The system has infinitely many solutions

$$\Rightarrow \begin{vmatrix} 1 & 1 & -3 \\ 1+\lambda & 2+\lambda & -8 \\ 1 & -(1+\lambda) & 2+\lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1 & 0 & 0 \\ 1+\lambda & 1 & 3\lambda-5 \\ 1 & -2-\lambda & 5+\lambda \end{vmatrix} = 0$$

Therefore $3\lambda^2 + 2\lambda - 5 = 0$ and hence

$$\lambda = 1, \frac{-5}{3}$$

Answer: (D) → (p), (s)

3. Match the items of Column I with those of Column II. Let

$$S = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \text{ and } A = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Column I	*Column II*
(A) $S^2 =$	(p) $\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$
(B) $2S^{-1} =$	(q) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$
(C) $1/2(SA) =$	(r) $\begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & -1 & 3 \end{bmatrix}$
(D) $SAS^{-1} =$	(s) $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

Solution:

(A) $$S^2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Answer: (A) → (s)

(B) $$\text{adj } S = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

Therefore

$$S^{-1} = \frac{\text{adj } S}{\det S} = \frac{1}{2}\text{adj } S$$

Answer: (B) → (p)

(C) $$\frac{1}{2}SA = \frac{1}{2}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$

Answer: (C) → (q)

(D) $$SAS^{-1} = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 4 & 0 \end{bmatrix} \times \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix} \times \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & -1 & 3 \end{bmatrix}$$

Answer: (D) → (r)

4. Match the items of Column I with those of Column II.

Column I	*Column II*
(A) If $w \neq 1$ is a cube root of unity, then the value of the determinant $\begin{vmatrix} 1 & w^3 & w^2 \\ w^3 & 1 & w \\ w^2 & w & 1 \end{vmatrix}$	(p) 0
(B) $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$	(q) 2
(C) $\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix}$	(r) 3
(D) If $\begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix} = k(3abc-a^3-b^3-c^3)$, then k is	(s) −3

Solution:

(A) $$\Delta = \text{Given determinant} = \begin{vmatrix} 1 & 1 & w^2 \\ 1 & 1 & w \\ w^2 & w & 1 \end{vmatrix}$$

$$\Delta = \begin{vmatrix} 0 & 1 & w^2 \\ 0 & 1 & w \\ w^2-w & w & 1 \end{vmatrix} = (w^2-w)(w-w^2) \quad (\text{by } C_1 - C_2)$$

$$= -(w^4 - 2w^3 + w^2)$$

$$= -[w - 2 + w^2]$$

$$= -[-1-2] = 3$$

Answer: (A) → (r)

(B) Given

$$\text{determinant} = \begin{vmatrix} 0 & 0 & 0 \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} (\text{on adding } R_2 + R_3 \text{ to } R_1) = 0$$

Answer: (B) → (p)

(C) Given

$$\text{determinant} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \frac{1}{abc}\begin{vmatrix} a & a^2 & abc \\ b & b^2 & abc \\ c & c^2 & abc \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} = 0$$

Answer: (C) → (p)

(D) Given determinant equals

$$2\begin{vmatrix} a+b+c & b+c & c+a \\ a+b+c & c+a & a+b \\ a+b+c & a+b & b+c \end{vmatrix} (\text{by } C_1 + C_2 + C_3)$$

$$= 2\begin{vmatrix} a+b+c & -a & -b \\ a+b+c & -b & -c \\ a+b+c & -c & -a \end{vmatrix} = 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 1 & b & c \\ 1 & c & a \end{vmatrix}$$

$$= 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 0 & b-a & c-b \\ 0 & c-a & a-b \end{vmatrix}$$

$$= 2(a+b+c)[-(a-b)^2 - (c-a)(c-b)]$$
$$= -2(a+b+c)[a^2+b^2+c^2-ab-bc-ca]$$
$$= 2[3abc - a^3 - b^3 - c^3]$$

Answer: (D) → (q)

5. Match the items of Column I with those of Column II.

Column I	*Column II*
(A) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $A^3 = O$, then A^2 is	(p) orthogonal matrix
(B) The matrix $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is	(q) zero matrix
(C) If A and B are symmetric matrices, then $AB + BA$ is	(r) idempotent matrix
(D) The matrix $\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ is	(s) symmetric matrix

Solution:

(A) $A^3 = O \Rightarrow \det A = 0 \Rightarrow ad - bc = 0 \Rightarrow ad = bc$

Now A satisfies its characteristic equation $|A = xI| = 0$ where I is a 2×2 unit matrix. This implies

$$A^2 - (a+d)A = 0 \qquad (\because ad = bc)$$
$$A^2 = (a+d)A$$

(1) If $a + d = 0$, then $A^2 = 0$.
(2) If $a + d \neq 0$, then

$$0 = A^3 = (a+d)A^2 \text{ and hence } A^2 = O$$

Answer: (A) → (q), (r), (s)

(B) Let

$$P = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Therefore

$$PP^{\mathrm{T}} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Therefore P is orthogonal.

Answer: (B) → (p)

(C) By hypothesis $A^{\mathrm{T}} = A$ and $B^{\mathrm{T}} = B$. Then

$$(AB + BA)^{\mathrm{T}} = (AB)^{\mathrm{T}} + (BA)^{\mathrm{T}}$$
$$= B^{\mathrm{T}}A^{\mathrm{T}} + A^{\mathrm{T}}B^{\mathrm{T}} = BA + AB = AB + BA$$

Hence $AB + BA$ is symmetric.

Answer: (C) → (s)

(D) Let $Q = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$

Now

$$Q^2 = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$
$$= \begin{bmatrix} 4+2-4 & -4-6+8 & -8-8+12 \\ -2-3+4 & 2+9-8 & 4+12-12 \\ 2+2-3 & -2-6+6 & -4-8+9 \end{bmatrix} = Q$$

Answer: (D) → (r)

6. Match the items of Column I with those of Column II.

Column I	*Column II*
(A) The system of equations $\lambda x + y + z = 0, -x + \lambda y + z = 0, -x - y + \lambda z = 0$ will have non-zero solution, if real value of λ is	(p) 1
(B) In the system of equations given in (A) if $\lambda = 1$, then the number of solutions is	(q) -1
(C) If the system of equations $x = cy + bz, y = az + cx, z = bx + ay$ has non-zero solution in x, y and z, then $a^2 + b^2 + c^2 + 2abc$ is equal to	(r) ± 1
(D) If P is a matrix of order 3×3 such that $P^{\mathrm{T}}P = I$ (unit matrix of order 3×3) and $\det P = 1$ then $\det (P - I)$ equals	(s) 0

Solution:

(A) The system has non-zero solution, if

$$\begin{vmatrix} \lambda & 1 & 1 \\ -1 & \lambda & 1 \\ -1 & -1 & \lambda \end{vmatrix} = 0$$

Solving this we get

$$\lambda(\lambda^2 + 3) = 0 \Rightarrow \lambda = 0$$

Answer: (A) → (s)

(B) In the above system, if $\lambda = 1$, then the matrix

$$\begin{bmatrix} \lambda & 1 & 1 \\ -1 & \lambda & 1 \\ -1 & -1 & \lambda \end{bmatrix}$$

is non-singular and hence $x = 0, y = 0, z = 0$ is the only solution. This is called trivial solution.

Answer: (B) → (p)

(C) The system has non-zero solution. This implies

$$\begin{vmatrix} -1 & c & b \\ c & -1 & a \\ b & a & -1 \end{vmatrix} = c$$

$$-1(1-a^2) - c(-c-ab) + b(ca+b) = 0$$
$$a^2 + b^2 + c^2 + 2abc = 1$$

Answer: (C) → (p)

(D) $|P - I| = |P^{\mathrm{T}}(P - I)| = |P^{\mathrm{T}}P - P^{\mathrm{T}}| = |I - P^{\mathrm{T}}|$

$$= |(I-P)^{\mathrm{T}}| + |I - P| = -|P - I|$$

Therefore $|P - I| = 0$.

Answer: (D) → (s)

Comprehension-Type Questions

1. Passage: If A is a square matrix, then the polynomial equation $f(x) \equiv |A - xI| = 0$ is called characteristic equation of the matrix A. It is given that every square matrix satisfies its characteristic equation, that is $f(A) = O$. If

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

then answer the following questions:

(i) The characteristic equation of the matrix A is

(A) $x^3 - 3x^2 - 8x + 2 = 0$ (B) $x^3 + 3x^2 - 8x + 2 = 0$

(C) $x^3 + 3x^2 - 8x - 2 = 0$ (D) $x^3 + 3x^2 + 8x + 2 = 0$

(ii) A^{-1} is equal to

(A) $\dfrac{1}{2}\begin{bmatrix} \frac{-1}{8} & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$ (B) $-\dfrac{1}{2}\begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$

(C) $\begin{bmatrix} -1 & -1 & -1 \\ 8 & -6 & 2 \\ -3 & 3 & -1 \end{bmatrix}$ (D) $\begin{bmatrix} -1 & 1 & -1 \\ -8 & 6 & 2 \\ 5 & -3 & 1 \end{bmatrix}$

(iii) $(A^{-1})^2$ equals

(A) $4I + \dfrac{1}{2}A^{-1} + \dfrac{3}{2}I$ (B) $-\dfrac{1}{2}(A - 8A^{-1} - 3I)$

(C) $\dfrac{1}{2}(A + 8A^{-1} + 3I)$ (D) $\dfrac{1}{2}(A + 8A^{-1} - 3I)$

Solution:

(i) Characteristic equation of A is

$$\begin{vmatrix} -x & 1 & 2 \\ 1 & 2-x & 3 \\ 3 & 1 & 1-x \end{vmatrix} = 0$$

Solving we get

$$-x[(2-x)(1-x)-3] - 1(1-x-9) + 2(1-6+3x) = 0$$
$$-x[2-3x+x^2-3] + 8 + x + 6x - 10 = 0$$
$$-x^3 + 3x^2 + x + 8 + x + 6x - 10 = 0$$
$$x^3 - 3x^2 - 8x + 2 = 0$$

Answer: (A)

(ii) From the given information

$$A^3 - 3A^2 - 8A + 2I = O$$
$$-\frac{1}{2}(A^2 - 3A - 8I)A = I$$
$$A^{-1} = -\frac{1}{2}(A^2 - 3A - 8I)$$

Answer: (B)

(iii) $(A^{-1})^2 = A^{-1}A^{-1} = -\dfrac{1}{2}(A^2 - 3A - 8I)A^{-1}$

$$= -\frac{1}{2}(A - 3I - 8A^{-1})$$

Answer: (B)

2. Passage: Let A be a 3×3 matrix,

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Using elementary row operations on the matrix equation $AX = B$, we obtain an equation of the form $A'X = B'$ which is equivalent to the system $AX = B$. That is either both systems are inconsistent or both have the same set of solutions in x, y and z. Consider the following system of equations

$$x + y + z = 6,\quad x + 2y + 2z = 10 \quad \text{and} \quad x + 2y + \lambda z = \mu$$

Answer the following questions

(i) The number of values of λ for which the system has unique solution is

(A) only one value

(B) all real values except two values

(C) only two real values

(D) all real values except one value

(ii) The system has no solution if

(A) $\lambda = 2, \mu \neq 10$

(B) $\lambda = 3, \mu = 10$

(C) $\lambda = -3, \mu = 10$

(D) $\lambda = -3, \mu = -10$

(iii) The system has infinitely many solutions, if

(A) $\lambda \neq 3, \mu \neq 10$

(B) $\lambda = 2, \mu = 10$

(C) $\lambda \neq 3, \mu = 10$

(D) $\lambda = 0, \mu = 10$

Solution:

(i) The given system is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & \lambda \end{bmatrix} X = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & \lambda - 1 \end{bmatrix} X = \begin{bmatrix} 6 \\ 4 \\ \mu - 6 \end{bmatrix} \quad (\text{by } R_2 \to R_2 - R_1, R_3 \to R_3 - R_1)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & \lambda - 2 \end{bmatrix} X = \begin{bmatrix} 2 \\ 4 \\ \mu - 10 \end{bmatrix} \quad (\text{by } R_1 \to R_1 - R_2, R_3 \to R_3 - R_2)$$

Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & \lambda \end{bmatrix}$$

then det $A = \lambda - 2$. Therefore $\lambda \neq 2 \Rightarrow A$ is non-singular matrix and hence

$$AX = B \Rightarrow X = A^{-1}B$$

is the unique solution.

Answer: (D)

(ii) If $\lambda = 2$ and $\mu \neq 10$, then the system reduces to

$$x = 6, \quad y + z = 4, \quad 0x + 0y + 0z = \mu - 10 \neq 0$$

Hence, no solution, if $\lambda = 2$ and $\mu \neq 10$.

Answer: (A)

(iii) For $\lambda = 2, \mu = 10$; the system has infinite number of solutions

Answer: (B)

3. Passage: Suppose A is a square matrix of order 3×3, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and B is column matrix of 3×1 order. It is given that, applying elementary row operations on the equation $AX = B$, we get a system of the form $A'X = B'$ such that both systems are equivalent. Based on this information, answer the following questions for the equations

$$x + y + z = 1$$
$$x + 2y + 4z = \lambda$$
and
$$x + 4y + 10z = \lambda^2$$

(i) The system is consistent for

(A) only one value of λ

(B) only two values of λ

(C) all real values except two values

(D) infinite number of values

(ii) The system is inconsistent for

(A) only one value of λ

(B) only two values of λ

(C) only three values of λ

(D) infinite number of values of λ

(iii) When $\lambda = 1$, solution of the given system is given by

(A) $x = 2k + 1, y = -3k, z = k$

(B) $x = 2k, y = -3k + 1, z = k$

(C) $x = k, y = -3k, z = k$

(D) $x = 2k - 1, y = -3k, z = k$

where k is any real number.

Solution: Given system is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 10 \end{bmatrix} X = \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 3 & 9 \end{bmatrix} X = \begin{bmatrix} 1 \\ \lambda - 1 \\ \lambda^2 - 1 \end{bmatrix} \quad (\text{by } R_2 \to R_2 - R_1, R_3 \to R_3 - R_1)$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} X = \begin{bmatrix} 2 - \lambda \\ \lambda - 1 \\ \lambda^2 - 3\lambda + 2 \end{bmatrix} \quad (\text{by } R_1 \to R_1 - R_2, R_3 \to R_3 - 3R_2)$$

(i) The system is inconsistent if $\lambda \neq 1$ and 2.

Answer: (B)

(ii) The system is consistent, if $\lambda = 1, 2$.

Answer: (D)

(iii) If $\lambda = 1$, the given system is equivalent to the system $x - 2z = 1, y + 3z = 0$ whose solution is $x = 2k + 1$, $y = -3k$ and $z = k$.

Answer: (A)

Assertion–Reasoning Type Questions

In each of the following, two statements, I and II, are given and one of the following four alternatives has to be chosen.

(A) Both I and II are correct and II is a correct reasoning for I.

(B) Both I and II are correct but II is not a correct reasoning for I.

(C) I is true, but II is not true.

(D) I is not true, but II is true.

1. Statement I: There exit matrices B and C of order 2×2 with integer elements such that

$$B^3+C^3=\begin{bmatrix}-1 & 1\\ 0 & -2\end{bmatrix}$$

Statement II: Every square matrix satisfies its characteristic equation. That is, if A is a square matrix, then A satisfies the polynomial equation det $(A - xI) = 0$ where I is a unit matrix of same order as that of A.

Solution: Let

$$A=\begin{bmatrix}-1 & 1\\ 0 & -2\end{bmatrix}$$

Therefore

$$|A-xI|=\begin{vmatrix}-1-x & 1\\ 0 & -2-x\end{vmatrix}=(1+x)(2+x)$$

The characteristic equation is

$$f(x)\equiv(1+x)(2+x)=0$$

that is

$$f(x)\equiv x^2+3x+2=0$$

Statement II is true by Cayley–Hamilton theorem.

$$\begin{aligned}f(A)=0 &\Rightarrow A^2+3A+2I=0\\ &\Rightarrow A^3+3A^2+2A=0\\ &\Rightarrow (A+1)^3-I=A\end{aligned}$$

Take

$$B=A+I=\begin{bmatrix}0 & 1\\ 0 & -1\end{bmatrix} \text{and } C=-I$$

We get $A=B^3+C^3$.

2. Statement I: If

$$\begin{bmatrix}2 & 1\\ 3 & 2\end{bmatrix}A\begin{bmatrix}-3 & 2\\ 5 & -3\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

then A is equal to

$$\begin{bmatrix}1 & -1\\ 1 & 0\end{bmatrix}$$

Statement II: If A and B are square matrices of the same order and P, Q are non-singular matrices compatible for multiplication with A such that $PAQ = B$, then $A = P^{-1}BQ^{-1}$.

Solution: Clearly Statement II is true. Now let

$$P=\begin{bmatrix}2 & 1\\ 3 & 2\end{bmatrix}, \text{ and } Q=\begin{bmatrix}-3 & 2\\ 5 & -3\end{bmatrix}$$

$$P^{-1}=\begin{bmatrix}2 & -1\\ -3 & 2\end{bmatrix} \text{ and } Q^{-1}=\begin{bmatrix}3 & 2\\ 5 & 3\end{bmatrix}$$

$$\begin{aligned}A=P^{-1}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}Q^{-1}=P^{-1}Q^{-1}&=\begin{bmatrix}2 & -1\\ -3 & 2\end{bmatrix}\begin{bmatrix}3 & 2\\ 5 & 3\end{bmatrix}\\ &=\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}\neq\begin{bmatrix}1 & -1\\ 1 & 0\end{bmatrix}\end{aligned}$$

Therefore statement I is false.

3. Statement I: If A and B are square matrices of order 3×3 then adj (AB) = (adj B)(adj A).

Statement II: For a square matrix of order 3×3,

$$P(\text{adj } P)=(\text{adj } P)P=(\det P)I$$

where I is the third order unit matrix.

Solution: $P(\text{adj } P)=(\text{adj } P)P=(\det P)I$

Therefore Statement II is true. Now

$$\begin{aligned}AB(\text{adj } B)\cdot(\text{adj } A)&=A(B\,\text{adj}\,B)(\text{adj}\,A)\\ &=A(\det B)I(\text{adj}\,A)\\ &=(\det B)A(\text{adj } A)\\ &=(\det B)(\det A)I\\ &=[\det(AB)]I\end{aligned}$$

Similarly (adj B)(adj A)AB = [det (AB)]I

Therefore adj (AB) = (adj B)(adj$A$).

Answer: (A)

4. Statement I: If A is a nonsingular matrix, then adj $(A^{-1}) = (\text{adj } A)^{-1}$.

Statement II: If P and Q are square matrices then adj (PQ) = (adj Q)(adj P).

Solution: Statement II is clear from Q3 above.

$$\begin{aligned}\text{Now } (\text{adj}\,A)(\text{adj}\,A^{-1})&=\text{adj}\,(A^{-1}A)\\ &=\text{adj}\,(I)=I\end{aligned}$$

Also

$$(\text{adj}\,A^{-1})(\text{adj}\,A)=\text{adj}(AA^{-1})=\text{adj}\,I=I$$

Therefore $(\text{adj}\,A)^{-1}=\text{adj}\,(A^{-1})$.

Answer: (A)

5. Statement I: Let B be a matrix of 3×3 order and adj $B = A$. If P and Q are matrices of 3×3 order such that $|P| = 1 = |Q|$ then $\text{adj}\,(Q^{-1}BP^{-1}) = PAQ$.

Statement II: If M is non-singular square matrix of order 3×3, then $\text{adj}\,(M^{-1}) = (\text{adj}\,M)^{-1}$.

Solution: Statement II is true is clear from Q4 above.

$$\begin{aligned}\text{adj}\,(Q^{-1}BP^{-1}) &= (\text{adj}\,P^{-1})(\text{adj}\,B)(\text{adj}\,Q^{-1})\\ &= (\text{adj}\,P^{-1})(\text{adj}\,B)(\text{adj}\,Q)^{-1}\\ &= P(\text{adj}\,B)\,Q\\ &= PAQ\end{aligned}$$

since $|P| = 1 = |Q| \Rightarrow (\text{adj}\,P)^{-1} = P$ and $(\text{adj}\,Q)^{-1} = Q$.

Answer: (A)

6. Statement I: If

$$A = \begin{bmatrix} 1 & 2 & 0\\ 2 & -1 & 0\\ 0 & 0 & -1\end{bmatrix}$$

then

$$(A^{-1})^2 = \frac{1}{5}[A + I - 5\text{A}^{-1}]$$

Statement II: If P is a square matrix of order 3×3, then P satisfies the polynomial equation $|P - xI| = 0$.

Solution: Statement II is Cayley–Hamilton theorem.

$$|A - xI| = \begin{vmatrix} 1-x & 2 & 0\\ 2 & -1-x & 0\\ 0 & 0 & -1-x\end{vmatrix} = -x^3 - x^2 + 5x + 5$$

Since A satisfies its characteristic equation we get

$$\begin{aligned}-A^3 - A^2 + 5A + 5I &= O\\ A^3 + A^2 - 5A - 5I &= O\\ \frac{1}{5}A(A^2 + A - 5I) &= I\\ A^{-1} &= \frac{1}{5}(A^2 + A - 5I)\\ (A^{-1})^2 &= \frac{1}{5}(A + I - 5A^{-1})\end{aligned}$$

Answer: (A)

7. Statement I: If $2s = a + b + c$, then

$$\begin{vmatrix} a^2 & (s-a)^2 & (s-a)^2\\ (s-b)^2 & b^2 & (s-b)^2\\ (s-c)^2 & (s-c)^2 & c^2\end{vmatrix} = 2s^3(s-a)(s-b)(s-c)$$

Statement II:

$$\begin{vmatrix} (b+c)^2 & a^2 & a^2\\ b^2 & (c+a)^2 & b^2\\ c^2 & c^2 & (a+b)^2\end{vmatrix} = 2abc(a+b+c)^3$$

Solution: Let

$$\Delta = \begin{vmatrix} (b+c)^2 & a^2 & a^2\\ b^2 & (c+a)^2 & b^2\\ c^2 & c^2 & (a+b)^2\end{vmatrix}$$

$$\Delta = \begin{vmatrix} (b+c)^2 & a^2-(b+c)^2 & a^2-(b+c)^2\\ b^2 & (c+a)^2-b^2 & 0\\ c^2 & 0 & (a+b)^2-c^2\end{vmatrix}$$

(by $C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$)

Taking $a + b + c$ common from C_2 and C_3 we get

$$\Delta = (a+b+c)^2\begin{vmatrix} (b+c)^2 & a-b-c & a-b-c\\ b^2 & c+a-b & 0\\ c^2 & 0 & a+b-c\end{vmatrix}$$

$$\Delta = (a+b+c)^2\begin{vmatrix} 2bc & -2c & -2b\\ b^2 & c+a-b & 0\\ c^2 & 0 & a+b-c\end{vmatrix}$$

[by $R_1 \to R_1 - (R_2 + R_3)$]

$$\Delta = \begin{vmatrix} 2bc & 0 & 0\\ b^2 & c+a & \dfrac{b^2}{c}\\ c^2 & \dfrac{c^2}{b} & a+b\end{vmatrix}(a+b+c)^2$$

$\left(\text{by } C_2 \to C_2 + \frac{1}{b}C_1, C_3 \to C_3 + \frac{1}{c}C_1\right)$

$$\begin{aligned}&= 2bc[(c+a)(a+b) - bc](a+b+c)^2\\ &= 2abc(a+b+c)^3\end{aligned}$$

Therefore Statement II is true. Now put $s - a = x, s - b = y, s - c = z$ so that

$$x + y + z = s,\ y + z = a,\ z + x = b,\ x + y = c$$

Therefore

$$\begin{vmatrix} a^2 & (s-a)^2 & (s-a)^2\\ (s-b)^2 & b^2 & (s-b)^2\\ (s-c)^2 & (s-c)^2 & c^2\end{vmatrix} = \begin{vmatrix} (y+z)^2 & x^2 & x^2\\ y^2 & (z+x)^2 & y^2\\ z^2 & z^2 & (x+y)^2\end{vmatrix}$$

Use Statement II. Hence, Statement I is correct.

Answer: (A)

Integer Answer Type Questions

1. If A is a square matrix, then the number of ordered pairs of matrices (P, Q) where P is a symmetric matrix and Q is a skew-symmetric matrix such that $A = P + Q$ is ______.

Solution: If A is any square matrix, then $1/2(A + A^{\mathrm{T}})$ is symmetric and $1/2(A - A^{\mathrm{T}})$ is skew-symmetric and

$$A = \frac{1}{2}(A + A^{\mathrm{T}}) + \frac{1}{2}(A - A^{\mathrm{T}})$$

Suppose $A = P + Q$ where P is symmetric matrix and Q is skew-symmetric matrix. Then

$$A^{\mathrm{T}} = P^{\mathrm{T}} + Q^{\mathrm{T}} = P - Q$$

But

$$P = \frac{1}{2}(A + A^{\mathrm{T}})\, Q = \frac{1}{2}(A - A^{\mathrm{T}})$$

Therefore (P, Q) is a unique pair.

Answer: 1

2. If

$$\begin{vmatrix} x^2 + 3x & x - 1 & x + 3 \\ x + 1 & -2x & x - 4 \\ x - 3 & x + 4 & 3x \end{vmatrix} = a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4$$

then the value of a_4 is ______.

Solution: For $x = 0$ we get

$$a_4 = \begin{vmatrix} 0 & -1 & 3 \\ 1 & 0 & -4 \\ -3 & 4 & 0 \end{vmatrix}$$
$$= 1(0 - 12) + 3(4 - 0) = 0$$

Answer: 0

3. If

$$\begin{vmatrix} 6i & -3i & 1 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix} = x + iy$$

where $i = \sqrt{-1}$, given $x^2 + y^2$ is equal to ______.

Solution: The given determinant is equal to

$$6i(-3+3) + 3i(4i + 20) + 1(12 - 60i) =$$
$$-12 + 60i + 12 - 60i = 0 = 0 + i0$$

Therefore $x = 0$, $y = 0$.

Answer: 0

4. The number of pairs (A, B) where A and B are 3×3 matrices such that $AB - BA = I$ (I is the unit matrix of 3×3 order) is ______.

Solution: Let

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \text{ and } B = \begin{bmatrix} a_1^1 & b_1^1 & c_1^1 \\ a_2^1 & b_2^1 & c_2^1 \\ a_3^1 & b_3^1 & c_3^1 \end{bmatrix}$$

such that $AB - BA = I$. Principal diagonal elements of $AB - BA$ are equal to 1.

$$(a_2^1b_1 - a_2b_1^1) + (c_1a_3^1 - c_1^1a_3) = 1 \quad (8.25)$$
$$(a_2b_1^1 - a_2^1b_1) + (c_2b_3^1 - c_2^1b_3) = 1 \quad (8.26)$$
$$(a_3c_1^1 - a_3^1c_1) + (b_3c_2^1 - b_3^1c_2) = 1 \quad (8.27)$$

Adding Eqs. (8.26) and (8.27) we get

$$(a_2b_1^1 - a_2^1b_1) + (a_3c_1^1 - a_3^1c_1) = 2$$

which is impossible according to Eq. (8.25). Hence, there exist no such matrices.

Answer: 0

5. Let S be the set of all symmetric matrices of order 3×3, all of whose elements are either 0 or 1. If five of these elements are 1 and four of them are 0, then the number of matrices in S is ______.

Solution: Let $A \in S$. In a symmetric matrix the (i, j)th element is same as (j, i)th element for $i \neq j$. That is, upper and lower parts of the principal diagonal are reflections of each other through the principal diagonal. Hence the principal diagonal of A must have three 1's or two 0's and a single because A has five 1's and four 0's. If all the three diagonal elements are 1, the number of such matrices is ${}^3C_1$. If two diagonal elements are zeros and one is 1, then the number of such matrices is ${}^3C_1 \times {}^3C_1$. Therefore, total number of matrices in $S = {}^3C_1 + {}^3C_1 \times {}^3C_1 = 3 + 3 \times 3 = 12$.

Answer: 12

6. Let a, b, c be real positive numbers such that $abc = 1$ and

$$A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$$

Let S be the set of all such matrices A such that $A^{\mathrm{T}}A = I$. Then, the number of matrices in S is ______.

Solution:

$$A^{\mathrm{T}}A = I \Rightarrow \begin{bmatrix} a^2+b^2+c^2 & \sum ab & \sum ab \\ \sum ab & a^2+b^2+c^2 & \sum ab \\ \sum ab & \sum ab & a^2+b^2+c^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now $\sum ab = 0$ which is not possible as a, b, c are positive. Hence S is an empty set.

Note: In 2003 (JEE), under the same hypothesis, it was asked to find the value of $a^3 + b^3 + c^3$ for which many authors gave $a^3 + b^3 + c^3$ value as 4, without verifying the fact whether such matrices exist or not.

Answer: 0

7. Consider the 8×8 square matrix filled with the natural numbers from 1 to 64 as is given below.

$$\begin{bmatrix} 1 & 2 & 3 & 4 & \cdots & 8 \\ 9 & 10 & 11 & 12 & \cdots & 16 \\ 17 & 18 & 19 & 13 & \cdots & 24 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 57 & 58 & 59 & 60 & \cdots & 64 \end{bmatrix}$$

A number is selected from the board and the corresponding row and column are deleted. Again another number is selected and the row and column are deleted. The process is continued upto 8 times so that no row and no column is left. Then the sum of the numbers so selected is ______.

Solution: Observe that the element in the ith row and jth column is $(i-1)8 + j$. Let $a_{1j_1}, a_{2j_2}, \ldots, a_{8j_8}$ be the numbers so selected where $j_1, j_2, \ldots, j_8$ are different. Therefore

$$\sum_{i=1}^{8} a_{ij_i} = \sum_{i=1}^{8} [(i-1)8 + j_i]$$
$$= 8(0+1+2+3+\cdots+7) + (1+2+3+\cdots+8)$$
$$= 8 \times 28 + 36$$
$$= 224 + 36$$
$$= 260$$

Answer: 260

8. Let

$$A_k = \begin{bmatrix} \cos^2 \alpha_k & \cos \alpha_k \sin \alpha_k \\ \cos \alpha_k \sin \alpha_k & \sin^2 \alpha_k \end{bmatrix} \quad (k = 1, 2)$$

If the difference between α_1 and α_2 is an odd multiple of $\pi/2$, then A_1A_2 is a matrix whose sum of all its elements is equal to

Solution: We have

$$A_1A_2 = \begin{bmatrix} \cos \alpha_1 \cos \alpha_2 & \cos \alpha_1 \sin \alpha_2 \\ \sin \alpha_1 \cos \alpha_2 & \sin \alpha_1 \sin \alpha_2 \end{bmatrix} \cos(\alpha_1 - \alpha_2)$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} \because \alpha_1 - \alpha_2 \text{ is an odd mutiple} \\ \text{of } \pi/2, \cos(\alpha_1 - \alpha_2) = 0 \end{bmatrix}$$

Sum of the elements of A_1A_2 is 0.

Answer: 0

9. If x, y, z are non-zero real numbers and

$$\begin{vmatrix} 1+x & 1 & 1 \\ 1+y & 1+2y & 1 \\ 1+z & 1+z & 1+3z \end{vmatrix} = 0$$

then $-\left(\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z}\right)$ is equal to ______.

Solution: Let Δ be the given determinant. Then apply row transformation $R_1 \to R_1 - R_3, R_2 \to R_2 - R_3$ we get

$$\Delta = \begin{vmatrix} x & 0 & 1 \\ y & 2y & 0 \\ -2z & -2z & 1+3z \end{vmatrix} = 0$$

$$\Delta = \begin{vmatrix} x & 0 & 0 \\ y & 2y & 1 - \dfrac{y}{x} \\ -2z & -2z & 1 + 3z + \dfrac{2z}{x} \end{vmatrix} = 0 \quad \left(\text{by } C_3 \to C_3 - \frac{1}{x}C_1\right)$$

Solving we get

$$x\left[2y\left(1 + 3z + \frac{2z}{x}\right) + 2z\left(1 - \frac{y}{x}\right)\right] = 0$$
$$[2xy + 6zxy + 4yz + 2zx - 2yz] = 0$$
$$2(xyz)\left[\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + 3\right] = 0$$
$$-\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = 3$$

Answer: 3

10. If

$$f(x) = \begin{vmatrix} x & 2 & x \\ x^2 & x & 6 \\ x & x & 6 \end{vmatrix} = ax^4 + bx^3 + cx^2 + dx + e$$

then the absolute value of $5a + 4b + 3c + 2d + e$ is ______.

Solution: We have

$$f(x) = x(6x - 6x) - 2(6x^2 - 6x) + x(x^3 - x^2)$$
$$= x^4 - x^3 - 12x^2 + 12x$$

Therefore $a = 1, b = -1, c = -12, d = 12, e = 0$. Also

$$5a + 4b + 3c + 2d + e = 5 - 4 - 36 + 24 + 0 = -11$$

So that absolute value is 11.

Answer: 11

SUMMARY

8.1 ***Matrix:*** Let a_{ij} ($1 \le i \le m$ and $1 \le j \le n$; m and n are positive integers) be real numbers or complex numbers or functions or any kind of expressions. Then the arrangement of these a_{ij} in the shape of a rectangle enclosed by two brackets is called a rectangular matrix of order $m \times n$.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \cdots a_{1n} \\ a_{21} & a_{22} & a_{23} \cdots a_{2n} \\ \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdots \\ a_{m1} & a_{m2} & a_{m3} \cdots a_{mn} \end{bmatrix}$$

is an $m \times n$ matrix

An $m \times n$ matrix in which the (i, j)th element is a_{ij} will be written as $[a_{ij}]_{m\times n}$ or $[a_{ij}]_{m\times n}$.

Horizontal lines are called rows and vertical lines are called columns a_{ij} is the element in the ith row and jth column position.

8.2 ***Vertical and horizontal matrices:*** If the number of rows is greater than the number of columns, it is called vertical matrix.

If the number of rows is less than the number of columns, it is called horizontal matrix.

QUICK LOOK

Rectangular matrix means either vertical or horizontal matrix.

8.3 ***Square matrix:*** If the number of rows is same as the number of columns, then the matrix is called a square matrix.

8.4 ***Principal diagonal and trace:*** In a square matrix $[a_{ij}]_{n\times n}$, the elements $a_{11}, a_{22}, a_{33}, \ldots, a_{nn}$ are called principal diagonal elements and their sum is called Trace of the matrix and is denoted by *Trace* A where A is a given square matrix.

8.5 ***Zero (null) matrix and unit matrix:*** In a matrix, if all the elements are zeros, then it is called zero matrix.

In a *square matrix*, if the principal diagonal elements are equal to 1 and the rest are zeros, it is called unit matrix.

8.6 ***Upper and lower triangular matrices:*** A square matrix $A = [a_{ij}]_{n\times n}$ is called upper triangular matrix, if $a_{ij} = 0$ for $i > j$ (i.e., the elements below the principal diagonal are zeros).

It is called lower triangular, if $a_{ij} = 0$ for $i < j$ (i.e., the elements above the principal diagonal are zeros).

8.7 ***Diagonal matrix:*** A matrix which is both upper and lower triangular is a diagonal matrix or a square matrix $A = [a_{ij}]_{n\times n}$ is called diagonal matrix, if $a_{ij} = 0$ for $i \ne j$.

8.8 ***Scalar matrix:*** In a diagonal matrix, if all the principal diagonal elements are equal, it is called scalar matrix. That is in a square matrix $A = [a_{ij}]_{n\times n}$, if

$$a_{ij} \begin{cases} 0 & \text{for } i \ne j \\ \lambda & \text{(real or complex) for } i = j \end{cases}$$

then A is called scalar matrix.

8.9 ***Transpose of a matrix:*** The matrix obtained from a given matrix by changing its rows in to columns is called transpose of the given matrix. If A is a matrix, its transpose is denoted by A^{T} or A^1.

QUICK LOOK

About transpose:

1. If A is of order $m \times n$, then A^{T} is of order $n \times m$.
2. The $(i\text{-}j)$th element of A is equal to $(j\text{-}i)$th element of A^{T}.
3. $(A^{\mathrm{T}})^{\mathrm{T}} = A$.

8.10 ***Addition of matrices:*** If $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ are two matrices of same order $m \times n$, then the matrix whose $(i\text{-}j)$th element is $a_{ij} + b_{ij}$ called sum of A and B and is denoted by $A + B = [a_{ij} + b_{ij}]_{m\times n}$.

8.11 ***Scalar multiplication:*** If $A = [a_{ij}]_{m\times n}$ is a matrix and k is a scalar (i.e., real or complex) then kA is the matrix $[ka_{ij}]_{m\times n}$.

In particular, if $k = -1$, then (-1) A is denoted by $-A$.

8.12 ***Difference of matrices:*** If A and B are two matrices of same order, then $A - B$ is defined as $A + (-B)$. That is, if $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$, then $A - B = [a_{ij} - b_{ij}]_{m\times n}$.

8.13 ***Theorem:*** The following hold for any matrices of same order.

(1) If A and B are matrices of same order, then $A + B = B + A$. (commutative law)

(2) If A, B and C are matrices of same order, then $(A + B) + C = A + (B + C)$ (Associative law).

(3) $A + O = O + A = A$ for any matrix of order $m \times n$ where O is a zero matrix of $m \times n$ order.

(4) $A + (-A) = (-A) + A = O$.

(5) If A and B are of same order, and λ is a scalar, then $\lambda(A + B) = \lambda A + \lambda B$.

(6) If λ and μ are any two scalars, then $(\lambda + \mu)A = \lambda A + \mu A$ for any matrix A.

(7) If A is an $m \times n$ matrix and $\lambda = 0$ is the usual zero scalar and 0 is the $m \times n$ zero matrix, then $0A = 0$.

(8) $(A^{\mathrm{T}})^{\mathrm{T}} = A$.

(9) $(A \pm B)^{\mathrm{T}} = A^{\mathrm{T}} \pm B^{\mathrm{T}}$.

(10) $(\lambda A)^{\mathrm{T}} = \lambda A^{\mathrm{T}}$ where λ is a scalar.

8.14 ***Matrix multiplication:*** Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{jk}]_{n\times p}$ be two matrices of orders $m \times n$ and $n \times p$ respectively. Let

$$C_{ik} = \sum_{j=1}^{n} (a_{ij} b_{jk}) = a_{i1}b_{1k} + a_{i2}b_{2k} + a_{i3}b_{3k} + \cdots + a_{in}b_r$$

Then the matrix $[c_{ik}]_{m\times p}$ (order $m \times p$) is the product AB.

QUICK LOOK

1. For convenience and easy to remember, the general element of A is taken as (i-j)th element a_{ij} and that of B as (j-k)th element b_{jk} and written (i-k)th element as the general element of the product AB. In fact to write some rth row and sth column element of the product AB, take the rth row of A and sth column of B, multiply the corresponding elements and add.
2. The product AB is defined only when the number of columns of A is same as the number of rows of B.
3. In general AB and BA are not equal, even though when both products are defined.

8.15 ***Theorem:*** Let A, B, C be $m \times n$, $n \times p$ and $p \times q$ matrices and λ, any scalar. Then, the following hold.

(1) $(AB)C = A(BC)$ (Associative law).

(2) $(\lambda A)B = \lambda(AB) = A(\lambda B)$.

(3) $0A = A0 = 0$ where 0 is the zero matrices of appropriate orders.

(4) If I_m and I_n are unit matrices of orders m and n respectively, then $I_m A = A = AI_n$.

8.16 ***Distributive laws:***

(1) Let A, B be matrices of same order $m \times n$ and C be any matrix of order $n \times p$. Then

$$(A + B)C = AC + BC$$

(2) Let A be of order $m \times n$ and B, C be of order $n \times p$. Then

$$A(B + C) = AB + AC$$

8.17 ***Important feature of a scalar matrix:*** Square matrix A is a scalar matrix if and only if A commutes with every matrix of the same order.

8.18 ***Transpose of a product:*** Let A and B be two matrices of $m \times n$ and $n \times p$ orders, respectively. Then

$$(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$$

8.19 ***Inverse of a matrix:*** Let A be a square matrix of order $n \times n$. If B is a square matrix of the same order $n \times n$ such that $AB = BA = I_n$ (unit matrix of order n), then B is called inverse of A and is denoted by A^{-1}. If a matrix has inverse, then it is called invertible matrix.

8.20 ***Inverse***

(1) $(A^{-1})^{-1} = A$.

(2) If A and B are square matrices of same order having inverses, then AB has also inverse and $(AB)^{-1} = B^{-1}A^{-1}$.

(3) If A is an invertible matrix, then

$$(A^{-1})^{\mathrm{T}} = (A^{\mathrm{T}})^{-1}$$

8.21 ***Some kinds of matrices:***

(1) ***Symetric matrix:*** A square matrix A is called symmetric matrix, if $A^{\mathrm{T}} = A$.

(2) ***Skew-symmetric matrix:*** A square matrix A is called skew-symmetric if $A^{\mathrm{T}} = -A$.

(3) ***Orthogonal matrix:*** Square matrix A is called orthogonal matrix, if $A^{\mathrm{T}}A = I$.

(4) ***Idempotent matrix:*** Square matrix A is called idempotent matrix if $A^2 = A$.

QUICK LOOK

In a skew-symmetric matrix, all the principal diagonal elements must be zeros. The converse is not true.

(5) ***Nilpotent matrix:*** Square matrix A is called nilpotent matrix, if $A^m = O$ for some positive integer m. The least positive integer m such that $A^m = O$ is called the *index* of the nilpotent matrix A.

(6) ***Periodic matrix:*** Square matrix A is called periodic matrix, if $A^{p+1} = A$ for some positive integer p. The least such positive integer p is called the period of A.

QUICK LOOK

An idempotent matrix is a periodic matrix of period 1.

(7) ***Involutary matrix:*** Square matrix A is called involuntary matrix, if $A^2 = I$.

QUICK LOOK

Every involuntary matrix is a periodic matrix of period 2.

8.22 ***Theorem:*** Let A and B be square matrices of order $n \times n$. Then the following hold.

(1) If A and B are symmetric matrices then so is $A \pm B$.

(2) If A and B are skew-symmetric matrices, then so is $A \pm B$.

(3) If $AB = BA$ and A and B are symmetric (skew-symmetric) then AB is symmetric.

(4) If A is symmetric, then for any scalar λ, λA is also symmetric. If A is skew-symmetric, then λA is also skew-symmetric.

(5) If $AB = BA$, then AB is skew-symmetric provided one of A and B is symmetric and the other is skew-symmetric.

8.23 ***Theorem:*** If A is any square matrix, then $A + A^{\mathrm{T}}$ is symmetric and $A - A^{\mathrm{T}}$ is skew-symmetric.

8.24 ***Representing square matrix in terms of symmetric and skew-symmetric matrices:*** Every square matrix A can be expressed as a sum of symmetric and skew-symmetric matrices uniquely and the representation is

$$A = \frac{1}{2}(A + A^{\mathrm{T}}) + \frac{1}{2}(A - A^{\mathrm{T}})$$

8.25 ***Conjugate of a matrix:*** It A is a matrix whose elements (i.e., entries) are complex numbers, then the matrix obtained from A by replacing its elements with their corresponding complex conjugates is called conjugate of A and is denoted by $\bar{A}$.

For example, if

$$A = \begin{bmatrix} 1+i & -i & 2 \\ \frac{1}{2} - i & \sqrt{2}i & a+ib \end{bmatrix}$$

then

$$\bar{A} = \begin{bmatrix} 1-i & i & 2 \\ \frac{1}{2} + i & -\sqrt{2}i & a-ib \end{bmatrix}$$

8.26 ***Some properties of conjugate:***

(1) $\overline{(\bar{A})} = A$.

(2) $\overline{(\lambda A)} = \bar{\lambda}\,\bar{A}$ for any scalar λ.

(3) $\overline{(A \pm B)} = \bar{A} \pm \bar{B}$.

(4) $\overline{AB} = \bar{A}\,\bar{B}$.

8.27 ***Theorem:*** If A is any matrix, then

$$(\bar{A})^{\mathrm{T}} = \overline{(A^{\mathrm{T}})}$$

8.28 ***Notation:*** $(\bar{A})^{\mathrm{T}}$ is denoted by A^*.

8.29 ***Hermitian and skew-Hermitian matrices:*** Square matrix A is called Hermitian or skew-Hermitian according as $A^* = A$ [i.e., $(\bar{A})^{\mathrm{T}} = A$] or

$$A^* = -A \text{ [i.e., } (\bar{A})^{\mathrm{T}} = -A]$$

8.30 ***Theorem:*** Let A and B be matrices. Then the following hold.

(1) $(A^*)^* = A$.

(2) $(\lambda A)^* = \bar{\lambda} A^*$ for any scalar λ where $\bar{\lambda}$ is the complex conjugate of λ.

(3) $(A \pm B)^* = A^* \pm B^*$.

(4) $(AB)^* = B^*A^*$ when A and B are compatible for multiplication.

8.31 ***On Hermitian and skew-Hermitian matrices:***

(1) If A is any square matrix, then $A + A^*$ is Hermitian and $A - A^*$ is skew-Hermitian.

(2) If λ is real and A is Hermitian (skew-Hermitian) then λA is Hermitian (skew-Hermitian).

(3) If A is Hermitian, then iA is skew-Hermitian and iA is Hermitian, if A is skew-Hermitian.

(4) If A and B are Hermitian, then so is $A \pm B$.

(5) If A and B are skew-Hermitian, then so is $A \pm B$.

(6) If $AB = BA$ and A, B are Hermitian, then AB is also Hermitian.

(7) If A and B are skew-Hermitian and $AB = BA$, then AB is Hermitian.

(8) If $AB = BA$ and one of A and B is Hermitian while the other is skew-Hermitian, then AB is skew-Hermitian.

8.32 ***Decomposition of a square matrix in terms of Hermitian and skew-Hermitian matrices:*** Let A be a square matrix. Then A can be expressed as sum of Hermitian and skew-Hermitian matrices in one and only one way and the representation is

$$A = \frac{1}{2}(A + A^*) + \frac{1}{2}(A - A^*)$$

QUICK LOOK

In the above, if we take

$$P = \frac{1}{2}(A + A^*) \quad \text{and} \quad Q = \frac{i(A^* - A)}{2}$$

where $i = \sqrt{-1}$, then $A = P + iQ$, where both P and Q are Hermitian matrices.

Determinants

Even though the determinant of any square matrix whose elements are real (complex) numbers may be defined, our main focus is on determinants of 2×2 matrix or 3×3 matrix. We begin with minor and cofactors of the elements of a matrix.

Throughout this summary on determinants our matrices are 3×3 matrices and in some cases 2×2 matrices.

8.33 Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then the number $ad - bc$ is called the determinant of A and is denoted by det A or $|A|$.

8.34 ***Minor:*** Let A be a 3×3 matrix. Then, the determinant of the 2×2 matrix obtained from A, by deleting the ith row and jth column of A is called minor of A with respect to $(i–j)$th element ($i = 1, 2, 3$ and $j = 1, 2, 3$). $(i–j)$th minor is denoted by M_{ij}.

8.35 ***Cofactor:*** $(-1)^{i+j} M_{ij}$ is called the cofactor of the element a_{ij} with respect to the matrix A. The cofactor of the element a_{ij} is denoted by A_{ij} (capital letter).

QUICK LOOK

In the formation of A_{ij}, the ith row and jth column will not participate.

8.36 ***Determinant of 3 × 3 matrix:***
Let

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

Then det A (or $|A|$) $= a_1A_1 + b_1B_1 + c_1C_1$, where A_1, B_1 and C_1 are the cofactors of a_1, b_1 and c_1, respectively.

Expansion:

$$\det A = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2)$$
$$+ c_1(a_2b_3 - a_3b_2) = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$$

8.37 ***Properties of determinants:***
Let

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

and capital letters A_i, B_i and C_i respectively denote cofactors of a_i, b_i and c_i for $i = 1, 2, 3$. Then

(1) $a_2A_2 + b_2B_2 + c_2C_2 = a_3A_3 + b_3B_3 + c_3C_3 = a_1A_1 + a_2A_2 + a_3A_3 = b_1B_1 + b_2B_2 + b_3B_3 = c_1C_1 + c_2C_2 + c_3C_3 = \det A$

That is, we can expand the determinant in any row or any column.

(2) In a matrix A, if any two rows (columns), are interchanged, then the sign of the determinant will change.

(3) In a matrix, if two rows (columns) are identical, then the value of the determinant is zero.

(4) det A = det (A^{T}).

(5) The elements of a row (column) are multiplied by some non-zero constant λ amounts that, the determinant is multiplied with the same constant λ.

In other words, if λ is a common factor of all the elements of a row (column), then the determinant of the matrix is equal to λ times the determinant of the matrix obtained after taking away λ from the elements of that row (column).

For example

$$\begin{vmatrix} \lambda a_1 & \lambda b_1 & \lambda c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \lambda\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

(6) det $(\lambda A) = \lambda^3$ det A. In general, if A is a square matrix of order n, then det $(\lambda A) = \lambda^n$ det A.

(7) $a_iA_j + b_iB_j + c_iC_j = 0$ for $i \neq j$.

That is, the sum of the products of the elements of a row with cofactors of the corresponding elements of another row is always zero. The same is true for columns also.

(8) The determinant of a matrix is unaltered by adding constant times the elements of a row to the corresponding elements of another row. The same is true for columns also.

(9) If each element of a row (column) is sum of two elements, then the determinant of the matrix can be expressed as sum of two determinants.

(10) If A and B are two square matrices of same order, then det $(AB) = (\det A)\,(\det B)$.

8.38 ***Adjoint of a matrix:*** Let A be a square matrix and B is the matrix obtained from A, by replacing its elements with their corresponding cofactors. Then B^{T} is called adjoint of A. Adjoint of A is written as adj A.

For example, let

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

and A_1, B_1, C_1, etc. denote the cofactors of a_1, b_1, c_1 etc. Then

$$\operatorname{adj} A = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$$

8.39 ***Theorem:*** If A is a square matrix, then $A(\operatorname{adj} A) = (\det A)\, I = (\operatorname{adj} A)\, A$, where I is the corresponding unit matrix.

8.40 ***Existence of inverse and formula for inverse:*** If A is a square matrix, then A^{-1} exists if and only if det $A \neq 0$ and in such a case

$$A^{-1} = \frac{\operatorname{adj} A}{\det A}$$

8.41 ***Non-singular and singular matrices:*** Square matrix A is called non-singular or singular according as det $A \neq 0$ or det $A = 0$.

QUICK LOOK

A^{-1} exists if and only if A is a non-singular matrix. That is, non-singular matrices only, will have inverses.

Elementary Row (Column) Operations

8.42 The following are called elementary row (column) operations on a matrix.

(1) Interchanging of two rows (columns) denoted by $R_{ij}(C_{ij})$. That is, interchanging of ith and jth rows (ith and jth columns).

(2) Multiplication of the elements of a row (column) by a non-zero constant k denoted by $R_i(k)(C_i(k))$.

(3) Multiplying the elements of a row (column) with a non-zero constant k and adding to the corresponding elements of another row (column) denoted by $R_s + R_r(k)$ or $R_s{:}R_s + R_r(k)$.

That is multiplying the elements of rth row by k and adding to the corresponding elements of sth row. Same is $C_s : C_s + C_r(k)$.

Elementary transformation means either row or column transformation.

8.43 ***Elementary matrix:*** Matrix obtained from unit matrix by applying elementary transformations. Every elementary matrix is invertible.

8.44 ***Theorem:*** Every non-singular matrix can be expressed as a product of elementary matrices.

Systems of Linear Equations

8.45 ***Homogeneous system:*** Let a_{ij} $(1 \le i \le m, 1 \le j \le n)$ be mn real numbers. Then the system of equations

$$\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{array}$$

is called homogeneous system of m equations in n unknowns.

Matrix equation: If

$$A = [a_{ij}]_{m\times n},\ X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$$

and O is $m \times 1$ zero matrix, then the above homogeneous system of equations can be represented by the matrix equation $AX = O$.

Non-homogeneous system: $AX = B$ where

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ b_m \end{bmatrix}$$

is an $m \times 1$ matrix and atleast one $b_i \neq 0$.

In both systems A is called coefficient matrix.

8.46 ***Zero solution or trivial solution:*** For $AX = O$, $t_1 = x_2 = x_3 = \cdots = x_n = 0$ is always solution and this solution is called zero solution or trivial solution.

8.47 ***Non-zero solution (non-trivial solution)***

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}$$

is called a non-zero solution, if atleast one $x_i \neq 0$ and satisfies the equation $AX = O$.

8.48 ***Existence of non-zero solution:*** Suppose A is a non-zero square matrix. A is non-singular if and only if $X = O$ is the only solution of $AX = O$ and hence $AX = O$ has non-zero solution if and only if A is a singular matrix (i.e., det $A = O$) and in such a case, the system has infinitely many solutions.

8.49 ***About AX = B (unique solution):*** If A is non-singular matrix and B is non-zero column matrix, then $AX = B$ has unique solution, viz., $X = A^{-1}B$.

8.50 ***Crammer's rule:*** Consider the system of simultaneous equations $a_1x + b_1y + c_1z = d_1, a_2x + b_2y + c_2z = d_2$ and $a_3x + b_3y + c_3z = d_2$ where atleast one $d_i \neq 0$. Let

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

and Δ_k is the determinant obtained from Δ by replacing its kth column with

$$\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Then

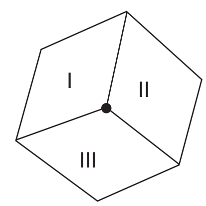
$$x_1 = \frac{\Delta_1}{\Delta}, \quad x_2 = \frac{\Delta_2}{\Delta}, \quad x_3 = \frac{\Delta_3}{\Delta}$$

is the solution for the given simultaneous system of equations.

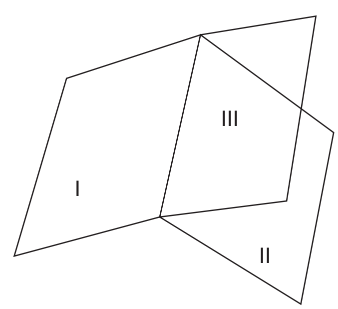
8.51 ***Consistency and inconsistency system:*** If AX = B has a solution, then it is called consistent system, otherwise inconsistent system.

8.52 ***Geometrical interpretation:*** Consider the system of three simultaneous equations $a_1x + b_1y + c_1z = d_1$, $a_2x + b_2y + c_2z = d_2$, $a_3x + b_3y + c_3z = d_3$ which represent planes in the three dimensional space. Then

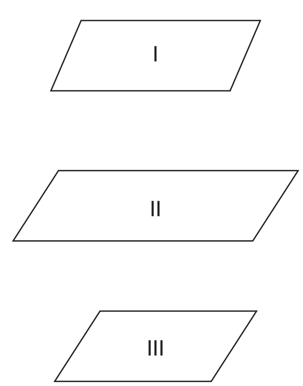
(1) Unique solution means, all the three planes are concurrent at a single point.

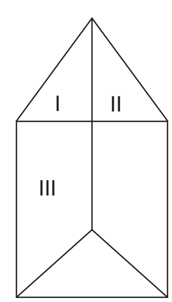


(2) Infinite number of solutions means, all the three planes pass through a single straight line.



(3) Inconsistent means either all the three planes are parallel to each other or form a triangular prism.





8.53 Let $f(x) \equiv a_0x^m + a_1x^{m-1} + \cdots + a_m$ where $a_0, a_1, a_2, \ldots, a_m$ are real (complex) numbers. If A is a square matrix, then $f(A)$ means, the matrix $a_0A^m + a_1A^{m-1} + a_2A^{m-2} + \cdots + a_{m-1}A + a_mI$ where I is the unit matrix of order same as A.

8.54 ***Characteristic polynomial (equation of a matrix):*** If A is a square matrix, then $|A - xI|$ which is a polynomial with real or complex coefficients is called characteristic polynomial of the matrix A and the equation $|A - xI| = 0$ is called characteristic equation of the matrix A.

8.55 ***Cayley–Hamilton theorem:*** Every square matrix statisfies its characteristic equation. That is, if A is a square matrix and $f(x) = |A - xI|$, then $f(A) = O$ (zero matrix).

8.56 ***Condition for a non-singular matrix:*** Let A be square matrix of order n $f(x) = |A - xI| = a_0x^n + a_1x^{n-1} + a_1x^{n-2} + \cdots + a_n$. Then A is non-singular if and only if $a_n \neq 0$.

EXERCISES

Single Correct Choice Type Questions

1. The number of 2×2 matrices with real entries which commute with the matrix $\begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$ is

(A) 1 (B) 2 (C) 4 (D) infinite

2. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $abcd \neq 0$, then $AA^T - A^TA$ is equal to

(A) $(c-b)\begin{bmatrix} b+c & d-a \\ d-a & -b-c \end{bmatrix}$

(B) $(b-c)\begin{bmatrix} b+c & d-a \\ d-a & -b-c \end{bmatrix}$

(C) $(c-d)\begin{bmatrix} d-a & -b-c \\ b+c & d-a \end{bmatrix}$

(D) $(a+b-c-d)\begin{bmatrix} a-d & b+c \\ b+c & a-d \end{bmatrix}$

3. If $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, then $\sum_{k=1}^{n} A^k$ is equal to

(A) $\begin{bmatrix} n & n+1 \\ 0 & n \end{bmatrix}$

(B) $\begin{bmatrix} n-1 & n \\ 0 & n-1 \end{bmatrix}$

(C) $\begin{bmatrix} n & \dfrac{n(n+1)}{2} \\ 0 & n \end{bmatrix}$

(D) $\begin{bmatrix} n+1 & \dfrac{(n+1)(n+2)}{2} \\ 0 & n+1 \end{bmatrix}$

4. If

$$F(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

then $F(x)F(y)$ is equal to

(A) $F(xy)$ (B) $F\left(\dfrac{x}{y}\right)$

(C) $F(x+y)$ (D) $F(x-y)$

5. Let $a \neq -1, b \neq -1, c \neq -1$, be real numbers. If the equations $a(y+z) = x$, $b(z+x) = y$, and $c(x+y) = z$ has non-zero solution, then the value of

$$\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} \text{ is}$$

(A) 2 (B) 1 (C) 1/2 (D) −2

6. The value of $x \in [0, \pi/2]$ such that the matrix

$$\begin{bmatrix} 2\sin x - 1 & \sin x & \cos x \\ -\sin x & 2\cos - \sqrt{3} & \tan x \\ -\cos x & -\tan x & 0 \end{bmatrix}$$

is skew-symmetric is

(A) $\pi/2$ (B) $\pi/3$ (C) $\pi/4$ (D) $\pi/6$

7. If w is a non-real cube root of unity and $i = \sqrt{-1}$, then the value of the determinant

$$\begin{vmatrix} 1 & w^2 & 1+i+w^2 \\ -i & -1 & -1-i+w \\ 1-i & w^2-1 & -1 \end{vmatrix}$$

is

(A) 1 (B) i (C) w (D) 0

8. Let a, b, c be positive and x any real number. Then, the value of the determinant

$$\begin{vmatrix} 1 & 1 & 1 \\ (a^x+a^{-x})^2 & (b^x+b^{-x})^2 & (c^x+c^{-x})^2 \\ (a^x-a^{-x})^2 & (b^x-b^{-x})^2 & (c^x-c^{-x})^2 \end{vmatrix}$$

is

(A) 0 (B) $(a+b+c)^2$
(C) $(a+b+c)^x$ (D) $(a+b+c)^{-x}$

9. If $\begin{vmatrix} a & a+d & a+2d \\ a^2 & (a+d)^2 & (a+2d)^2 \\ 2a+3d & 2a+2d & 2a+d \end{vmatrix} = 0,$

then

(A) $d=0$ (B) $d=-a$
(C) $a=0$ or $d=-a$ (D) $d=0$ or $d=-a$

10. Let $1 \le x, y, z \le 9$ be integers which are in AP. If $x51$, $y41$ and $z31$ are three digit numbers, then the value of the determinant

$$\begin{vmatrix} 5 & 4 & 3 \\ x51 & y41 & z31 \\ x & y & z \end{vmatrix}$$

is equal to

(A) $x+y+z$ (B) $x-y+z$ (C) xyz (D) 0

11. Which one of the following systems of equations has unique solution?

(A) $3x-y+4z=3$
$x+2y-3z=-2$
$6x+5y-5z=-3$

(B) $x+y-2z=0$
$2x-3y+z=0$
$x-5y+4z=1$

(C) $x+y+z=9$
$2x+5y+7z=52$
$2x+y-z=0$

(D) $y+z=1$
$z-x=1$
$x+y=1$

12. The value of the determinant

$$\begin{vmatrix} {}^{11}C_4 & {}^{11}C_5 & {}^{12}C_m \\ {}^{12}C_6 & {}^{12}C_7 & {}^{13}C_{m+2} \\ {}^{13}C_8 & {}^{13}C_9 & {}^{14}C_{m+4} \end{vmatrix}$$

is equal to zero when the value of m is

(A) 6 (B) 5 (C) 4 (D) 1

13. The determinant $\begin{vmatrix} bc & ab & ca \\ ab & ca & bc \\ ca & bc & ab \end{vmatrix}$ is equal to

(A) $\begin{vmatrix} bc & a^2 & a^2 \\ b^2 & ca & b^2 \\ c^2 & c^2 & ab \end{vmatrix}$ (B) $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

(C) $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ (D) $\begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix}$

14. $\begin{vmatrix} 29 & 26 & 22 \\ 25 & 31 & 27 \\ 63 & 54 & 46 \end{vmatrix}$ equals

(A) 122 (B) 132 (C) 0 (D) 1

15. $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$ is equal to

(A) $abc(a+b+c)^3$ (B) $(ab+bc+ca)(a+b+c)^2$
(C) $(a+b+c)^3$ (D) $(ab+bc+ca)(a+b+c)$

16. If the system of equations

$ax+4y+z=0, \quad bx+3y+z=0, \quad cx+2y+z=0$

has non-zero solution, then a, b, c are in

(A) AP (B) GP (C) HP (D) AGP

17. For a fixed positive integer n, let

$$D = \begin{vmatrix} n! & (n+1)! & (n+2)! \\ (n+1)! & (n+2)! & (n+3)! \\ (n+2)! & (n+3)! & (n+4)! \end{vmatrix}$$

Then $(D/(n!)^3 - 4)$ is divisible by

(A) n (B) $(n!)^2+4$ (C) $n!+4$ (D) $n+4$

18. If A and B are symmetric matrices of same order, then the matrix $AB-BA$ is

(A) symmetric matrix (B) skew-symmetric matrix
(C) diagonal matrix (D) null matrix

19. $\begin{vmatrix} 0 & ab^2 & ac^2 \\ a^2b & 0 & bc^2 \\ a^2c & b^2c & 0 \end{vmatrix}$ is equal to

(A) $a^3b^3c^3$ (B) $2a^3b^3c^3$
(C) $2(a^2+b^2+c^2)(a+b+c)$ (D) $4a^2b^2c^2$

20. Which one of the following matrices is non-singular?

(A) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -5 \\ 1 & 2 & 4 \end{bmatrix}$ (B) $\begin{bmatrix} 4 & -5 & -2 \\ 5 & -4 & 2 \\ 2 & 2 & 8 \end{bmatrix}$

(C) $\begin{bmatrix} 2 & 5 \\ 6 & 15 \end{bmatrix}$ (D) $\begin{bmatrix} 2 & -7 & -6 \\ 3 & 5 & -2 \\ 4 & -2 & -7 \end{bmatrix}$

21. If $A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$, then A^{-1} equals

(A) $A^2 + 6A - 11I$ (B) $A^2 + 6A + 11I$
(C) $A^2 - 6A + 11I$ (D) $A^2 - 6A - 6I$

22. Let $w \neq 1$ be a cube root of unity and

$$A = \begin{bmatrix} 1 & w & w^2 \\ w & w^2 & 1 \\ w^2 & 1 & w \end{bmatrix}$$

Then $A^{-1} =$

(A) $\begin{bmatrix} 1 & w^2 & w \\ w & 1 & w^2 \\ w^2 & w & 1 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (D) does not exist

23. If $A = \begin{bmatrix} 5 & 3 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 4 \end{bmatrix}$, then adj A is equal to

(A) $\begin{bmatrix} -2 & -10 & 8 \\ -5 & 19 & -13 \\ 3 & -7 & -1 \end{bmatrix}$ (B) $\begin{bmatrix} 2 & -10 & 8 \\ -5 & 19 & -13 \\ 3 & -7 & -1 \end{bmatrix}$

(C) $\begin{bmatrix} 2 & -10 & 8 \\ -5 & -19 & 13 \\ 3 & -7 & -1 \end{bmatrix}$ (D) $\begin{bmatrix} -2 & -10 & 8 \\ 5 & -19 & 13 \\ 3 & -7 & -1 \end{bmatrix}$

24. Value of the determinant

$$\begin{vmatrix} 1 + a^2 - b^2 & 2ab & -2b \\ 2ab & 1 - a^2 + b^2 & 2a \\ 2b & -2a & 1 - a^2 - b^2 \end{vmatrix}$$

is

(A) $(1 + a^2 + b^2)^3$ (B) $(1 + a^3 + b^3)^2$
(C) $(a^2 + b^2)(1 + a^2 + b^2)^2$ (D) $2(1 + a^2 + b^2)^3$

25. Let

$$A = \begin{bmatrix} 13 & b_1 & c_1 \\ 5 & b_2 & 15 \\ x & b_3 & c_3 \end{bmatrix}$$

If the sum of the elements of each row, each column and each of the diagonals of A are equal, then value of x is

(A) 9 (B) 10
(C) 12 (D) cannot be determined

Multiple Correct Answer Type Questions

1. Consider the following system of equations:

$$x - 2y + z = -4, \quad x + y + \lambda z = 4, \quad 2x - y + 2z = 2$$

Which of the following statements are true?
(A) System has infinitely many solutions when $\lambda = 2$
(B) Unique solution when $\lambda \neq 1$
(C) Has no solution when $\lambda = 1$
(D) Unique solution when $\lambda = 1$

2. If the determinant

$$\begin{vmatrix} a & b & a\alpha + b \\ b & c & b\alpha + c \\ a\alpha + b & b\alpha + c & 0 \end{vmatrix} = 0$$

then which of the following may be true?
(A) a, b, c are in AP
(B) a, b, c are in GP
(C) a, b, c are in HP
(D) $x - \alpha$ is a factor of $ax^2 + 2bx + c$

3. Let $A = \begin{bmatrix} 4 & -5 & -2 \\ 5 & -4 & 2 \\ 2 & 2 & 8 \end{bmatrix}$, then

(A) A is non-singular (B) A^{-1} does not exist
(C) det (adj A) = 0 (D) A is idempotent

4. Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

and the system of equations

$$y + 2z + 8 = 0, \quad x + 2y + 3z + 14 = 0$$

and $\quad 3x + y + z + 8 = 0$

Then

(A) A is non-singular
(B) The system has unique solution
(C) A is singular
(D) The system has infinitely many solutions

5. Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. Then

(A) $A^2 = A$ (B) $A^3 = I$
(C) $A^2 = I$ (D) $A^2 = A^{-1}$

6. Let $A = \begin{bmatrix} 3 & 8 \\ 2 & 1 \end{bmatrix}$. Then

(A) $13A^{-1} = A - 4I$ (B) adj $A = \begin{bmatrix} 1 & -8 \\ -2 & 3 \end{bmatrix}$
(C) $|\text{adj } A| = 13$ (D) $A^3 = A$

7. Let A be a matrix whose elements are real or complex. A matrix is obtained from A whose elements are the complex conjugates of the corresponding elements of A is denoted by $\bar{A}$. That is, if $A = [a_{ij}]_{m\times n}$, then $\bar{A} = [\bar{a}_{ij}]_{m\times n}$. In such case

(A) $(\bar{\bar{A}}) = A$
(B) If λ is a scalar, then $(\overline{\lambda A}) = \bar{\lambda}\bar{A}$
(C) $\overline{AB} = \bar{A}\bar{B}$
(D) $(\overline{AB})^{\mathrm{T}} = (\overline{B^{\mathrm{T}}})(\overline{A^{\mathrm{T}}})$

8. If A is any matrix, then $(\bar{A})^{\mathrm{T}} = (\overline{A^{\mathrm{T}}})$ and we denote $(\bar{A})^{\mathrm{T}}$ by A^*. Which of the following are true?

(A) $(A^*)^* = A$
(B) $(A+B)^* = A^* + B^*$
(C) $(AB)^* = B^*A^*$
(D) If λ is a scalar, then $(\lambda A)^* = \bar{\lambda}A^*$

9. A square matrix A is called Hermitian or skew-Hermitian according as $A^* = A$ or $A^* = -A$ where A^* is $(\bar{A})^{\mathrm{T}}$. Which of the following are true?

(A) In a skew-Hermitian matrix, each principal diagonal element is either zero or pure imaginary.
(B) If A and B are Hermitian matrices and $AB = BA$, then AB is also Hermitian matrix.
(C) If A and B are Hermitian matrices, then $AB - BA$ is skew-Hermitian.
(D) If A is Hermitian and $i = \sqrt{-1}$, then iA is skew-Hermitian.

10. If $A = \begin{bmatrix} x & 2 \\ 2 & x \end{bmatrix}$ and $|A^3| = 125$, then x may be

(A) 5 (B) 3 (C) –5 (D) –3

11. Let $D = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$ which of the following are true?

(A) D is a symmetric matrix
(B) If $d_1, d_2, d_3 \neq 0$, then

$$D^{-1} = \begin{bmatrix} \frac{1}{d_1} & 0 & 0 \\ 0 & \frac{1}{d_2} & 0 \\ 0 & 0 & \frac{1}{d_3} \end{bmatrix}$$

(C) Trace of $D = d_1 + d_2 + d_3$
(D) D commutes with every 3×3 order matrix

12. Let $A = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$, $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$, where $i = \sqrt{-1}$. Then

(A) $A^2 = B^2 = C^2 = -I$ (B) $-B = A^{-1}BA$
(C) $A = -CB^{-1}$ (D) $C^{-1} = -B^{-1}A^{-1}$

13. Let $a > b > c$. If the system of equations $ax + by + cz = 0$, $bx + cy + az = 0$ and $cx + ay + bz = 0$ has non-zero solution, then the quadratic equation $at^2 + bt + c = 0$ has

(A) real roots
(B) one positive root
(C) one positive and one negative root
(D) non-real roots

14. Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. Then

(A) $A^{3n} = I$ for all positive integers n
(B) $A^{-1} = A^2$
(C) A is a periodic matrix with least period 3
(D) $|\text{adj } A| = 1$

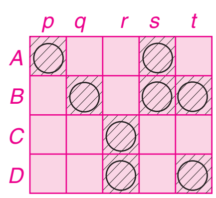
15. If A and B are square matrices of same order such that $AB = A$ and $BA = B$, then

(A) $A^2 = A$ and $B^2 = B$
(B) $A^2 = B$ and $B^2 = A$
(C) $AB = BA$
(D) A and B are periodic matrices

Matrix-Match Type Questions

In each of the following questions, statements are given in two columns, which have to be matched. The statements in Column I are labeled as **(A)**, **(B)**, **(C)** and **(D)**, while those in Column II are labeled as **(p)**, **(q)**, **(r)**, **(s)** and **(t)**. Any given statement in Column I can have correct matching with *one or more* statements in Column II. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following example.

Example: If the correct matches are (A) → (p), (s); (B) → (q), (s), (t); (C) → (r); (D) → (r), (t); that is if the matches are (A) → (p) and (s); (B) → (q), (s) and (t); (C) → (r); and (D) → (r), (t); then the correct darkening of bubbles will look as follows:



1. Column I contains some matrices while Column II contains their corresponding determinants values. Match them.

Column I	*Column II*
(A) $\begin{bmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{bmatrix}$	(p) $a^3+b^3+c^3-3abc$
(B) $\begin{bmatrix} b & c & a \\ a & b & c \\ c & a & b \end{bmatrix}$	(q) $3abc-a^3-b^3-c^3$
(C) $\begin{bmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{bmatrix}$	(r) $4a^2b^2c^2$
(D) $\begin{bmatrix} b^2+c^2 & a^2 & a^2 \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{bmatrix}$	(s) $(a+b+c)^3$

2. Let $w \neq 1$ be a cube root of unity and

$$A = \begin{bmatrix} w & 0 \\ 0 & w \end{bmatrix}$$

Match the items of Column I with the items of Column II.

Column I	*Column II*
(A) A^2	(p) $\begin{bmatrix} w^2 & 0 \\ 0 & w^2 \end{bmatrix}$
(B) A^3	(q) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
(C) A^{-1}	(r) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
(D) A^{2010}	(s) $\begin{bmatrix} w & 0 \\ 0 & w \end{bmatrix}$

3. $w \neq 1$ is a cube root of unity. Column I consists of some matrices and Column II consists of their corresponding determinant values. Match them.

Column I	*Column II*
(A) $\begin{bmatrix} 1 & w & w^2 \\ w & w^2 & 1 \\ w^2 & 1 & w \end{bmatrix}$	(p) $-3w^2$
(B) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1-w^2 & w^2 \\ 1 & w^2 & w^4 \end{bmatrix}$	(q) 0
(C) $\begin{bmatrix} 1+w & w^2 & -w \\ 1+w^2 & w & -w^2 \\ w^2+w & w & -w^2 \end{bmatrix}$	(r) -3
(D) $\begin{bmatrix} 1 & 1 & w \\ 1 & 1 & w^2 \\ w^2 & w & 1 \end{bmatrix}$	(s) $3w(w-1)$

4. Match the items of Column II with those of Column II.

Column I	Column II
(A) $\begin{vmatrix} 1 & yz & y+z \\ 1 & zx & z+x \\ 1 & xy & x+y \end{vmatrix} =$	(p) $(x-y)(y-z)(z-x)(xy+yz+zx)$
(B) $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} =$	(q) $-(x+y+z)(x-y)(y-z)(z-x)$
(C) $\begin{vmatrix} x & y+z & x^2 \\ y & z+x & y^2 \\ z & x+y & z^2 \end{vmatrix} =$	(r) $(x-y)(y-z)(z-x)$
(D) $\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix} =$	(s) $(x+y+z)(x-y)(y-z)(z-x)$

5. Match the items of Column I with those in Column II.

Column I	Column II
(A) The matrix $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is	(p) Nilpotent matrix
(B) The matrix $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ is	(q) Diagonal matrix
(C) Matrix $\begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix}$ is	(r) Idempotent matrix
(D) If $F(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then the matrix $\left(F\left(\frac{\pi}{2}\right)\right)^2$ is	(s) Symmetric matrix

Comprehension-Type Questions

1. Passage: Let A be a square matrix. Then

(A) A is called idempotent matrix, if $A^2 = A$.

(B) A is called nilpotent matrix of index k, if $A^k = O$ and $A^{k-1} \neq O$.

(C) A is called involutory matrix if $A^2 = I$.

(D) A is called periodic matrix with least periodic k, if $A^{k+1} = A$ and $A^k \neq A$.

Answer the following questions:

(i) The matrix $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ is

(A) idempotent (B) involutory
(C) nilpotent (D) skew-symmetric

(ii) If A is an idempotent matrix, then $I - A$ is

(A) idempotent
(B) nilpotent
(C) involutory
(D) periodic matrix with least period 4

(iii) The matrix $A = \begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix}$ is

(A) idempotent matrix
(B) involutory
(C) nilpotent matrix of index 2
(D) $AA^{\mathrm{T}} = I$.

2. Passage: Let A be 3×3 matrix and B is adj A. Answer the following questions:

(i) If $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 3 & -1 & 4 \end{bmatrix}$, then A^{-1} is equal to

(A) $\frac{1}{11}\begin{bmatrix} 8 & -5 & -2 \\ -4 & -3 & 1 \\ -7 & 3 & -1 \end{bmatrix}$ (B) $\frac{1}{11}\begin{bmatrix} -8 & 5 & 2 \\ 4 & 3 & -1 \\ 7 & -3 & 1 \end{bmatrix}$

(C) $\frac{-1}{11}\begin{bmatrix} -7 & 3 & -1 \\ -4 & -3 & 1 \\ 8 & -5 & -2 \end{bmatrix}$ (D) $\frac{1}{11}\begin{bmatrix} -8 & 5 & 2 \\ 7 & -3 & 1 \\ 4 & 3 & -1 \end{bmatrix}$

(ii) If $A = \begin{bmatrix} 4 & -5 & -2 \\ 5 & -4 & 2 \\ 2 & 2 & 8 \end{bmatrix}$, then adj B is equal to

(A) 0 (B) I

(C) $\begin{bmatrix} -36 & 36 & -18 \\ -36 & 36 & 18 \\ 18 & -18 & 0 \end{bmatrix}$ (D) $\begin{bmatrix} -36 & -36 & 18 \\ 36 & 36 & -18 \\ 18 & -18 & 0 \end{bmatrix}$

(iii) If det $A \neq 0$, then B^{-1} is

(A) A (B) $|A|\,A$

(C) $\dfrac{A}{|A|}$ (D) $\dfrac{A^{-1}}{|A|}$

3. Passage: Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix}$ and X_1, X_2, X_3 be column matrices such that

$$AX_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, AX_2 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \text{ and } AX_3 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

Let X be the 3×3 matrix whose first, second and third columns are, respectively, X_1, X_2 and X_3. Answer the following questions:

(i) The value of det(X) is

(A) 3 (B) –3 (C) 3/2 (D) 2

(ii) The sum of all the elements of X^{-1} is

(A) –1 (B) 0 (C) 1 (D) 3

(iii) The matrix $[3 \quad 2 \quad 0]_{1\times 3}\, X \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$ is

(A) $[5]_{1\times1}$ (B) $\left[\dfrac{5}{2}\right]_{1\times1}$

(C) $[4]_{1\times1}$ (D) $\left[\dfrac{3}{2}\right]_{1\times1}$

4. Passage: Let X_1, X_2 and X_3 be column matrices such that

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 2 & 1 \end{bmatrix} X_1 = \begin{bmatrix} 6 \\ 7 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 2 & 9 \end{bmatrix} X_2 = \begin{bmatrix} 6 \\ 17 \\ 2 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix} X_3 = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$$

A is the 3×3 matrix whose first, second and third rows are respectively X_1^{T}, X_2^{T} and X_3^{T}. Answer the following questions:

(i) det A is equal to

(A) 2 (B) 0 (C) –8 (D) 8

(ii) Sum of all the elements of A is

(A) an even number

(B) a number of the form $4k + 3$

(C) a prime number

(D) a perfect square of an integer

(iii) t_r (adj A) is

(A) even number

(B) number of the form $3k + 2$

(C) perfect cube of an integer

(D) a prime number

Assertion–Reasoning Type Questions

In each of the following, two statements, I and II, are given and one of the following four alternatives has to be chosen.

(A) Both I and II are correct and II is a correct reasoning for I.

(B) Both I and II are correct but II is not a correct reasoning for I.

(C) I is true, but II is not true.

(D) I is not true, but II is true.

1. Statement I: The determinant of the matrix $\begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$ is zero.

Statement II: The determinant of a skew-symmetric matrix of odd order is zero.

2. Statement I: The system of equations $x + y + z = 4$, $2x - y + 2x = 5$, $x - 2y - z = -3$ has unique solution.

Statement II: If A is a 3×3 matrix and B is a 3×1 non-zero column matrix, then the equation $AX = B$ has unique solution if A is non-singular.

3. Statement I: The matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ is an idempotent matrix.

Statement II: If A is an idempotent matrix, then $A^4 = A$.

4. Statement I: The inverse of the matrix in the above problem 3 is itself.

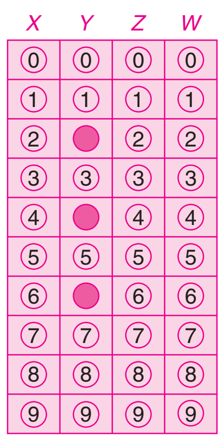
Statement II: The inverse of any idempotent matrix is itself.

5. Statement I: If A and B are symmetric matrices of same order, then $AB + BA$ is symmetric and $AB - BA$ is skew-symmetric.

Statement II: If P and Q are matrices of same order, then $(P \pm Q)^{\mathrm{T}} = P^{\mathrm{T}} \pm Q^{\mathrm{T}}$ and if P, Q are compatible for multiplication, then $(PQ)^{\mathrm{T}} = Q^{\mathrm{T}} P^{\mathrm{T}}$.

Integer Answer Type Questions

The answer to each of the questions in this section is a non-negative integer. The appropriate bubbles below the respective question numbers have to be darkened. For example, as shown in the figure, if the correct answer to the question number Y is 246, then the bubbles under Y labeled as 2, 4, 6 are to be darkened.



1. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{bmatrix}$, then the least positive integer k such that $A^k = 0$ is ______.

2. Let Z_1, Z_2, Z_3, be non-zero complex numbers and $|Z_1| = a$, $|Z_2| = b$, $|Z_3| = c$. If the matrix $\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$ is singular and Δ is the area of the triangle whose vertices are at Z_1, Z_2 and Z_3 and R is its circumradius, then $\dfrac{4\Delta}{R^2\sqrt{3}}$ is equal to ______.

3. Let S be the set of all 2×2 matrices whose elements are 0 or 1. Then the number of non-singular matrices belonging to S is ______.

4. If A is 3×3 matrix and $|A| = 2$, then $|\text{adj}(\text{adj } A)|$ is ______.

5. If $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$, then $\det(A + B)$ is ______.

ANSWERS

Single Correct Choice Type Questions

1. (D)
2. (B)
3. (C)
4. (C)
5. (A)
6. (D)
7. (D)
8. (A)
9. (D)
10. (D)
11. (C)
12. (B)
13. (A)
14. (B)
15. (C)
16. (A)
17. (A)
18. (B)
19. (B)
20. (D)
21. (C)
22. (D)
23. (A)
24. (A)
25. (C)

Multiple Correct Choice Type Questions

1. (B), (C)
2. (B), (D)
3. (B), (C)
4. (A), (B)
5. (B), (D)
6. (A), (B), (C)
7. (A), (B), (C), (D)
8. (A), (B), (C), (D)
9. (A), (B), (C), (D)
10. (B), (D)
11. (A), (B), (C)
12. (A), (B), (C), (D)
13. (A), (B)
14. (A), (B), (C), (D)
15. (A), (D)

Matrix-Match Type Questions

1. (A) $\to$ (p), (B) $\to$ (p), (C) $\to$ (r), (D) $\to$ (r)
2. (A) $\to$ (p), (B) $\to$ (q), (C) $\to$ (p), (D) $\to$ (q)
3. (A) $\to$ (q), (B) $\to$ (s), (C) $\to$ (p), (D) $\to$ (r)
4. (A) $\to$ (r), (B) $\to$ (r), (C) $\to$ (q), (D) $\to$ (p)
5. (A) $\to$ (q), (r), (s) (B) $\to$ (s), (C) $\to$ (p), (D) $\to$ (q), (s)

Comprehension-Type Question

1. (i) (B); (ii) (A); (iii) (C)
2. (i) (B); (ii) (A); (iii) (C)
3. (i) (A); (ii) (B); (iii) (A)
4. (i) (D); (ii) (C); (iii) (D)

Assertion–Reasoning Type Questions

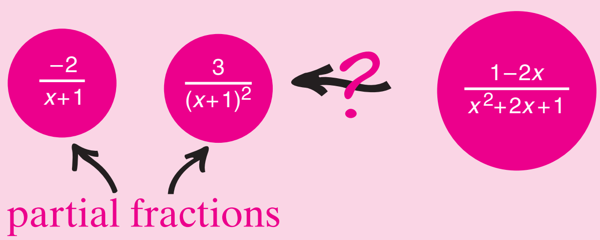
1. (A)
2. (A)
3. (D)
4. (C)
5. (A)

Integer Answer Type Questions

1. 2
2. 3
3. 6
4. 16
5. 0

Partial Fractions

9



Partial Fractions

Contents

The **partial fraction** decomposition or **partial fraction expansion** is used to reduce the degree of either the numerator or the denominator of a rational function. The partial fraction decomposition may be seen as the inverse procedure of the more elementary operation of addition of fractions, that produces a single rational fraction with a numerator and denominator usually of high degree.

It is well known that a polynomial in x is an expression of the form $a_0 + a_1 x + \cdots + a_n x^n$, where $a_0, a_1, \ldots, a_n$ are real numbers or complex numbers. Polynomials are usually denoted by the symbols $f(x)$, $g(x)$, etc. The degree of a polynomial $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ is defined to be n if $a_n \neq 0$. The a_i are called the coefficients of x^i in $f(x)$. A polynomial $f(x)$ is said to be the *zero polynomial* if each $a_i = 0$. A zero polynomial can have any degree and every non-zero real (complex) number is considered to be a polynomial of degree zero. Zero polynomial is denoted by usual 0 (zero). If two polynomials $f(x)$ and $g(x)$ are equal, then we write $f(x) = g(x)$. Two polynomials are said to be *equal* if their corresponding coefficients are equal. We are familiar with adding and multiplying two polynomials. In this chapter, we confine to polynomials whose coefficients are all real numbers.

9.1 | Rational Fractions

DEFINITION 9.1 An expression of the form $f(x)/g(x)$, where $f(x)$ and $g(x)$ are polynomials and $g(x) \neq 0$, is called a *rational fraction*.

DEFINITION 9.2 A rational fraction $f(x)/g(x)$ is called a *proper fraction* if either $f(x) = 0$ or degree of $f(x)$ is less than degree of $g(x)$. If $\deg f(x) \geq \deg g(x)$, then $f(x)/g(x)$ is called an *improper fraction.*

Examples

(1) $\dfrac{1+x}{1+x+x^2}$, $\dfrac{2+3x+4}{3+2x+3x^2}$, $\dfrac{2}{1+x}$, 0 and $\dfrac{1}{1+x^2}$ are all proper fractions.

(2) $\dfrac{1+x+x^2}{3+2x+4x^2}$, $\dfrac{1+2x+x^2}{2+x}$, $\dfrac{3}{2}$, and $\dfrac{1+x^8}{1+x^7}$ are all improper fractions.

Note that 3/2 is an improper fraction, since 3 and 2 are both polynomials of degree 0.

THEOREM 9.1 Let $f(x)/g(x)$ be a rational fraction. Then there exist unique polynomials $q(x)$ and $r(x)$ such that

$$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}$$

and $r(x)/g(x)$ is a proper fraction.

PROOF Since $f(x)/g(x)$ is a rational fraction, $f(x)$ and $g(x)$ are polynomials and $g(x) \neq 0$. If $f(x)/g(x)$ is a proper fraction, then we can take $q(x) = 0$ and $r(x) = f(x)$. Suppose that $f(x)/g(x)$ is an improper fraction. Then $\deg f(x) \geq \deg g(x)$.

Let

$$f(x) = a_0 + a_1 x + \cdots + a_m x^m, \; a_m \neq 0$$

and

$$g(x) = b_0 + b_1 x + \cdots + b_n x^n, \; b_n \neq 0$$

Then $\deg f(x) = m \geq n = \deg g(x)$. We shall apply induction on m.

If $m = 0$, then $n = 0$ and $f(x)/g(x) = a_0/b_0$, which is a real number. In this case, we can take $q(x) = a_0/b_0$ and $r(x) = 0$. Let $m > 0$ and suppose that the theorem is true for all rational fractions $h(x)/g(x)$ with $\deg(h(x)) < m$. Then put

$$h(x) = f(x) - \frac{a_m}{b_n} x^{m-n} g(x)$$

It can be easily seen that $\deg h(x) \leq m - 1 < m$ and therefore, by the induction hypothesis, we can write

$$\frac{h(x)}{g(x)} = q_1(x) + \frac{r(x)}{g(x)}$$

for some polynomials $q_1(x)$ and $r(x)$ such that $r(x)/g(x)$ is a proper fraction. Now, consider

$$\begin{aligned}\frac{f(x)}{g(x)} &= \frac{h(x)+a_m b_n^{-1} x^{m-n} g(x)}{g(x)} \\ &= a_m b_n^{-1} x^{m-n} + \frac{h(x)}{g(x)} \\ &= a_m b_n^{-1} x^{m-n} + q_1(x) + \frac{r(x)}{g(x)} \\ &= q(x) + \frac{r(x)}{g(x)}\end{aligned}$$

where $q(x) = a_m b_n^{-1} x^{m-n} + q_1(x)$ and $r(x)/g(x)$ is a proper fraction. Therefore, we have proved that there exist polynomials $q(x)$ and $r(x)$ such that

$$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}$$

and $r(x)/g(x)$ is a proper fraction.

To prove uniqueness of $q(x)$ and $r(x)$, suppose $q'(x)$ and $r'(x)$ are also polynomials such that

$$\frac{f(x)}{g(x)} = q'(x) + \frac{r'(x)}{g(x)}$$

where $r'(x)/g(x)$ is a proper fraction. Then

$$q(x)\, g(x) + r(x) = f(x) = q'(x)\, g(x) + r'(x)$$

and hence

$$[q(x) - q'(x)]\, g(x) = r'(x) - r(x)$$

If $r(x) - r'(x) \neq 0$, then

$$\deg\, [r'(x) - r(x)] = \deg\, [\{q(x) - q'(x)\}\, g(x)] \geq \deg\, g(x)$$

which is a contradiction, since $\deg r(x) < \deg g(x)$ and $\deg r'(x) < \deg g(x)$. Therefore $r'(x) - r(x) = 0$ and hence $[q(x) - q'(x)]\, g(x) = 0$, so that $q(x) - q'(x) = 0$ (since $g(x) \neq 0$). Thus

$$q(x) = q'(x) \quad \text{and} \quad r(x) = r'(x)$$

The unique polynomials $q(x)$ and $r(x)$ found above are called the *quotient* and the *remainder*, respectively, and the algorithm to find $q(x)$ and $r(x)$ is called the *division algorithm for polynomials*. Note that $q(x)$ and $r(x)$ are unique polynomials satisfying the property

$$f(x) = q(x)\, g(x) + r(x)$$

such that $r(x) = 0$ or $\deg r(x) < \deg g(x)$.

The reader might be familiar with the algorithm to find the quotient and remainder. If $r(x) = 0$ in the above, then we say that $g(x)$ divides $f(x)$. Further, any polynomial $f(x)$ of degree greater than one can be uniquely expressed as

$$f(x) = q(x)(x - a) + r$$

for some real number r, where a is a given real number, and therefore, we have $r = f(a)$ so that

$$f(x) = q(x)(x - a) + f(a)$$

Also, $x - a$ divides $f(x)$ if and only if $f(a) = 0$; this is popularly known as the *factorization theorem*. ■

Examples

(1) $\dfrac{1+x^2}{1+x} = (-1+x) + \dfrac{2}{1+x}$

(2) $\dfrac{1+x+x^2}{1+x} = x + \dfrac{1}{1+x}$

COROLLARY 9.1 Let $q(x)$, $q'(x)$, $r(x)$, $r'(x)$, $g(x)$ and $g'(x)$ be polynomials in x such that $r(x)/g(x)$ and $r'(x)/g'(x)$ are proper fractions and

$$q(x) + \frac{r(x)}{g(x)} = q'(x) + \frac{r'(x)}{g'(x)}$$

Then

$$q(x) = q'(x) \quad \text{and} \quad \frac{r(x)}{g(x)} = \frac{r'(x)}{g'(x)}$$

DEFINITION 9.3 Two polynomials $f(x)$ and $g(x)$ are said to be *relatively prime* (or *prime to each other*) if there is no polynomial of positive degree dividing both $f(x)$ and $g(x)$.

Examples

(1) The polynomials $1+x$ and $1-x$ are relatively prime, since any common divisor of these must divide their sum, which is a polynomial of degree 0.

(2) $1+2x+x^2$ and $1-x^2$ are not relatively prime since $1+x$ is a common divisor of these.

We assume the following theorem whose proof is beyond the scope of this book.

THEOREM 9.2 Two polynomials $f(x)$ and $g(x)$ are relatively prime if and only if there exist polynomials $p(x)$ and $q(x)$ such that

$$f(x)\,p(x) + g(x)\,q(x) = 1$$

9.2 | Partial Fractions

In this section we discuss several methods of expressing a rational fraction as a sum of similar fractions; such a rational fraction is known as *partial fraction.* First, we have the following main result.

THEOREM 9.3 (FUNDAMENTAL THEOREM) Let $f(x)$, $g(x)$ and $h(x)$ be polynomials such that $h(x)/f(x)g(x)$ is a proper fraction. Suppose that $f(x)$ and $g(x)$ are relatively prime. Then there exist proper fractions $\alpha(x)/f(x)$ and $\beta(x)/g(x)$ such that

$$\frac{h(x)}{f(x)g(x)} = \frac{\alpha(x)}{f(x)} + \frac{\beta(x)}{g(x)}$$

PROOF Since $f(x)$ and $g(x)$ are relatively prime, there exist polynomials $p(x)$ and $q(x)$ such that

$$f(x)\,q(x) + g(x)\,p(x) = 1$$

Now,

$$\frac{h(x)}{f(x)g(x)} = \frac{h(x)[f(x)q(x) + g(x)p(x)]}{f(x)g(x)} = \frac{h(x)p(x)}{f(x)} + \frac{h(x)q(x)}{g(x)}$$

If $h(x)p(x)/f(x)$ and $h(x)q(x)/g(x)$ are not proper fractions, then by Theorem 9.1, we can write

$$\frac{h(x)p(x)}{f(x)} = t(x) + \frac{\alpha(x)}{f(x)} \quad \text{and} \quad \frac{h(x)q(x)}{g(x)} = s(x) + \frac{\beta(x)}{g(x)}$$

for some polynomials $t(x), s(x), \alpha(x)$ and $\beta(x)$ such that $\alpha(x)/f(x)$ and $\beta(x)/g(x)$ are proper fractions and, therefore, we have

$$\frac{h(x)}{f(x)g(x)} = \frac{h(x)p(x)}{f(x)} + \frac{h(x)q(x)}{g(x)} = s(x) + t(x) + \frac{\alpha(x)}{f(x)} + \frac{\beta(x)}{g(x)} = s(x) + t(x) + \frac{\alpha(x)g(x) + \beta(x)f(x)}{f(x)g(x)}$$

Since $\deg \alpha(x) < \deg f(x)$ and $\deg \beta(x) < \deg g(x)$, we get that $\deg[\alpha(x)g(x) + \beta(x)f(x)] < \deg[f(x)g(x)]$ which implies that $[\alpha(x)g(x) + \beta(x)f(x)]/f(x)g(x)]$ is a proper fraction. Hence $h(x)/f(x)g(x)$ and therefore

$$s(x) + t(x) = 0 \quad \text{and} \quad \frac{h(x)}{f(x)g(x)} = \frac{\alpha(x)}{f(x)} + \frac{\beta(x)}{g(x)}$$

■

In the following theorems, we will prove that a proper fraction can be resolved into sum of its simplest partial fractions, in various cases. First, we have the following.

DEFINITION 9.4 If a proper fraction is expressed as the sum of two or more proper fractions, then each of these is called a *partial fraction* of the given proper fraction. The process of finding partial fractions of a given proper fraction is known as resolving the proper fraction into partial fractions.

Example 9.1

Find the partial fractions of

(a) $\dfrac{1}{6 - 5x + x^2}$

(b) $\dfrac{2}{x^2 - 1}$

Solution:

(a) The given fraction can be simplified as

$$\frac{1}{6 - 5x + x^2} = \frac{1}{(x-3)(x-2)} = \frac{1}{x-3} + \frac{-1}{x-2}$$

Therefore

$$\frac{1}{x-3} \quad \text{and} \quad \frac{-1}{x-2}$$

are called the partial fractions of the given fraction.

(b) The given fraction can be simplified as

$$\frac{2}{x^2 - 1} = \frac{1}{x-1} + \frac{-1}{x+1}$$

Therefore

$$\frac{1}{x-1} \quad \text{and} \quad \frac{-1}{x+1}$$

are partial fractions of $2/(x^2 - 1)$.

DEFINITION 9.5 A polynomial of positive degree is said to be *irreducible* if it cannot be expressed as a product of two or more polynomials of positive degree.

Examples

(1) Any polynomial of degree 1 is irreducible (such polynomials are called *linear polynomials*).

(2) $1 + x + x^2$ is an irreducible polynomial.

Example 9.2

Show that the following polynomials are not irreducible.

(a) $1 + x + x^2 + x^3$

(b) $x^3 - 6x^2 + 11x - 6$

Solution:

(a) $1 + x + x^2 + x^3$ is not an irreducible polynomial, since $(1 + x)(1 + x^2) = 1 + x + x^2 + x^3$.

(b) $x^3 - 6x^2 + 11x - 6$ is not an irreducible polynomial, since $x^3 - 6x^2 + 11x - 6 = (x - 3)(x - 2)(x - 1)$.

THEOREM 9.4 Let $f(x)/g(x)$ be a proper fraction and $ax + b$ a non-repeated factor of $g(x)$. Then $f(x)/g(x)$ has a partial fraction of the form

$$\frac{A}{ax + b}$$

where A is a constant.

PROOF Since $ax + b$ is a non-repeated factor of $g(x)$, we can write

$$g(x) = (ax + b)h(x)$$

where $ax + b$ and $h(x)$ are relatively prime. Then by the fundamental theorem (Theorem 9.3), we can write

$$\frac{f(x)}{g(x)} = \frac{\alpha(x)}{ax + b} + \frac{\beta(x)}{h(x)}$$

where $\alpha(x)/(ax + b)$ and $\beta(x)/h(x)$ are proper fractions.

In particular, deg $\alpha(x) <$ deg $(ax + b) = 1$ and hence $\alpha(x)$ is a constant, say A. Therefore

$$\frac{A}{ax + b} \text{ is a partial fraction of } \frac{f(x)}{g(x)}$$

■

THEOREM 9.5 If $f(x)/g(x)$ is a proper fraction and $ax^2 + bx + c$ $(a \neq 0)$ is a non-repeated irreducible factor of $g(x)$, then $f(x)/g(x)$ has a partial fraction of the form

$$\frac{Ax + B}{ax^2 + bx + c}$$

PROOF Same as in the above theorem. ■

In the following example, we demonstrate a method of finding partial fractions.

Example 9.3

Resolve the following fraction into partial fractions:

$$\frac{3x + 2}{(x - 3)(2x + 1)}$$

Solution: Write

$$\frac{3x + 2}{(x - 3)(2x + 1)} = \frac{A}{x - 3} + \frac{B}{2x + 1}$$

Then

$$\frac{3x + 2}{(x - 3)(2x + 1)} = \frac{A(2x + 1) + B(x - 3)}{(x - 3)(2x + 1)}$$

Therefore

$$3x + 2 = A(2x + 1) + B(x - 3)$$

By taking $x = 3$, we get

$$3 \cdot 3 + 2 = A(2 \cdot 3 + 1) + B(3 - 3)$$
$$A = \frac{11}{7}$$

Similarly, by taking $x = -1/2$, we get $B = -1/7$. Therefore

$$\frac{3x+2}{(x-3)(2x+1)} = \frac{11/7}{x-3} + \frac{-1/7}{2x+1} = \frac{11}{7(x-3)} - \frac{1}{7(2x+1)}$$

THEOREM 9.6 Let $f(x)/(ax+b)^n$ be a proper fraction. Then, there exist constants $A_1, A_2, \ldots, A_n$ such that

$$\frac{f(x)}{(ax+b)^n} = \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}$$

PROOF This is a repeated application of Theorem 9.1. Since $f(x)/(ax+b)^n$ is a proper fraction, $\deg f(x) < n$. First use Theorem 9.1 to write

$$\frac{f(x)}{ax+b} = q_1(x) + \frac{r_1(x)}{ax+b} \tag{9.1}$$

where $q_1(x)$ and $r_1(x)$ are some polynomials and $\deg r_1(x) < \deg(ax+b) = 1$. Therefore $r_1(x)$ is a constant. Put $B_1 = r_1(x)$. Again, we can write (using Theorem 9.1),

$$\frac{q_1(x)}{ax+b} = q_2(x) + \frac{B_2}{ax+b} \tag{9.2}$$

where $q_2(x)$ is a polynomial and B_2 is a constant. From Eqs. (9.1) and (9.2), we get that

$$\frac{f(x)}{(ax+b)^2} = \frac{q_1(x)}{ax+b} + \frac{r_1(x)}{(ax+b)^2} = q_2(x) + \frac{B_2}{ax+b} + \frac{B_1}{(ax+b)^2}$$

Again,

$$\frac{q_2(x)}{(ax+b)} = q_3(x) + \frac{B_3}{ax+b}$$

for some polynomial $q_3(x)$ and a constant B_3. Substituting this in the above, we get

$$\frac{f(x)}{(ax+b)^3} = q_3(x) + \frac{B_3}{ax+b} + \frac{B_2}{(ax+b)^2} + \frac{B_1}{(ax+b)^3}$$

The process can be performed n times to get

$$\frac{f(x)}{(ax+b)^n} = q_n(x) + \frac{B_n}{ax+b} + \frac{B_{n-1}}{(ax+b)^2} + \cdots + \frac{B_1}{(ax+b)^n}$$

where $B_1, B_2, \ldots, B_n$ are constants. Since $\deg f(x) < n$, we get $q_n(x) = 0$. Now, put $A_i = B_{n-i+1}$ to get

$$\frac{f(x)}{(ax+b)^n} = \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}$$

■

Example 9.4

Resolve the following fraction into partial fractions

$$\frac{1+x+x^2}{(x+2)^3}$$

Solution: By Theorem 9.6, we can write

$$\frac{1+x+x^2}{(x+2)^3} = \frac{A_1}{x+2} + \frac{A_2}{(x+2)^2} + \frac{A_3}{(x+2)^3} \tag{9.3}$$

We have to find the values of the constants A_1, A_2, and A_3. From Eq. (9.3), we have

$$1 + x + x^2 = A_1(x+2)^2 + A_2(x+2) + A_3 \qquad (9.4)$$

By substituting –2 for x, we get

$$1 + (-2) + (-2)^2 = 0 + 0 + A_3$$

$$A_3 = 3$$

Taking $x = 0$ and $x = 1$ in Eq. (9.4), we get

$$-2 = 4A_1 + 2A_2 \quad \text{or} \quad 2A_1 + A_2 = -1$$

and $\qquad 0 = 9A_1 + 3A_2 \quad \text{or} \quad 3A_1 + A_2 = 0$

By solving these two, we get that $A_1 = 1$ and $A_2 = -3$ from Eq. (9.3), we have

$$\frac{1+x+x^2}{(x+2)^3} = \frac{1}{(x+2)} - \frac{3}{(x+2)^2} + \frac{3}{(x+2)^3}$$

Example 9.5

Resolve the following into partial fractions:

$$\frac{1+x+x^2+x^3}{(1+2x+x^2)(1-2x+x^2)}$$

Solution: First, note that $(1+x)(1+x^2) = 1+x+x^2+x^3$. Therefore, the given fraction is

$$\frac{(1+x)(1+x^2)}{(1+x)^2(1-x)^2} = \frac{1+x^2}{(1+x)(1-x)^2}$$

Let

$$\frac{1+x^2}{(1+x)(1-x)^2} = \frac{A_1}{1+x} + \frac{A_2}{1-x} + \frac{A_3}{(1-x)^2}$$

Then

$$1 + x^2 = A_1(1-x)^2 + A_2(1+x)(1-x) + A_3(1+x)$$

(a) Taking $x = 0$, $A_1 + A_2 + A_3 = 1$
(b) Taking $x = 1$, $2A_3 = 2$ or $A_3 = 1$
(c) Taking $x = -1$, $4A_1 = 2$ or $A_1 = 1/2$

Therefore

$$A_2 = 1 - A_1 - A_3 = 1 - \frac{1}{2} - 1 = -\frac{1}{2}$$

and so

$$\frac{1+x+x^2+x^3}{(1+2x+x^2)(1-2x+x^2)} = \frac{1}{2(1+x)} - \frac{1}{2(1-x)} + \frac{1}{(1-x)^2}$$

Example 9.6

Resolve the following into partial fractions:

$$\frac{42-19x}{(x-4)(x^2+1)}$$

Solution: Let

$$\frac{42-19x}{(x-4)(x^2+1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-4}$$

Therefore

$$42 - 19x = (Ax+B)(x-4) + C(x^2+1) \qquad (9.5)$$

Put $x = 4$. Then $C = -2$. Equating coefficient of x^2 on both sides of Eq. (9.5), we get

$$0 = A + C \Rightarrow A = 2$$

Equating the coefficient of x on both sides of Eq. (9.5), we get

$$-4A + B = -19 \quad \text{so that} \quad B = -11$$

Therefore

$$\frac{42-19x}{(x-4)(x^2+1)} = \frac{2x-11}{x^2+1} - \frac{2}{x-4}$$

WORKED-OUT PROBLEMS

Single Correct Choice Type Questions

1. If

$$\frac{mx+n}{(x-a)(x+b)} = \frac{A}{x-a} + \frac{B}{x+b}$$

then $A + B$ is equal to

(A) m (B) n (C) $m + n$ (D) mn

Solution: We have

$$mx + n = A(x+b) + B(x-a)$$

Now

$$x = a \Rightarrow A = \frac{ma+n}{a+b}$$

$$x = -b \Rightarrow B = \frac{-mb+n}{-(a+b)} = \frac{mb-n}{a+b}$$

Adding we get

$$A + B = \frac{m(a+b)}{a+b} = m$$

Answer: (A)

2. If

$$\frac{7x-1}{6x^2-5x+1} = \frac{A}{3x-1} + \frac{B}{2x-1}$$

then $B - A$ equals

(A) 1 (B) 7 (C) 8 (D) 9

Solution: We have

$$7x - 1 = A(2x-1) + B(3x-1)$$

Now

$$x = \frac{1}{3} \Rightarrow \frac{7}{3} - 1 = A\left(\frac{2}{3} - 1\right) \Rightarrow A = -4$$

$$x = \frac{1}{2} \Rightarrow \frac{7}{2} - 1 = B\left(\frac{3}{2} - 1\right) \Rightarrow B = 5$$

Therefore

$$B - A = 5 + 4 = 9$$

Answer: (D)

3. If

$$\frac{2x^2+3x+1}{(1-2x)(1-x^2)} = \frac{A}{1-x} + \frac{B}{1-2x}$$

then A, B are, respectively,

(A) $-3, 4$ (B) $-3, -4$ (C) $3, 4$ (D) $3, -4$

Solution: Clearly

$$\frac{2x^2+3x+1}{(1-2x)(1-x)(1+x)} = \frac{(2x+1)(x+1)}{(1-2x)(1-x)(1+x)}$$

$$= \frac{2x+1}{(1-2x)(1-x)}$$

Therefore

$$2x + 1 = A(1-2x) + B(1-x)$$

Now

$$x = 1 \Rightarrow A = -3$$

$$x = \frac{1}{2} \Rightarrow B = 4$$

Answer: (A)

4. Let

$$\frac{9}{(x+1)(x-2)^2} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

then the ordered triple (A, B, C) is

(A) $(1, 3, -1)$ (B) $(1, -1, 3)$

(C) $(3, -1, 1)$ (D) $(1, 1, -3)$

Solution: We have

$$9 = A(x-2)^2 + B(x+1)(x-2) + C(x+1)$$

Now for

$$x = -1 \Rightarrow A = 1$$

$$x = 2 \Rightarrow C = 3$$

Now $A + B =$ Coefficient of $x^2 = 0$. Therefore $B = -1$ and so

$$A = 1, B = -1, C = 3$$

Answer: (B)

5. Suppose

$$\frac{3x^3-8x^2+10}{(x-1)^4} = \frac{a}{x-1} + \frac{b}{(x-1)^2} + \frac{c}{(x-1)^3} + \frac{d}{(x-1)^4}$$

Then $a + b + c + d$ is equal to

(A) 1 (B) 2 (C) 4 (D) -5

Solution: Put $x - 1 = y$. Therefore we have

$$\frac{3(y+1)^3 - 8(y+1)^2 + 10}{y^4}$$

$$= \frac{3(y^3+3y^2+3y+1) - 8(y^2+2y+1) + 10}{y^4}$$

$$\frac{3y^3+y^2-7y+5}{y^4} = \frac{5}{y^4} - \frac{7}{y^3} + \frac{1}{y^2} + \frac{3}{y}$$

$$= \frac{5}{(x-1)^4} - \frac{7}{(x-1)^3} + \frac{1}{(x-1)^2} + \frac{3}{x-1}$$

Answer: (B)

6. If

$$\frac{x^2+4}{(x^2+1)(2x^2+3)} = \frac{A}{x^2+1} + \frac{B}{2x^2+3}$$

then $A - B$ is

(A) 6 (B) 8 (C) 9 (D) 11

Solution: Put $y = x^2$ in the given expression. We get the fraction

$$\frac{y+4}{(y+1)(2y+3)} = \frac{A}{y+1} + \frac{B}{2y+3}$$
$$y + 4 = A(2y+3) + B(y+1)$$

Solving we get

$$y = -1 \Rightarrow A = 3$$
$$y = \frac{-3}{2} \Rightarrow B = -5$$

Therefore $A - B = 3 - (-5) = 8$.

Note that the substitutions $y = -1$ and $-3/2$ are for the fraction in y, but not for the fraction in x.

Answer: (B)

7. Consider the series

$$\frac{1}{(1+x)(1+x^2)} + \frac{x}{(1+x^2)(1+x^3)} + \frac{x^2}{(1+x^3)(1+x^4)} + \cdots$$

If $x > 1$, then sum to infinity of the series is

(A) $\frac{1}{1-x^2}$ (B) $\frac{1}{x^2-1}$

(C) $\frac{1}{x(1-x^2)}$ (D) $\frac{1}{x(x^2-1)}$

Solution: Let

$$u_k = \frac{x^{k-1}}{(1+x^k)(1+x^{k+1})}$$
$$= \frac{1}{x(x-1)}\left[\frac{1}{1+x^k} - \frac{1}{1+x^{k+1}}\right]$$

Let s_n be the sum to n terms of the series. Then

$$s_n = \sum_{k=1}^{n} u_k = \frac{1}{x(x-1)}\left[\frac{1}{1+x} - \frac{1}{1+x^{n+1}}\right]$$

Therefore

$$\operatorname*{Lt}_{n\to\infty} s_n = \frac{1}{x(x^2-1)}$$

Answer: (D)

8. When

$$\frac{x^3}{(x-a)(x-b)(x-c)}$$

is resolved into partial fractions, then

$$\sum \frac{a^3}{(a-b)(a-c)(a-d)}$$

where d is any real number not equal to a, b and c is

(A) 0 (B) 1

(C) $a+b+c+d$ (D) $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}$

Solution: We have

$$\frac{x^3}{(x-a)(x-b)(x-c)} = 1 + \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$$
$$x^3 = (x-a)(x-b)(x-c) + A(x-b)(x-c) + B(x-a)(x-c) + C(x-a)(x-b)$$

Therefore

$$A = \frac{a^3}{(a-b)(a-c)},\ B = \frac{b^3}{(b-a)(b-c)},\ C = \frac{c^3}{(c-a)(c-b)}$$

which gives

$$\frac{x^3}{(x-a)(x-b)(x-c)} = 1 + \frac{A}{(x-a)(a-b)(a-c)} + \frac{B}{(x-b)(b-a)(b-c)} + \frac{C}{(x-c)(c-a)(c-b)}$$

Put $x = d$ on both sides. We get

$$\sum \frac{a^3}{(a-b)(a-c)(a-d)} = 1$$

Answer: (B)

9. When $x^4/[(x-a)(x-b)(x-c)]$ is resolved into partial fractions and d is any real number different from a, b and c, then

$$\sum \frac{a^4}{(a-b)(a-c)(a-d)} =$$

(A) 1 (B) 0

(C) $a+b+c+d$ (D) $abc+abd+acd+bcd$

Solution: We have

$$\frac{x^4}{(x-a)(x-b)(x-c)} = (x+a+b+c) + \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$$

Therefore as in the above, we have

$$\sum \frac{a^4}{(a-b)(a-c)(a-d)} = a+b+c+d$$

Answer: (C)

Multiple Correct Choice Type Questions

1. If

$$\frac{13x+46}{12x^2-11x-15}=\frac{A}{3x-5}+\frac{B}{4x+3}$$

then

(A) $A=-5$ (B) $B=7$
(C) $A=7$ (D) $B=-5$

Solution: We have

$$13x+46=A(4x+3)+B(3x-5)$$

Now

$$x=\frac{5}{3}\Rightarrow A\left(\frac{29}{3}\right)=\frac{203}{3}\Rightarrow A=7$$

$$x=\frac{-3}{4}\Rightarrow B\left(\frac{-29}{4}\right)=\frac{145}{4}\Rightarrow B=-5$$

Answers: (C), (D)

2. If

$$\frac{2x+1}{(x-1)(x^2+1)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+1}$$

then

(A) $A=\frac{3}{2}$ (B) $B=\frac{3}{2}, C=\frac{1}{2}$
(C) $A=\frac{-3}{2}$ (D) $B=\frac{-3}{2}, C=\frac{1}{2}$

Solution: From the given expression we have

$$2x+1=A(x^2+1)+(Bx+C)(x-1)$$

For $x=1$ we get $A=3/2$.

Also $0=$ coefficienct of $x^2=A+B$. Therefore

$$B=\frac{-3}{2}$$

For $x=0$

$$A-C=1\Rightarrow C=A-1=\frac{3}{2}-1=\frac{1}{2}$$

Answers: (A), (D)

3. If

$$\frac{3-2x^2}{(x^2-3x+2)^2}=\frac{A}{1-x}+\frac{B}{(1-x)^2}+\frac{C}{2-x}+\frac{D}{(2-x)^2}$$

then

(A) $A+B=3$ (B) $A+D=-3$
(C) $B+C=3$ (D) $A+B+C+D=-4$

Solution: Simplifying the given expression we get

$$3-2x^2=A(1-x)(2-x)^2+B(2-x)^2$$
$$+C(2-x)(1-x)^2+D(1-x)^2$$

Now for

$$x=1\Rightarrow B=1$$

$$x=2\Rightarrow D=-5$$

$$0=\text{coefficient of } x^3=A+C \tag{9.6}$$

$$x=0\Rightarrow 4A+4B+2C+D=3$$

$$\Rightarrow 4A+4+2C-5=3$$

Therefore

$$2A+C=2 \tag{9.7}$$

From Eqs. (9.6) and (9.7) we get $A=2$, $C=-2$. Hence

$$\frac{3-2x^2}{(1-x)^2(2-x)^2}=\frac{2}{1-x}+\frac{1}{(1-x)^2}-\frac{2}{2-x}-\frac{5}{(2-x)^2}$$

Answers: (A), (B), (D)

4. If

$$\frac{(x+1)^2}{x(x^2+1)}=\frac{A}{x}+\frac{Bx+C}{x^2+1}$$

then

(A) $A=1$ (B) $B=1$ (C) $C=2$ (D) $B=0$

Solution: Simplifying the given expression we get

$$(x+1)^2=A(x^2+1)+(Bx+C)\,x$$

Now

$$x=0\Rightarrow A=1$$

$$2=\text{coefficient of } x=C$$

$$1=\text{coefficient of } x^2=A+B=1+B\Rightarrow B=0$$

Hence

$$\frac{(x+1)^2}{x(x^2+1)}=\frac{1}{x}+\frac{2}{x^2+1}$$

Answers: (A), (C), (D)

5. If

$$\frac{3x+4}{(x+1)(x^2-1)}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$$

then

(A) $A = \frac{7}{4}$ (B) $A + B = 0$

(C) $C = \frac{1}{2}$ (D) $C = \frac{-1}{2}$

Solution: We have

$$\frac{3x+4}{(x+1)(x^2-1)} = \frac{3x+4}{(x+1)^2(x-1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$3x+4 = A(x+1)^2 + B(x^2-1) + C(x-1)$$

Now

$$x = 1 \Rightarrow 7 = 4A \quad \text{or} \quad A = \frac{7}{4}$$

$$x = -1 \Rightarrow -2C = 1 \quad \text{or} \quad C = \frac{-1}{2}$$

$$0 = \text{coefficient of } x^2 = A + B \Rightarrow B = \frac{-7}{4}$$

Answers: (A), (B), (D)

6. If

$$\frac{4x}{x^4+x^2+1} = \frac{A}{x^2-x+1} + \frac{B}{x^2+x+1}$$

then

(A) $A + B = 0$ (B) $AB = -4$

(C) $A = 2, B = -2$ (D) $\frac{A}{B} = -1$

Solution: We have

$$\frac{4x}{x^4+x^2+1} = \frac{4x}{(x^2-x+1)(x^2+x+1)}$$

$$= 2\left[\frac{1}{x^2-x+1} - \frac{1}{x^2+x+1}\right]$$

Therefore, we get $A = 2, B = -2$

Answers: (A), (B), (C), (D)

SUMMARY

9.1 ***Polynomial:*** If $a_0, a_1, a_2, \ldots, a_n$ are real or complex numbers, $a_0 \neq 0$ and n is a positive integer, thus an expression of the form $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n$ is called polynomial of degree n.

$a_0, a_1, a_2, \ldots, a_n$ are called coefficients of the polynomial. If $a_0, a_1, a_2, \ldots, a_n$ are real numbers, then the polynomial is called polynomial with real coefficients and if $a_0, a_1, a_2, \ldots, a_n$ are complex numbers, then it is called polynomial with complex coefficients. Generally polynomials are denoted by $f(x)$, $g(x)$, $h(x)$, etc.

9.2 ***Constant polynomial:*** Every non-zero number is considered as a polynomial of zero degree and it is called constant polynomial.

9.3 ***Zero (null) polynomial:*** A polynomial is called zero polynomial if all of its coefficients are zeros.

9.4 ***Degree of a zero polynomial:*** Any positive integer can be considered to be the degree of zero polynomial.

9.5 ***Division algorithm (or Euclid's algorithm):*** If $f(x)$ and $g(x)$ are two polynomials and $g(x) \not\equiv 0$, then there exist unique polynomials $q(x)$ and $r(x)$ such that $f(x) = q(x)\, g(x) + r(x)$ where either $r(x) \equiv 0$ (i.e., zero polynomial) or degree of $r(x)$ is less than the degree of $g(x)$. If $r(x) \equiv 0$, then $g(x)$ is called factor of $f(x)$.

9.6 ***Proper and improper fractions:*** Let $f(x)$ and $g(x)$ be two polynomials and $g(x) \not\equiv 0$. Then $f(x)/g(x)$ is called proper or improper fraction according as the degree of $f(x)$ is less than or greater than the degree of $g(x)$.

9.7 ***Relatively prime polynomials:*** Two polynomials are said to be relatively prime to each other (or coprime to each other) if they do not have a common factor of positive degree.

Two polynomials $f(x)$ and $g(x)$ are relatively prime to each other if and only if there exist polynomials $p(x)$ and $q(x)$ such that

$$f(x)\, p(x) + g(x)\, q(x) = 1$$

9.8 ***Fundamental theorem:*** Let $f(x)$, $g(x)$ and $h(x)$ be polynomials such that $h(x)/f(x)\, g(x)$ is a proper fraction. If $f(x)$ and $g(x)$ are relatively prime to each other, then there exist proper fractions $p(x)/f(x)$ and $q(x)/g(x)$ such that

$$\frac{h(x)}{f(x)g(x)} = \frac{p(x)}{f(x)} + \frac{q(x)}{g(x)}$$

9.9 ***Partial fractions:*** If a proper fraction of two polynomials is expressed as sum of two or more proper fractions, then each of these proper fractions is called partial fraction of the given proper fraction.

9.10 ***Irreducible polynomial:*** A polynomial of positive degree is said to be irreducible if it cannot be expressed as a product of two or more polynomials of positive degrees.

9.11 ***Cases of partial fractions:*** Let $f(x)/g(x)$ be a proper fraction [i.e., degree of $f(x)$ is less than the degree of $g(x)$]. Then

(1) If $g(x)$ has a non-repeated linear factor $ax + b$, then $A/ax + b$ is a partial fraction of $f(x)/g(x)$ where A is a constant which can be determined.

(2) If $g(x)$ has a non-repeated irreducible quadratic factor $ax^2 + bx + c$. Then

$$\frac{Ax+B}{ax^2+bx+c}$$

is a partial fraction of $f(x)/g(x)$ for some real constant A and B.

(3) If $g(x)$ has a repeated linear factor of the form $(ax + b)^n$, then

$$\frac{A_1}{ax+b}, \frac{A_2}{(ax+b)^2}, \ldots, \frac{A_n}{(ax+b)^n}$$

where $A_1, A_2, \ldots, A_n$ are constants occur as partial fractions in $f(x)/g(x)$.

(4) If $(ax^2 + bx + c)^k$, where $ax^2 + bx + c$ is irreducible, is a repeated factor of $g(x)$, then the fractions of the form

$$\frac{A_1x+B_1}{ax^2+bx+c}, \frac{A_2x+B_2}{(ax^2+bx+c)^2}, \ldots, \frac{A_kx+B_k}{(ax^2+bx+c)^k}$$

occur as partial fraction for $f(x)/g(x)$.

EXERCISES

Single Correct Choice Type Questions

1. If

$$\frac{5x+6}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$$

then $A + B$ is equal to

(A) 4 (B) 3 (C) 5 (D) –5

2. If

$$\frac{x^3}{(x-1)(x+2)} = (x-1) + \frac{A}{x-1} + \frac{B}{x+2}$$

then $A + B$ is

(A) –3 (B) 3 (C) 5 (D) –5

3. If

$$\frac{x^3}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$$

then $A - C$ is

(A) 3
(B) 1
(C) an even prime number
(D) an odd prime number of the form $4n + 1$

4. If

$$\frac{5x^2+2}{x^3+x} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

then the ordered triple (A, B, C) is

(A) (2, 3, 1) (B) (1, 2, 3)
(C) (0, 2, 3) (D) (2, 3, 0)

5. If

$$\frac{x^2+5}{(x^2+2)^2} = \frac{1}{x^2+2} + \frac{A}{(x^2+2)^2}$$

then

(A) $A = 3$ (B) $2A = 5$
(C) $3A = 1$ (D) $A = -1$

6. Let

$$\frac{x^2+5x+1}{(x+1)(x+2)(x+3)} = \frac{a}{x+1} + \frac{b}{(x+2)(x+1)} + \frac{c}{(x+1)(x+2)(x+3)}$$

and

$$P = \begin{bmatrix} a & b \\ c & -1 \end{bmatrix}$$

then P is

(A) idempotent matrix (B) involuntary matrix
(C) symmetric matrix (D) scalar matrix

7. If

$$\frac{3x}{(x-a)(x-b)} = \frac{2}{x-a} + \frac{1}{x-b}$$

then the relation between a and b is

(A) $a = 2b$ (B) $b = 2a$
(C) $a = -2b$ (D) $b = -2a$

8. If

$$\frac{x^2+1}{(2+x)(2-x)(x-1)} = \frac{a}{3(x-1)} + \frac{b}{4(2-x)} - \frac{c}{12(x+2)}$$

and $$A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$$

then $|A|$ is equal to

(A) –34 (B) 34
(C) –24 (D) –108

9. If

$$\frac{x^4}{(x-1)(x-2)} = f(x) + \frac{A}{x-1} + \frac{B}{x-2}$$

then

(A) $f(x) > 0$ for all real x
(B) $f(x) = 0$ has distinct real roots
(C) $f(x) = 0$ has equal roots
(D) Range of $f(x)$ is $\mathbb{R}$

10. If

$$\frac{3x}{(x-6)(x+3)} = \frac{a}{x-6} + \frac{b}{x+3} \quad \text{and} \quad A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

then A^{-1} is equal to

(A) $\begin{bmatrix} \frac{2}{3} & \frac{-1}{3} \\ \frac{-1}{3} & \frac{2}{3} \end{bmatrix}$ (B) $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

(C) $\begin{bmatrix} \frac{-1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{-1}{3} \end{bmatrix}$ (D) $\begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$

Multiple Correct Choice Type Questions

1. If

$$\frac{x^2-3}{(x+2)(x^2+1)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+1}$$

then

(A) $5B = 4$ (B) $5A = 1$
(C) $5C = -8$ (D) $ABC = -32$

2. Let

$$\frac{x^2+1}{(2+x)(2-x)(x-1)} = \frac{a}{3(x-1)} + \frac{b}{4(2-x)} - \frac{c}{12(x+2)}$$

and $Z_1 = a + bi$, $Z_2 = b + ci$ and $Z_3 = c + ai$ where $i = \sqrt{-1}$. Then Z_1, Z_2, Z_3 represent

(A) collinear points
(B) vertices of an equilateral triangle
(C) vertices of an isosceles triangle
(D) vertices of a right-angled triangle

3. Let

$$\frac{x^4}{x^2-3x+2} = f(x) + \frac{A}{x-1} + \frac{B}{x-2}$$

Then

(A) $f(x) = 0$ has irrational roots
(B) $f(x) = 0$ has no real roots
(C) $f(x) + A + B = 0$ has integer roots
(D) $f(x) + A + B = 0$ has no real roots

4. If

$$\frac{1}{x^3(x+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x+2}$$

then

(A) $A + D = 0$
(B) $C + B = 0$
(C) the number of permutations of the values of A, B, C and $-D$ is 12
(D) $A + B = \frac{1}{8}$

ANSWERS

Single Correct Choice Type Questions

1. (C)
2. (B)
3. (C)
4. (D)
5. (A)
6. (B)
7. (C)
8. (D)
9. (A)
10. (A)

Multiple Correct Choice Type Questions

1. (A), (B), (C)
2. (C), (D)
3. (B), (D)
4. (A), (B), (C)

Index

M

N

O

P

Q

R

S

T

U

V

Z

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